Dynamical Pose Estimation

Supplementary Material

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A1. Semantic Uncertainty Ellipsoid

The idea of a semantic uncertainty ellipsoid (SUE) is borrowed from the *error ellipsoid* that is commonly used in statistics, but we apply it to category-level pose estimation for the first time. Given a library of K shapes of a category, $\mathcal{B}_k, k = 1, \dots, K$, where each $\mathcal{B}_k \in \mathbb{R}^{3\times N}$ contains a list of N semantic keypoints. For example, in the category of car, $\mathcal{B}_k$ can be different CAD models from different car manufacturers, with annotations of certain semantic keypoints that exist for all CAD models, *e.g.*, wheels, mirrors. Then we build a SUE for the i-th semantic keypoint as follows. We first compute the average position of the semantic keypoint as

$$\boldsymbol{b}_i = \frac{1}{K}\sum_{k=1}^{K} \mathcal{B}_k(i), \tag{A1}$$

where $\mathcal{B}_k(i)$ denotes the location of the i-th keypoint in the k-th shape. We then compute the covariance matrix for the i-th keypoint

$$\boldsymbol{C}_i = \frac{1}{K}\sum_{k=1}^{K} (\mathcal{B}_k(i) - \boldsymbol{b}_i)(\mathcal{B}_k(i) - \boldsymbol{b}_i)^{\mathsf{T}}. \tag{A2}$$

Using $\boldsymbol{b}_i$ and $\boldsymbol{C}_i$, we assume that the position of the i-th semantic keypoint, denoted as $\boldsymbol{x}_i$, satisfies the following multivariate Gaussian distribution:

$$p(\boldsymbol{x}_i) = \frac{\exp\left(-\frac{1}{2}(\boldsymbol{x}_i - \boldsymbol{b}_i)^{\mathsf{T}}\boldsymbol{C}_i^{-1}(\boldsymbol{x}_i - \boldsymbol{b}_i)\right)}{\sqrt{(2\pi)^3\,|\boldsymbol{C}_i|}}, \tag{A3}$$

where $|\boldsymbol{C}_i| \triangleq \det(\boldsymbol{C}_i)$ denotes the determinant of $\boldsymbol{C}_i$. Under this assumption, it is known that the square of the Mahalanobis distance, *i.e.*, $(\boldsymbol{x}_i - \boldsymbol{b}_i)^{\mathsf{T}}\boldsymbol{C}_i^{-1}(\boldsymbol{x}_i - \boldsymbol{b}_i)$ satisfies a chi-square distribution with three degrees of freedom:

$$(\boldsymbol{x}_i - \boldsymbol{b}_i)^{\mathsf{T}}\boldsymbol{C}_i^{-1}(\boldsymbol{x}_i - \boldsymbol{b}_i) \sim \chi_3^2. \tag{A4}$$

Therefore, given a confidence $\eta \in (0, 1)$, we have:

$$\mathbb{P}\left((\boldsymbol{x}_i - \boldsymbol{b}_i)^{\mathsf{T}}\boldsymbol{C}_i^{-1}(\boldsymbol{x}_i - \boldsymbol{b}_i) \leq \chi_3^2(\eta)\right) = \eta, \tag{A5}$$

where $\chi_3^2(\eta)$ corresponds to the probabilistic quantile of confidence η. This states that, with probability η, the point $\boldsymbol{x}_i$ lies inside the 3D ellipsoid

$$\frac{(\boldsymbol{x}_i - \boldsymbol{b}_i)^{\mathsf{T}}\boldsymbol{C}_i^{-1}(\boldsymbol{x}_i - \boldsymbol{b}_i)}{\chi_3^2(\eta)} \leq 1. \tag{A6}$$

We call this ellipsoid the SUE with confidence η, and in our experiments we choose $\eta = 0.5$. Fig. A1 shows two examples of category models with SUEs.

A2. Proof of Theorem 6

Proof. Results 1-6 are basic results in 3D geometry [2, 5] that can be verified by inspection. We now prove 7 and 8. The proof for 7 is based on [4], while the proof for 8 is new.

Point-Ellipsoid (PE). According to the definition of the shortest distance (2), the point in the ellipsoid $E(\boldsymbol{y}, \boldsymbol{A})$ that attains the shortest distance to $\boldsymbol{x}$ is the minimizer of the following optimization:

$$\min_{\boldsymbol{z}\in\mathbb{R}^3} \quad \|\boldsymbol{z} - \boldsymbol{x}\|^2 \tag{A7}$$

$$\text{subject to} \quad (\boldsymbol{z} - \boldsymbol{y})^{\mathsf{T}}\boldsymbol{A}(\boldsymbol{z} - \boldsymbol{y}) \leq 1. \tag{A8}$$

Problem (A7) has a single inequality constraint and hence satisfies the *linear independence constraint qualification* (LICQ) [1]. Therefore, any solution of (A7) must satisfy the KKT conditions, *i.e.* there exist $(\boldsymbol{z}, \lambda)$ such that:

$$(\boldsymbol{z} - \boldsymbol{y})^{\mathsf{T}}\boldsymbol{A}(\boldsymbol{z} - \boldsymbol{y}) - 1 \leq 0 \tag{A9}$$

$$\lambda \geq 0 \tag{A10}$$

$$\nabla_{\boldsymbol{z}}\mathcal{L} \triangleq 2(\boldsymbol{z} - \boldsymbol{x}) + 2\lambda\boldsymbol{A}(\boldsymbol{z} - \boldsymbol{y}) = \boldsymbol{0} \tag{A11}$$

$$\lambda((\boldsymbol{z} - \boldsymbol{y})^{\mathsf{T}}\boldsymbol{A}(\boldsymbol{z} - \boldsymbol{y}) - 1) = 0 \tag{A12}$$

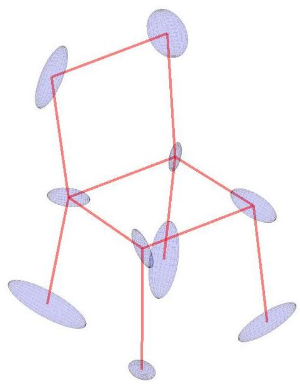
where $\mathcal{L} \triangleq \|\boldsymbol{z} - \boldsymbol{x}\|^2 + \lambda((\boldsymbol{z} - \boldsymbol{y})^{\mathsf{T}}\boldsymbol{A}(\boldsymbol{z} - \boldsymbol{y}) - 1)$ is the Lagrangian. Let $\boldsymbol{z}_y \triangleq \boldsymbol{z} - \boldsymbol{y}, \boldsymbol{x}_y \triangleq \boldsymbol{x} - \boldsymbol{y}$, the equations above can be written as:

$$\boldsymbol{z}_y^{\mathsf{T}}\boldsymbol{A}\boldsymbol{z}_y - 1 \leq 0 \tag{A13}$$

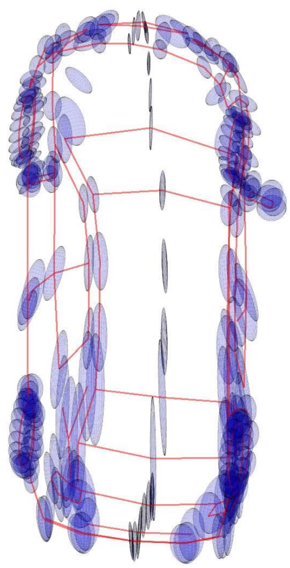
$$\lambda \geq 0 \tag{A14}$$

$$(\lambda\boldsymbol{A} + \mathbf{I})\boldsymbol{z}_y = \boldsymbol{x}_y \tag{A15}$$

$$\lambda(\boldsymbol{z}_y^{\mathsf{T}}\boldsymbol{A}\boldsymbol{z}_y - 1) = 0 \tag{A16}$$



(a) Chair in PASCAL3D+ [10].



(b) Car in FG3DCar [6].

Figure A1: Represent category models using a collection of semantic uncertainty ellipsoids (SUEs).

Now we can discuss two cases: (i) if $\lambda = 0$, then from (A15), we have $\boldsymbol{z}_y = \boldsymbol{x}_y$, thus $\boldsymbol{z} = \boldsymbol{x}$ attains the global minimum $\|\boldsymbol{z} - \boldsymbol{x}\| = 0$ (the objective function is lower bounded by 0). In order to satisfy feasibility (A13), $\boldsymbol{x}_y^\mathsf{T} \boldsymbol{A} \boldsymbol{x}_y \leq 1$ must hold and $\boldsymbol{x}$ has to belong to the ellipsoid; (ii) if $\lambda > 0$, then from (A16) we have $\boldsymbol{z}_y^\mathsf{T} \boldsymbol{A} \boldsymbol{z}_y = 1$ and the optimal $\boldsymbol{z}$ lies on the surface of the ellipsoid. Because $\lambda > 0$ and $\boldsymbol{A} \succ 0$, $\lambda \boldsymbol{A} + \mathbf{I}$ must be invertible and eq. (A15) yields:

$$\boldsymbol{z}_y(\lambda) = (\lambda \boldsymbol{A} + \mathbf{I})^{-1} \boldsymbol{x}_y, \tag{A17}$$

where we use $\boldsymbol{z}_y(\lambda)$ to indicate $\boldsymbol{z}_y$ as a function of λ. Substituting this expression into $\boldsymbol{z}_y^\mathsf{T} \boldsymbol{A} \boldsymbol{z}_y = 1$, we have that:

$$g(\lambda) \triangleq \boldsymbol{z}_y(\lambda)^\mathsf{T} \boldsymbol{A} \boldsymbol{z}_y(\lambda) - 1 = 0. \tag{A18}$$

To see how many roots $g(\lambda)$ has in the range $\lambda > 0$, we note:

$$g(\lambda = 0) = \boldsymbol{x}_y^\mathsf{T} \boldsymbol{A} \boldsymbol{x}_y - 1 \tag{A19}$$

$$g(\lambda \to +\infty) = -1 \tag{A20}$$

and compute the derivative of $g(\lambda)$:

$$\begin{aligned} g'(\lambda) &= 2\boldsymbol{z}_y'(\lambda)^\mathsf{T} \boldsymbol{A} \boldsymbol{z}_y(\lambda) \\ &= -2\boldsymbol{z}_y(\lambda)^\mathsf{T} \left(\boldsymbol{A} (\lambda \boldsymbol{A} + \mathbf{I})^{-1} \boldsymbol{A} \right) \boldsymbol{z}_y(\lambda) < 0 \end{aligned} \tag{A21}$$

where $\boldsymbol{z}_y'(\lambda)$, the derivative of $\boldsymbol{z}_y(\lambda)$ w.r.t. λ, can be obtained by differentiating both sides of eq. (A15) w.r.t. λ:

$$\begin{aligned} &\boldsymbol{A} \boldsymbol{z}_y(\lambda) + (\lambda \boldsymbol{A} + \mathbf{I}) \boldsymbol{z}_y'(\lambda) = 0 \Rightarrow \\ &\boldsymbol{z}_y'(\lambda) = -(\lambda \boldsymbol{A} + \mathbf{I})^{-1} \boldsymbol{A} \boldsymbol{z}_y(\lambda). \end{aligned} \tag{A22}$$

The last inequality in (A21) follows from the positive definiteness of the matrix $\boldsymbol{A}(\lambda \boldsymbol{A} + \mathbf{I})^{-1} \boldsymbol{A}$. Eqs. (A19)-(A21) show that the function $g(\lambda)$ is monotonically decreasing for $\lambda > 0$. Therefore, $g(\lambda)$ has a unique root in the range $\lambda > 0$ if and only if $g(0) > 0$, *i.e.* $\boldsymbol{x}_y^\mathsf{T} \boldsymbol{A} \boldsymbol{x}_y > 1$. Lastly, to see the solution is indeed a minimizer, observe that the Hessian of the Lagrangian w.r.t. $\boldsymbol{z}$ is:

$$\nabla_{\boldsymbol{z}\boldsymbol{z}} \mathcal{L} = 2(\lambda \boldsymbol{A} + \mathbf{I}) \succ 0, \tag{A23}$$

which is positive definite, a sufficient condition for $\boldsymbol{z}$ to be a global minimizer (because there is a single local minimizer, it is also global), concluding the proof of 7.

We note that the proof above also provides an efficient algorithm to numerically compute the root of $g(\lambda) = 0$ and find the optimal $\boldsymbol{z}$, using Newton's root finding method [8]. To do so, we initialize $\lambda_0 = 0$, and iteratively perform:

$$\lambda_k = \lambda_{k-1} - \frac{g(\lambda_{k-1})}{g'(\lambda_{k-1})}, \quad k = 1, \ldots \tag{A24}$$

until $g(\lambda_k) = 0$ (up to numerical accuracy). This algorithm has local quadratic convergence and typically finds the root within 20 iterations (as we will show in Section 4).

Ellipsoid-Line (EL). First we decide if the line intersects with the ellipsoid. Since any point on the line $L(\boldsymbol{y}, \boldsymbol{v})$ can be written as $\boldsymbol{y} + \alpha \boldsymbol{v}$ for some $\alpha \in \mathbb{R}$, the line intersects with the ellipsoid if and only if:

$$(\boldsymbol{y} + \alpha \boldsymbol{v} - \boldsymbol{x})^\mathsf{T} \boldsymbol{A} (\boldsymbol{y} + \alpha \boldsymbol{v} - \boldsymbol{x}) = 1 \tag{A25}$$

has real solutions. Let $\boldsymbol{y}_x \triangleq \boldsymbol{y} - \boldsymbol{x}$, eq (A25) simplifies as:

$$(\boldsymbol{v}^\mathsf{T} \boldsymbol{A} \boldsymbol{v}) \alpha^2 + (2\boldsymbol{y}_x^\mathsf{T} \boldsymbol{A} \boldsymbol{v}) \alpha + (\boldsymbol{y}_x^\mathsf{T} \boldsymbol{A} \boldsymbol{y}_x - 1) = 0, \tag{A26}$$

where $\boldsymbol{v}^\top \boldsymbol{A}\boldsymbol{v} > 0$ due to $\boldsymbol{A} \succ 0$. The discriminant of the quadratic polynomial is:

$$\Delta = 2\sqrt{(\boldsymbol{y}_x^\top \boldsymbol{A}\boldsymbol{v})^2 - (\boldsymbol{v}^\top \boldsymbol{A}\boldsymbol{v})(\boldsymbol{y}_x^\top \boldsymbol{A}\boldsymbol{y}_x - 1)}. \tag{A27}$$

Therefore, eq. (A26) has two roots (counting multiplicity):

$$\alpha_{1,2} = \frac{-\boldsymbol{y}_x^\top \boldsymbol{A}\boldsymbol{v} \pm \Delta}{\boldsymbol{v}^\top \boldsymbol{A}\boldsymbol{v}} \tag{A28}$$

if $\Delta \geq 0$, and zero roots otherwise. Accordingly, when $\Delta \geq 0$, the line intersects the ellipsoid and the entire line segment $\boldsymbol{y} + \alpha\boldsymbol{v} : \alpha \in [\alpha_1, \alpha_2]$ is inside the ellipsoid, hence the shortest distance is zero.

On the other hand, when $\Delta_\alpha < 0$, there is no intersection between the line and the ellipsoid, we seek to find the shortest distance pair by solving the following optimization:

$$\min_{\boldsymbol{z}\in\mathbb{R}^3, \alpha\in\mathbb{R}} \quad \|\boldsymbol{z} - (\boldsymbol{y} + \alpha\boldsymbol{v})\|^2 \tag{A29}$$

$$\text{subject to} \quad (\boldsymbol{z} - \boldsymbol{x})^\top \boldsymbol{A}(\boldsymbol{z} - \boldsymbol{x}) \leq 1 \tag{A30}$$

Similarly, problem (A29) satisfies LICQ and we write down the KKT conditions:

$$(\boldsymbol{z} - \boldsymbol{x})^\top \boldsymbol{A}(\boldsymbol{z} - \boldsymbol{x}) \leq 1 \tag{A31}$$

$$\lambda \geq 0 \tag{A32}$$

$$\nabla_{\boldsymbol{x}}\mathcal{L} \triangleq 2(\boldsymbol{z} - \boldsymbol{y} - \alpha\boldsymbol{v}) + 2\lambda\boldsymbol{A}(\boldsymbol{z} - \boldsymbol{x}) = \boldsymbol{0} \tag{A33}$$

$$\nabla_\alpha\mathcal{L} \triangleq 2\boldsymbol{v}^\top(\alpha\boldsymbol{v} + \boldsymbol{y} - \boldsymbol{z}) = 0 \tag{A34}$$

$$\lambda((\boldsymbol{z} - \boldsymbol{x})^\top \boldsymbol{A}(\boldsymbol{z} - \boldsymbol{x}) - 1) = 0. \tag{A35}$$

Let $\boldsymbol{z}_x \triangleq \boldsymbol{z} - \boldsymbol{x}$, we can simplify the equations above:

$$\boldsymbol{z}_x^\top \boldsymbol{A}\boldsymbol{z}_x - 1 \leq 0 \tag{A36}$$

$$\lambda \geq 0 \tag{A37}$$

$$\left(\lambda\boldsymbol{A} + (\mathbf{I} - \boldsymbol{v}\boldsymbol{v}^\top)\right)\boldsymbol{z}_x = (\mathbf{I} - \boldsymbol{v}\boldsymbol{v}^\top)\boldsymbol{y}_x \tag{A38}$$

$$\lambda(\boldsymbol{z}_x^\top \boldsymbol{A}\boldsymbol{z}_x - 1) = 0, \tag{A39}$$

where we have combined (A33) and (A34) by first obtaining:

$$\alpha = \boldsymbol{v}^\top(\boldsymbol{z}_x - \boldsymbol{y}_x), \tag{A40}$$

from (A34) and then inserting it to (A33). Now we can discuss two cases for the KKT conditions (A36)-(A39). (i) If $\lambda = 0$, then eq. (A38) reads:

$$(\mathbf{I} - \boldsymbol{v}\boldsymbol{v}^\top)(\boldsymbol{z} - \boldsymbol{y}) = \boldsymbol{0}, \tag{A41}$$

which indicates that either $\boldsymbol{z} = \boldsymbol{y}$ or $\boldsymbol{z} - \boldsymbol{y} = k\boldsymbol{v}$ for some $k \neq 0$ (note that $\boldsymbol{v}$ is the eigenvector of $\mathbf{I} - \boldsymbol{v}\boldsymbol{v}^\top$ with associated eigenvalue 0), which both mean that $\boldsymbol{z}$ lies on the line $L(\boldsymbol{y}, \boldsymbol{v})$. This is in contradiction with the assumption that there is no intersection between the line and the ellipsoid. (ii) Therefore, $\lambda > 0$ and $\boldsymbol{z}_x^\top \boldsymbol{A}\boldsymbol{z}_x = 1$. In this case, we write $\boldsymbol{V} \triangleq \mathbf{I} - \boldsymbol{v}\boldsymbol{v}^\top \succeq 0$. Since $\lambda > 0$, $\boldsymbol{A} \succ 0$, we have $\lambda\boldsymbol{A} + \boldsymbol{V} \succ 0$ is invertible, and we get from (A38) that:

$$\boldsymbol{z}_x(\lambda) = (\lambda\boldsymbol{A} + \boldsymbol{V})^{-1}\boldsymbol{V}\boldsymbol{y}_x. \tag{A42}$$

Substituting it back to $\boldsymbol{z}_x^\top \boldsymbol{A}\boldsymbol{z}_x = 1$, we have that λ must satisfy:

$$g(\lambda) \triangleq \boldsymbol{z}_x(\lambda)^\top \boldsymbol{A}\boldsymbol{z}_x(\lambda) - 1 = 0. \tag{A43}$$

To count the number of roots of $g(\lambda)$ within $\lambda > 0$, we note that:

$$g(\lambda \to 0_+) = \boldsymbol{y}_x^\top \boldsymbol{A}\boldsymbol{y}_x - 1 > 0 \tag{A44}$$

$$g(\lambda \to +\infty) = -1 < 0 \tag{A45}$$

where $\boldsymbol{y}_x^\top \boldsymbol{A}\boldsymbol{y}_x > 1$ because there is no intersection between the line and the ellipsoid and $\boldsymbol{y}$ must lie outside the ellipsoid. We then compute the derivative of $g(\lambda)$ w.r.t. λ:

$$\begin{aligned} g'(\lambda) &= 2\boldsymbol{z}_x'(\lambda)^\top \boldsymbol{A}\boldsymbol{z}_x(\lambda) \\ &= -2\boldsymbol{z}_x(\lambda)^\top \boldsymbol{A}(\lambda\boldsymbol{A} + \boldsymbol{V})^{-1}\boldsymbol{A}\boldsymbol{z}_x(\lambda) < 0 \end{aligned} \tag{A46}$$

where $\boldsymbol{z}_x'(\lambda) = -(\lambda\boldsymbol{A} + \boldsymbol{V})^{-1}\boldsymbol{A}\boldsymbol{z}_x(\lambda)$ can be obtained by differentiating both sides of eq. (A38) w.r.t. λ. Eqs. (A44)-(A46) show that $g(\lambda)$ is a monotonically decreasing function in $\lambda > 0$, and a unique root exists in the range $\lambda > 0$. Finally, problem (A29) admits a global minimizer due to positive definiteness of the Hessian of the Lagrangian.

The proof above suggests that we can also use Newton's root finding algorithm as in (A24) to compute the root of $g(\lambda)$. To make sure $g'(\lambda)$ (eq. (A46)) is well defined at λ_0, we initialize $\lambda_0 = 10^{-6}$ instead of $\lambda_0 = 0$ in the PE case. □

A3. Proof of Lemma 10

Proof. Let $(\underline{\boldsymbol{x}}_i, \underline{\boldsymbol{y}}_i) \in (\boldsymbol{T} \otimes X_i, Y_i)_{\mathrm{p}}$ be the two endpoints of the shortest distance pair, we have that the cost function of (1) is $\sum_{i=1}^N \|\underline{\boldsymbol{x}}_i - \underline{\boldsymbol{y}}_i\|^2$. On the other hand, the potential energy of the system is stored in the virtual springs as $\sum_{i=1}^N \frac{k}{2}\|\underline{\boldsymbol{x}}_i - \underline{\boldsymbol{y}}_i\|^2$, which equates the cost if $k = 2$. □

A4. Proof of Theorem 11

Proof. Let $\mathcal{X} = \{P(\boldsymbol{x}_i)\}_{i=1}^N$ and $\mathcal{Y} = \{P(\boldsymbol{y}_i)\}_{i=1}^N$ be two sets of 3D points, and with slight abuse of notation, we will use $\boldsymbol{x}_i \in \mathbb{R}^3$ and $\boldsymbol{y}_i \in \mathbb{R}^3$ to denote the 3D points and their coordinates interchangeably. Under this setup of Example 1, problem (1) becomes

$$\min_{\boldsymbol{R}\in\mathrm{SO}(3), \boldsymbol{t}\in\mathbb{R}^3} \sum_{i=1}^N \|\boldsymbol{y}_i - \boldsymbol{R}\boldsymbol{x}_i - \boldsymbol{t}\|^2. \tag{A47}$$

Let

$$\bar{\boldsymbol{x}} = \frac{1}{N}\sum_{i=1}^N \boldsymbol{x}_i, \boldsymbol{x}_{\mathrm{r}_i} = \boldsymbol{x}_i - \bar{\boldsymbol{x}}, \boldsymbol{J} = -m\sum_{i=1}^N [\boldsymbol{x}_{\mathrm{r}_i}]_\times^2 \tag{A48}$$

be the (initial) center of mass of $\mathcal{X}$, relative positions of $\boldsymbol{x}_i$ w.r.t. center of mass, and moment of inertia of $\mathcal{X}$. According to eq. (13), the total external force is

$$\boldsymbol{f}_i' = k(\boldsymbol{y}_i - \boldsymbol{R}_q \boldsymbol{x}_{\mathrm{r}_i} - \boldsymbol{x}_{\mathrm{c}}) - \mu m(\boldsymbol{v}_{\mathrm{c}} + \boldsymbol{R}_q(\boldsymbol{\omega} \times \boldsymbol{x}_{\mathrm{r}_i})) \quad \boldsymbol{f} = \sum_{i=1}^{N} \boldsymbol{f}_i', \tag{A49}$$

where $k > 0$ is the constant spring coefficient. Similarly, according to eq. (14), the total torque in the body frame is

$$\boldsymbol{\tau} = \sum_{i=1}^{N} \boldsymbol{x}_{\mathrm{r}_i} \times (\boldsymbol{R}_q^{\mathsf{T}} \boldsymbol{f}_i'). \tag{A50}$$

Now we analyze how many equilibrium points eq. (12) has. Towards this goal, setting the first two equations of (12) to zero, we get

$$\boldsymbol{v}_{\mathrm{c}} = \boldsymbol{0}, \quad \boldsymbol{\omega} = \boldsymbol{0}, \tag{A51}$$

which implies that the system must have zero linear velocity and angular velocity at equilibrium. Substituting (A51) to the force and torque expressions in (A49) and (A50), we have

$$\boldsymbol{f}_i' = k(\boldsymbol{y}_i - \boldsymbol{R}_q \boldsymbol{x}_{\mathrm{r}_i} - \boldsymbol{x}_{\mathrm{c}}), \tag{A52}$$

$$\boldsymbol{f} = k \sum_{i=1}^{N} \boldsymbol{y}_i - \boldsymbol{R}_q \boldsymbol{x}_{\mathrm{r}_i} - \boldsymbol{x}_{\mathrm{c}}, \tag{A53}$$

$$\begin{aligned} \boldsymbol{\tau} &= k \sum_{i=1}^{N} \boldsymbol{x}_{\mathrm{r}_i} \times (\boldsymbol{R}_q^{\mathsf{T}} (\boldsymbol{y}_i - \boldsymbol{R}_q \boldsymbol{x}_{\mathrm{r}_i} - \boldsymbol{x}_{\mathrm{c}})) \\ &= k \sum_{i=1}^{N} \boldsymbol{x}_{\mathrm{r}_i} \times \boldsymbol{R}_q^{\mathsf{T}} (\boldsymbol{y}_i - \boldsymbol{x}_{\mathrm{c}}), \end{aligned} \tag{A54}$$

where we have used the equality that $\boldsymbol{x}_{\mathrm{r}_i} \times \boldsymbol{x}_{\mathrm{r}_i} = \boldsymbol{0}$. Now we set the last two equations of (12) to zero (*i.e.*, the system has no linear or angular acceleration), we have that

$$\boldsymbol{f} = \boldsymbol{0}, \quad \boldsymbol{\tau} = \boldsymbol{0}, \tag{A55}$$

which implies that external forces and torques must balance at an equilibrium point. From $\boldsymbol{f} = \boldsymbol{0}$ and the expression of $\boldsymbol{f}$ in (A53), we obtain

$$\sum_{i=1}^{N} \boldsymbol{y}_i - \boldsymbol{R}_q \boldsymbol{x}_{\mathrm{r}_i} - \boldsymbol{x}_{\mathrm{c}} = \boldsymbol{0} \Longrightarrow \tag{A56}$$

$$\boxed{\boldsymbol{x}_{\mathrm{c}} = \frac{1}{N} \left(\sum_{i=1}^{N} \boldsymbol{y}_i - \boldsymbol{R}_q \boldsymbol{x}_{\mathrm{r}_i} \right) = \frac{1}{N} \sum_{i=1}^{N} \boldsymbol{y}_i := \bar{\boldsymbol{y}}}, \tag{A57}$$

where we have used

$$\frac{1}{N} \sum_{i=1}^{N} \boldsymbol{R}_q \boldsymbol{x}_{\mathrm{r}_i} = \frac{1}{N} \boldsymbol{R}_q \sum_{i=1}^{N} \boldsymbol{x}_{\mathrm{r}_i} = \boldsymbol{0} \tag{A58}$$

from the definition of $\boldsymbol{x}_{\mathrm{r}_i}$ in (A48). Eq. (A57) states that $\mathcal{X}$ and $\mathcal{Y}$ must have their center of mass aligned at an equilibrium point. Now using a similar notation $\boldsymbol{y}_{\mathrm{r}_i} \triangleq \boldsymbol{y}_i - \bar{\boldsymbol{y}}$, $\boldsymbol{\tau} = \boldsymbol{0}$ from eq. (A54) implies that

$$\boxed{\sum_{i=1}^{N} \boldsymbol{x}_{\mathrm{r}_i} \times \boldsymbol{R}_q^{\mathsf{T}} \boldsymbol{y}_{\mathrm{r}_i} = \boldsymbol{0}}. \tag{A59}$$

Eq. (A57) and (A59) are the necessary and sufficient condition for an equilibrium point $\dot{\boldsymbol{s}} = \boldsymbol{0}$. Now we are ready to prove the four claims in Theorem 11. We first prove (ii).

(ii): Optimal solution is an equilibrium point. To show the optimal solution of problem (A47) is an equilibrium point, we will write down its closed-form solution and show that it satisfies (A57) and (A59).

Lemma A1 (Closed-form Point Cloud Registration)**.** *The global optimal solution to* (A47) *is*

$$\boldsymbol{t}^\star = \bar{\boldsymbol{y}} - \boldsymbol{R}^\star \bar{\boldsymbol{x}}, \tag{A60}$$

$$\boldsymbol{R}^\star = \boldsymbol{U}_+ \boldsymbol{V}_+^{\mathsf{T}}, \tag{A61}$$

where $\boldsymbol{U}_+, \boldsymbol{V}_+ \in \mathrm{SO}(3)$ *are obtained from the singular value decomposition:*

$$\boldsymbol{M} = \sum_{i=1}^{N} \boldsymbol{y}_{\mathrm{r}_i} \boldsymbol{x}_{\mathrm{r}_i}^{\mathsf{T}} = \boldsymbol{U}\boldsymbol{S}\boldsymbol{V}^{\mathsf{T}}, \quad \boldsymbol{U}, \boldsymbol{V} \in \mathrm{O}(3), \tag{A62}$$

$$\boldsymbol{U}_+ = \boldsymbol{U} \operatorname{diag}\left([1, 1, \det(\boldsymbol{U})]\right) \in \mathrm{SO}(3), \tag{A63}$$

$$\boldsymbol{V}_+ = \boldsymbol{V} \operatorname{diag}\left([1, 1, \det(\boldsymbol{V})]\right) \in \mathrm{SO}(3). \tag{A64}$$

Using $\boldsymbol{U}_+, \boldsymbol{V}_+$*, we have* $\boldsymbol{M} = \boldsymbol{U}_+ \boldsymbol{S}' \boldsymbol{V}_+^{\mathsf{T}}$*, with*

$$\boldsymbol{S}' = \operatorname{diag}\left([s_1, s_2, s_3 \det(\boldsymbol{U}\boldsymbol{V})]\right). \tag{A65}$$

Lemma A1 is a standard result in point cloud registration [3, 7]. Using $(\boldsymbol{t}^\star, \boldsymbol{R}^\star)$, one immediately sees that the center of mass of $\mathcal{X}$ is transformed to

$$\boldsymbol{x}_{\mathrm{c}} = \boldsymbol{R}^\star \bar{\boldsymbol{x}} + \boldsymbol{t}^\star = \boldsymbol{R}^\star \bar{\boldsymbol{x}} + \bar{\boldsymbol{y}} - \boldsymbol{R}^\star \bar{\boldsymbol{x}} = \bar{\boldsymbol{y}} \tag{A66}$$

and coincide with $\bar{\boldsymbol{y}}$, hence satisfies eq. (A57). Now replace $\boldsymbol{R}_q$ with $\boldsymbol{R}^\star = \boldsymbol{U}_+ \boldsymbol{V}_+^{\mathsf{T}}$ in eq. (A59), our goal is to show

$$\boldsymbol{\tau}(\boldsymbol{R}^\star) \triangleq \sum_{i=1}^{N} \boldsymbol{x}_{\mathrm{r}_i} \times (\boldsymbol{V}_+ \boldsymbol{U}_+^{\mathsf{T}} \boldsymbol{y}_{\mathrm{r}_i}) \tag{A67}$$

equal to zero. Towards this goal, we will show each entry of $\boldsymbol{\tau}(\boldsymbol{R}^\star)$ is zero, *i.e.*, $\boldsymbol{e}_j^{\mathsf{T}} \boldsymbol{\tau}(\boldsymbol{R}^\star) = 0$ for $j = 1, 2, 3$. Note that

$$e_1^\top \tau(R^\star) = \sum_{i=1}^N e_1^\top [x_{r_i}]_\times V_+ U_+^\top y_{r_i} \quad (A68)$$

$$= \sum_{i=1}^N x_{r_i}^\top (e_2 e_3^\top - e_3 e_2^\top) V_+ U_+^\top y_{r_i} \quad (A69)$$

$$= \sum_{i=1}^N \mathrm{tr}\left(U_+^\top y_{r_i} x_{r_i}^\top (e_2 e_3^\top - e_3 e_2^\top) V_+\right) \quad (A70)$$

$$= \mathrm{tr}\left(U_+^\top \left(\sum_{i=1}^N y_{r_i} x_{r_i}^\top\right)(e_2 e_3^\top - e_3 e_2^\top) V_+\right) \quad (A71)$$

$$= \mathrm{tr}\left(U_+^\top M (e_2 e_3^\top - e_3 e_2^\top) V_+\right) \quad (A72)$$

$$= \mathrm{tr}\left(U_+^\top U_+ S' V_+^\top (e_2 e_3^\top - e_3 e_2^\top) V_+\right) \quad (A73)$$

$$= \mathrm{tr}\left(V_+ S' V_+^\top (e_2 e_3^\top - e_3 e_2^\top)\right) \quad (A74)$$

$$= [V_+ S' V_+^\top]_{32} - [V_+ S' V_+^\top]_{23} = 0 \quad (A75)$$

where the last "$= 0$" holds because $V_+ S' V_+^\top$ is an symmetric matrix, and we have used the fact that

$$e_1^\top [x]_\times \equiv x^\top (e_2 e_3^\top - e_3 e_2^\top), \forall x \in \mathbb{R}^3. \quad (A76)$$

By the same token, one can verify that

$$e_2^\top \tau(R^\star) = [V_+ S' V_+^\top]_{13} - [V_+ S' V_+^\top]_{31} = 0, \quad (A77)$$

$$e_3^\top \tau(R^\star) = [V_+ S' V_+^\top]_{21} - [V_+ S' V_+^\top]_{12} = 0. \quad (A78)$$

Therefore, the optimal solution $(t^\star, R^\star)$ is an equilibrium point of the system (12).

(i) and (iii): Three spurious equilibrium points. We now show that besides the optimal equilibrium point $(t^\star, R^\star)$, the equation (A59) has three and only three different solutions if $s_1 > s_2 > s_3 > 0$, which we denote as *generic configuration*. Towards this goal, let us assume there is a rotation matrix R_q that satisfies (A59), and we write it as

$$R_q = U_+ \bar{R} V_+^\top. \quad (A79)$$

Note that such a parametrization is always possible with

$$\bar{R} = U_+^\top R_q V_+ \in \mathrm{SO}(3). \quad (A80)$$

Using this parametrization, $\tau(R_q) = \mathbf{0}$ is equivalent to

$$Z \triangleq V_+ \bar{R} S' V_+^\top \quad (A81)$$

being symmetric (using similar derivations as in (A68)-(A75)). Then it is easy to see that Z being symmetric is equivalent to $\bar{R} S'$ being symmetric because $\bar{R} S' = V_+^\top Z V_+$. Explicitly, we require

$$\bar{R} S' = (\bar{R} S')^\top = S' \bar{R}^\top. \quad (A82)$$

Since $s_1 > s_2 > s_3 > 0$, S' is invertible and $(S')^{-1} = \mathrm{diag}\left([1/s_1, 1/s_2, 1/s_3']\right)$ with $s_3' = s_3 \det(UV)$. Therefore, $\bar{R} S' = S' \bar{R}^\top$ is equivalent to

$$\bar{R} = S' \bar{R}^\top (S')^{-1} \Leftrightarrow \quad (A83)$$

$$\underbrace{\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}}_{\bar{R} \in \mathrm{SO}(3)} = \underbrace{\begin{bmatrix} r_{11} & \frac{s_1}{s_2} r_{21} & \frac{s_1}{s_3'} r_{31} \\ \frac{s_2}{s_1} r_{12} & r_{22} & \frac{s_2}{s_3'} r_{32} \\ \frac{s_3'}{s_1} r_{13} & \frac{s_3'}{s_2} r_{23} & r_{33} \end{bmatrix}}_{S' \bar{R}^\top (S')^{-1}} \quad (A84)$$

Now we use $\left|\frac{s_1}{s_2}\right|, \left|\frac{s_1}{s_3'}\right|, \left|\frac{s_2}{s_3'}\right| > 1$, and the fact that both sides of (A84) are rotation matrices:

$$r_{11}^2 + r_{21}^2 + r_{31}^2 = r_{11}^2 + \left(\frac{s_1}{s_2}\right)^2 r_{21}^2 + \left(\frac{s_1}{s_3'}\right)^2 r_{31}^2 = 1, \quad (A85)$$

$$r_{33}^2 + r_{32}^2 + r_{31}^2 = r_{33}^2 + \left(\frac{s_2}{s_3'}\right)^2 r_{32}^2 + \left(\frac{s_1}{s_3'}\right)^2 r_{31}^2 = 1, \quad (A86)$$

which implies that

$$\left(\left(\frac{s_1}{s_2}\right)^2 - 1\right) r_{21}^2 + \left(\left(\frac{s_1}{s_3'}\right)^2 - 1\right) r_{31}^2 = 0, \quad (A87)$$

$$\left(\left(\frac{s_2}{s_3'}\right)^2 - 1\right) r_{32}^2 + \left(\left(\frac{s_1}{s_3'}\right)^2 - 1\right) r_{31}^2 = 0, \quad (A88)$$

and hence $r_{21} = r_{31} = r_{32} = 0$. Substituting them back into (A84), we have $r_{12} = r_{13} = r_{23} = 0$. Therefore, we conclude that $\bar{R}$ is a diagonal matrix. However, there are only four rotation matrices that are diagonal:

$$\bar{R}_1 = \mathrm{diag}\left([1, 1, 1]\right), \quad (A89)$$

$$\bar{R}_2 = \mathrm{diag}\left([1, -1, -1]\right), \quad (A90)$$

$$\bar{R}_3 = \mathrm{diag}\left([-1, 1, -1]\right), \quad (A91)$$

$$\bar{R}_4 = \mathrm{diag}\left([-1, -1, 1]\right). \quad (A92)$$

As a result, the equation (A59) has four and only four solutions. Note that $\bar{R}_1 = \mathbf{I}_3$ corresponds to the optimal equilibrium point $R^\star$, and the angular distance between $R^\star$ and the other three spurious equilibrium points $U_+ \bar{R}_j V_+^\top, j = 2, 3, 4$ is:

$$\left|\arccos\left(\frac{\mathrm{tr}\left((R^\star)^\top U_+ \bar{R}_j V_+^\top\right) - 1}{2}\right)\right|$$

$$= \left|\arccos\left(\frac{\mathrm{tr}\left(V_+ U_+^\top U_+ \bar{R}_j V_+^\top\right) - 1}{2}\right)\right|$$

$$= \left|\arccos\left(\frac{\mathrm{tr}\left(\bar{R}_j\right) - 1}{2}\right)\right| = \pi. \quad (A93)$$

(iv): Locally unstable spurious equilibrium points. Lastly, we are ready to show that the three spurious equilibrium points are locally unstable. Let the system be at one of the three spurious equilibrium points $\boldsymbol{s} = (\bar{\boldsymbol{y}}, \boldsymbol{U}_+ \bar{\boldsymbol{R}}_j \boldsymbol{V}_+^\mathsf{T}, \boldsymbol{0}, \boldsymbol{0}), j = 2, 3, 4$ with zero translational and angular velocities (such that the total energy of the system equals the total potential energy of the system due to zero kinetic energy), and consider a small perturbation to the equilibrium point:

$$\boldsymbol{s}_\triangle = (\bar{\boldsymbol{y}}, \boldsymbol{R}_\triangle \boldsymbol{U}_+ \bar{\boldsymbol{R}}_j \boldsymbol{V}_+^\mathsf{T}, \boldsymbol{0}, \boldsymbol{0}), \tag{A94}$$

with a perturbing rotation $\boldsymbol{R}_\triangle \in \mathrm{SO}(3)$. The total (potential) energy of the system at $\boldsymbol{s}$ is

$$\begin{aligned} V(\boldsymbol{s}) &= \frac{k}{2} \sum_{i=1}^N \left\| \boldsymbol{y}_i - \boldsymbol{U}_+ \bar{\boldsymbol{R}}_j \boldsymbol{V}_+^\mathsf{T} \boldsymbol{x}_{\mathrm{r}_i} - \bar{\boldsymbol{y}} \right\|^2 && \text{(A95)} \\ &= \frac{k}{2} \sum_{i=1}^N \left\| \boldsymbol{y}_{\mathrm{r}_i} - \boldsymbol{U}_+ \bar{\boldsymbol{R}}_j \boldsymbol{V}_+^\mathsf{T} \boldsymbol{x}_{\mathrm{r}_i} \right\|^2 && \text{(A96)} \\ &= \overbrace{\frac{k}{2} \sum_{i=1}^N \|\boldsymbol{y}_{\mathrm{r}_i}\|^2 + \frac{k}{2} \sum_{i=1}^N \|\boldsymbol{x}_{\mathrm{r}_i}\|^2}^{:=E} - \\ &\qquad k \sum_{i=1}^N \operatorname{tr}\left(\boldsymbol{x}_{\mathrm{r}_i}^\mathsf{T} \boldsymbol{V}_+ \bar{\boldsymbol{R}}_j \boldsymbol{U}_+^\mathsf{T} \boldsymbol{y}_{\mathrm{r}_i} \right) && \text{(A97)} \\ &= E - k \operatorname{tr}\left(\left(\sum_{i=1}^N \boldsymbol{y}_{\mathrm{r}_i} \boldsymbol{x}_{\mathrm{r}_i}^\mathsf{T} \right) \boldsymbol{V}_+ \bar{\boldsymbol{R}}_j \boldsymbol{U}_+^\mathsf{T} \right) && \text{(A98)} \\ &= E - k \operatorname{tr}\left(\boldsymbol{U}_+ \boldsymbol{S}' \boldsymbol{V}_+^\mathsf{T} \boldsymbol{V}_+ \bar{\boldsymbol{R}}_j \boldsymbol{U}_+^\mathsf{T} \right) && \text{(A99)} \\ &= E - k \operatorname{tr}\left(\boldsymbol{S}' \bar{\boldsymbol{R}}_j \right). && \text{(A100)} \end{aligned}$$

The total energy of the system at $\boldsymbol{s}_\triangle$ is

$$\begin{aligned} V(\boldsymbol{s}_\triangle) &= \frac{k}{2} \sum_{i=1}^N \left\| \boldsymbol{y}_{\mathrm{r}_i} - \boldsymbol{R}_\triangle \boldsymbol{U}_+ \bar{\boldsymbol{R}}_j \boldsymbol{V}_+^\mathsf{T} \boldsymbol{x}_{\mathrm{r}_i} \right\|^2 && \text{(A101)} \\ &= E - k \sum_{i=1}^N \operatorname{tr}\left(\boldsymbol{x}_{\mathrm{r}_i}^\mathsf{T} \boldsymbol{V}_+ \bar{\boldsymbol{R}}_j \boldsymbol{U}_+^\mathsf{T} \boldsymbol{R}_\triangle^\mathsf{T} \boldsymbol{y}_{\mathrm{r}_i} \right) && \text{(A102)} \\ &= E - k \operatorname{tr}\left(\left(\sum_{i=1}^N \boldsymbol{y}_{\mathrm{r}_i} \boldsymbol{x}_{\mathrm{r}_i}^\mathsf{T} \right) \boldsymbol{V}_+ \bar{\boldsymbol{R}}_j \boldsymbol{U}_+^\mathsf{T} \boldsymbol{R}_\triangle^\mathsf{T} \right) && \text{(A103)} \\ &= E - k \operatorname{tr}\left(\boldsymbol{U}_+ \boldsymbol{S}' \bar{\boldsymbol{R}}_j \boldsymbol{U}_+^\mathsf{T} \boldsymbol{R}_\triangle^\mathsf{T} \right) && \text{(A104)} \\ &= E - k \operatorname{tr}\left(\boldsymbol{S}' \bar{\boldsymbol{R}}_j \boldsymbol{U}_+^\mathsf{T} \boldsymbol{R}_\triangle^\mathsf{T} \boldsymbol{U}_+ \right). && \text{(A105)} \end{aligned}$$

Therefore, we have that the difference of energy from $V(\boldsymbol{s})$ to $V(\boldsymbol{s}_\triangle)$ is

$$V(\boldsymbol{s}) - V(\boldsymbol{s}_\triangle) = k \left\langle \boldsymbol{S}' \bar{\boldsymbol{R}}_j, \boldsymbol{U}_+^\mathsf{T} \boldsymbol{R}_\triangle \boldsymbol{U}_+ - \mathbf{I}_3 \right\rangle. \tag{A106}$$

Using the Rodrigues' rotation formula on $\boldsymbol{R}_\triangle$:

$$\boldsymbol{R}_\triangle = \cos\theta \mathbf{I}_3 + \sin\theta [\boldsymbol{u}]_\times + (1 - \cos\theta) \boldsymbol{u}\boldsymbol{u}^\mathsf{T}, \tag{A107}$$

where θ is the rotation angle and $\boldsymbol{u} \in \mathbb{S}^2$ is the rotation axis, we have

$$\begin{aligned} &\boldsymbol{U}_+^\mathsf{T} \boldsymbol{R}_\triangle \boldsymbol{U}_+ - \mathbf{I}_3 = \\ &(\cos\theta - 1)\mathbf{I}_3 + (1 - \cos\theta) \boldsymbol{U}_+^\mathsf{T} \boldsymbol{u}\boldsymbol{u}^\mathsf{T} \boldsymbol{U}_+ + \\ &\qquad \sin\theta \boldsymbol{U}_+^\mathsf{T} [\boldsymbol{u}]_\times \boldsymbol{U}_+, \end{aligned} \tag{A108}$$

and the last term $\sin\theta \boldsymbol{U}_+^\mathsf{T} [\boldsymbol{u}]_\times \boldsymbol{U}_+$ is skew-symmetric. Since $\boldsymbol{S}' \bar{\boldsymbol{R}}_j$ is diagonal (and its inner product with any skew-symmetric matrix is zero), we have

$$\begin{aligned} V(\boldsymbol{s}) - V(\boldsymbol{s}_\triangle) &= k(1 - \cos\theta) \left\langle \boldsymbol{S}' \bar{\boldsymbol{R}}_j, \boldsymbol{z}\boldsymbol{z}^\mathsf{T} - \mathbf{I}_3 \right\rangle && \text{(A109)} \\ &= k(1 - \cos\theta) \left(\boldsymbol{z}^\mathsf{T} (\boldsymbol{S}' \bar{\boldsymbol{R}}_j) \boldsymbol{z} - \operatorname{tr}\left(\boldsymbol{S}' \bar{\boldsymbol{R}}_j \right) \right), && \text{(A110)} \end{aligned}$$

where we have denoted $\boldsymbol{z} \triangleq \boldsymbol{U}_+^\mathsf{T} \boldsymbol{u} \in \mathbb{S}^2$. Now using the expression for $\bar{\boldsymbol{R}}_j, j = 2, 3, 4$, we have:

$$\begin{aligned} \boldsymbol{S}' \bar{\boldsymbol{R}}_2 &= \operatorname{diag}\left([s_1, -s_2, -s_3'] \right), && \text{(A111)} \\ \boldsymbol{S}' \bar{\boldsymbol{R}}_3 &= \operatorname{diag}\left([-s_1, s_2, -s_3'] \right), && \text{(A112)} \\ \boldsymbol{S}' \bar{\boldsymbol{R}}_4 &= \operatorname{diag}\left([-s_1, -s_2, s_3'] \right). && \text{(A113)} \end{aligned}$$

Hence, when $j = 2$, we choose $\boldsymbol{z} = [1, 0, 0]^\mathsf{T}$, so that

$$V(\boldsymbol{s}) - V(\boldsymbol{s}_\triangle) = k(1 - \cos\theta)(s_2 + s_3') > 0; \tag{A114}$$

when $j = 3$, we choose $\boldsymbol{z} = [0, 1, 0]^\mathsf{T}$, so that

$$V(\boldsymbol{s}) - V(\boldsymbol{s}_\triangle) = k(1 - \cos\theta)(s_1 + s_3') > 0; \tag{A115}$$

when $j = 4$, we choose $\boldsymbol{z} = [0, 0, 1]^\mathsf{T}$, so that

$$V(\boldsymbol{s}) - V(\boldsymbol{s}_\triangle) = k(1 - \cos\theta)(s_1 + s_2) > 0. \tag{A116}$$

This implies that, in all three cases, there exist small rotational perturbations with angle θ along axis $\boldsymbol{U}_+ \boldsymbol{z}$ (recall that $\boldsymbol{z} = \boldsymbol{U}_+^\mathsf{T} \boldsymbol{u}$), such that this small perturbation will cause a strict decrease in the total energy of the system. As a result, the system is locally unstable at the three spurious equilibrium points. Using Lyapunov's local stability theory [9], we know that, unless starting exactly at one of the spurious equilibrium points, the system will never converge to these locally unstable equilibrium points. □

A5. Corner Cases of Point Cloud Registration

We show two examples of corner cases of point cloud registration where the configuration is not general and violates the $s_1 > s_2 > s_3 > 0$ assumption in Section A4, they correspond to when there is no noise between $\mathcal{X}$ and $\mathcal{Y}$ and both of them have symmetry.

When $N = 3$ (Fig. A2(a)), consider both $\mathcal{X}$ (blue) and $\mathcal{Y}$ (red) are equilateral triangles with l being the length from the vertex to the center. Assume the particles have equal masses such that the CM is also the geometric center O, and all virtual springs have equal coefficients. $\mathcal{X}$ is obtained from $\mathcal{Y}$ by first rotating counter-clockwise (CCW) around O with angle θ, and then flipped about the line that goes through point 1 and the middle point between point 2 and 3. We will show that this is an equilibrium point of the dynamical system for any θ. When the CM of $\mathcal{X}$ and the CM of $\mathcal{Y}$ aligns, we know the forces $\boldsymbol{f}_i, i = 1,2,3$ are already balanced. It remains to show that the torques $\boldsymbol{\tau}_i, i = 1,2,3$ are also balanced for any θ. $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_3$ applies clockwise (CW, cyan) and the value of their sum is:

$$\|\boldsymbol{\tau}_1 + \boldsymbol{\tau}_3\| = \|\boldsymbol{\tau}_1\| + \|\boldsymbol{\tau}_3\| \tag{A117}$$
$$= kl^2 (\sin\theta + \sin\beta) \tag{A118}$$
$$= kl^2 \left(\sin\theta + \sin\left(\theta + \frac{2\pi}{3}\right)\right) \tag{A119}$$
$$= kl^2 \sin\left(\theta + \frac{\pi}{3}\right), \tag{A120}$$

and τ_2 applies CCW (green) and its value is:

$$\|\boldsymbol{\tau}_2\| = kl^2 \sin\alpha = kl^2 \sin\left(\frac{2\pi}{3} - \theta\right) \tag{A121}$$
$$= kl^2 \sin\left(\theta + \frac{\pi}{3}\right). \tag{A122}$$

Therefore, the torques cancel with each other and the configuration in Fig. A2(a) is an equilibrium state for all θ. However, it is easy to observe that this type of equilibrium is unstable because any perturbation that drives point 2 out of the 2D plane will immediately drives the system out of this type of equilibrium. When $N = 4$, one can verify that same torque cancellation happens:

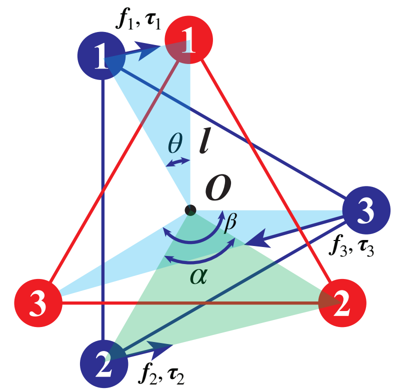
$$\|\boldsymbol{\tau}_1\| = kl^2 \sin\beta = kl^2 \sin\left(\theta + \frac{\pi}{2}\right) \tag{A123}$$
$$= kl^2 \sin\left(\frac{\pi}{2} - \theta\right) = kl^2 \sin\alpha = \|\boldsymbol{\tau}_3\|, \tag{A124}$$

and the system also has infinite locally unstable equilibria.

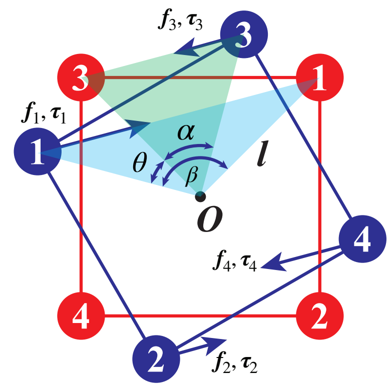
A6. Extra Experimental Results

Mesh registration. Fig. A3 shows the translation error of DAMP compared with SDR [2] on varying noise levels, as well as the relative duality gap of SDR. Because the relative duality gap of SDR is numerically zero, we can say that SDR finds the globally optimal solutions in all Monte Carlo runs. Then we look at the translation error boxplot and observe that DAMP always returns the same solution as SDR, which indicates that DAMP always converges to the optimal solution.

Robot primitive registration. Fig. A4 plots the rotation error, translation error and runtime of DAMP on registering a noisy point cloud observation to the robot primitive including planes, spheres, cylinders and cones, under increasing noise levels, where 1000 Monte Carlo runs are performed at each noise level. We find that DAMP always returns an accurate pose estimation, even when the noise standard deviation is 2 (note that the scene radius is 10), strongly suggesting that DAMP always converges to the optimal solution. Moreover, DAMP has a runtime that is below 1 second (re-



(a) Equilateral triangle.



(b) Square.

Figure A2: Examples of symmetric point clouds: (a) an equilateral triangle and (b) a square. The dynamical system has infinite equilibrium points.

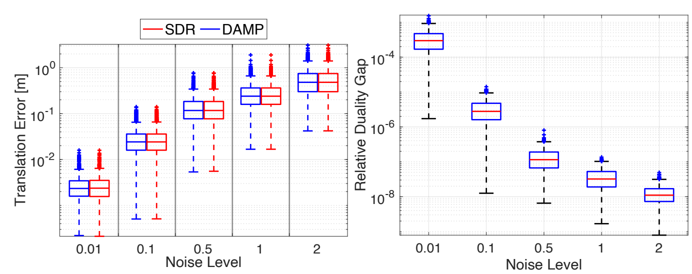


Figure A3: Translation estimation error of DAMP compared with the certifiably optimal SDR solver [2] on random primitive registration with increasing noise levels. Right plot shows the relative duality gap computed from SDR, which certifies that both DAMP and SDR attains the globally optimal solution.

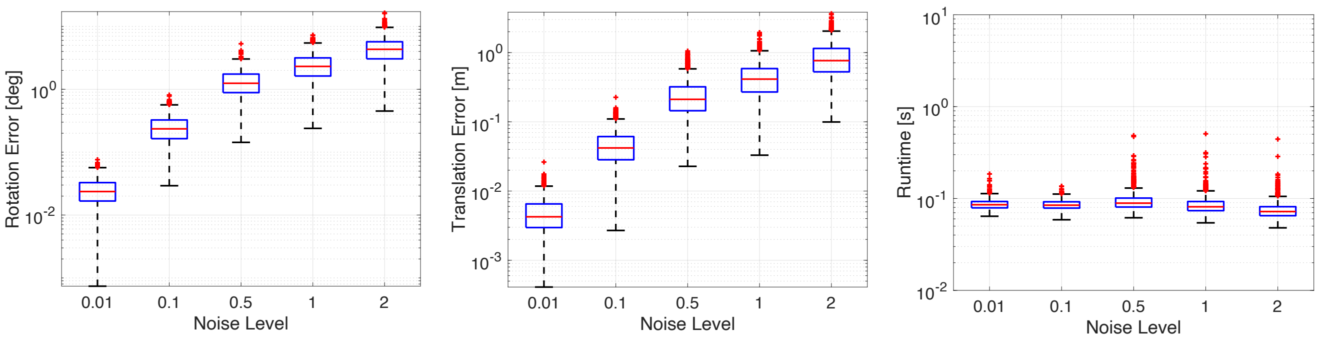


Figure A4: Rotation error, translation error and runtime of DAMP on aligning point cloud observation to the robot primitive model (*c.f*. Fig. 1(b) in main text) under increasing noise levels. Although there is no guaranteed globally optimal solver to verify DAMP's optimality, the accurate estimations strongly indicate DAMP's global convergence (1000 Monte Carlo runs per noise level).

call that our implementation is in Matlab with for loops, because DAMP is a general algorithm that checks the type of the primitive for each correspondence).

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