Mathematical derivations of the optimal formulae

Formula/equation numbers refer to the ones in the article.

In this file, components of the normal vector in (1) are denoted by $n = (n1, \ n2, \ n3)$; the components of the 4 vectors w_{ij} for $i = 1, \ 2$ and $j = 1, \ 2$ are denoted by (w11, w12, w13), (w21, w22, w23), (w31, w32, w33) and (w41, w42, w43); components of the vector w_c are $w_c = $ (w51, w52, w53); and entries of the matrix A are (a11, a12, a21, a22).

The following expression is $f(n)$, this expression is to be minimized:

Out[ ]=

$$\left(a11 - \frac{n1\ w11 + n2\ w12 + n3\ w13}{n1\ w51 + n2\ w52 + n3\ w53}\right)^2 + \left(a12 - \frac{n1\ w21 + n2\ w22 + n3\ w23}{n1\ w51 + n2\ w52 + n3\ w53}\right)^2 +$$

$$\left(a21 - \frac{n1\ w31 + n2\ w32 + n3\ w33}{n1\ w51 + n2\ w52 + n3\ w53}\right)^2 + \left(a22 - \frac{n1\ w41 + n2\ w42 + n3\ w43}{n1\ w51 + n2\ w52 + n3\ w53}\right)^2$$

Case A, n3=1

We make the substitution n3 = 1:

In[ ]:=

$$\left(a11 - \frac{n1\ w11 + n2\ w12 + n3\ w13}{n1\ w51 + n2\ w52 + n3\ w53}\right)^2 + \left(a12 - \frac{n1\ w21 + n2\ w22 + n3\ w23}{n1\ w51 + n2\ w52 + n3\ w53}\right)^2 +$$

$$\left(a21 - \frac{n1\ w31 + n2\ w32 + n3\ w33}{n1\ w51 + n2\ w52 + n3\ w53}\right)^2 + \left(a22 - \frac{n1\ w41 + n2\ w42 + n3\ w43}{n1\ w51 + n2\ w52 + n3\ w53}\right)^2 \ /.\ n3 \to 1$$

Out[ ]=

$$\left(a11 - \frac{n1\ w11 + n2\ w12 + w13}{n1\ w51 + n2\ w52 + w53}\right)^2 + \left(a12 - \frac{n1\ w21 + n2\ w22 + w23}{n1\ w51 + n2\ w52 + w53}\right)^2 +$$

$$\left(a21 - \frac{n1\ w31 + n2\ w32 + w33}{n1\ w51 + n2\ w52 + w53}\right)^2 + \left(a22 - \frac{n1\ w41 + n2\ w42 + w43}{n1\ w51 + n2\ w52 + w53}\right)^2$$

We compute its partial derivative w.r.t. the first variable (that is, $\partial_1 g(n_x, \ n_y)$ in the article) and take only the numerator (it is assumed that the denominator is non-zero):

$In[\circ]:=$ $\mathrm{D}\Big[\Big(\mathrm{a11} - \dfrac{\mathrm{n1\ w11} + \mathrm{n2\ w12} + \mathrm{w13}}{\mathrm{n1\ w51} + \mathrm{n2\ w52} + \mathrm{w53}}\Big)^2 + \Big(\mathrm{a12} - \dfrac{\mathrm{n1\ w21} + \mathrm{n2\ w22} + \mathrm{w23}}{\mathrm{n1\ w51} + \mathrm{n2\ w52} + \mathrm{w53}}\Big)^2 +$

$\quad \Big(\mathrm{a21} - \dfrac{\mathrm{n1\ w31} + \mathrm{n2\ w32} + \mathrm{w33}}{\mathrm{n1\ w51} + \mathrm{n2\ w52} + \mathrm{w53}}\Big)^2 + \Big(\mathrm{a22} - \dfrac{\mathrm{n1\ w41} + \mathrm{n2\ w42} + \mathrm{w43}}{\mathrm{n1\ w51} + \mathrm{n2\ w52} + \mathrm{w53}}\Big)^2 ,$

$\quad \mathrm{n1}\Big]$ // Together // Numerator // Simplify

$Out[\circ]=$

$2\ \big(-\mathrm{w13}^2\ \mathrm{w51} - \mathrm{w23}^2\ \mathrm{w51} - \mathrm{w33}^2\ \mathrm{w51} - \mathrm{w43}^2\ \mathrm{w51} +$

$\quad \mathrm{n2}^2\ \big(-\mathrm{w12}^2\ \mathrm{w51} - \mathrm{w22}^2\ \mathrm{w51} - \mathrm{w32}^2\ \mathrm{w51} - \mathrm{w42}^2\ \mathrm{w51} + \mathrm{w31\ w32\ w52} + \mathrm{w41\ w42\ w52} +$

$\qquad \mathrm{a21\ w32\ w51\ w52} + \mathrm{a22\ w42\ w51\ w52} + \mathrm{w12\ (w11 + a11\ w51)\ w52} +$

$\qquad \mathrm{w22\ (w21 + a12\ w51)\ w52} - \mathrm{a11\ w11\ w52}^2 - \mathrm{a12\ w21\ w52}^2 - \mathrm{a21\ w31\ w52}^2 - \mathrm{a22\ w41\ w52}^2\big) +$

$\quad \mathrm{w11\ w13\ w53} + \mathrm{w21\ w23\ w53} + \mathrm{w31\ w33\ w53} + \mathrm{w41\ w43\ w53} + \mathrm{a11\ w13\ w51\ w53} +$

$\quad \mathrm{a12\ w23\ w51\ w53} + \mathrm{a21\ w33\ w51\ w53} + \mathrm{a22\ w43\ w51\ w53} -$

$\quad \mathrm{a11\ w11\ w53}^2 - \mathrm{a12\ w21\ w53}^2 - \mathrm{a21\ w31\ w53}^2 - \mathrm{a22\ w41\ w53}^2 +$

$\quad \mathrm{n2}\ \big(-2\ \mathrm{w12\ w13\ w51} - 2\ \mathrm{w22\ w23\ w51} - 2\ \mathrm{w32\ w33\ w51} - 2\ \mathrm{w42\ w43\ w51} + \mathrm{w11\ w13\ w52} +$

$\qquad \mathrm{w21\ w23\ w52} + \mathrm{w31\ w33\ w52} + \mathrm{w41\ w43\ w52} + \mathrm{a11\ w13\ w51\ w52} + \mathrm{a12\ w23\ w51\ w52} +$

$\qquad \mathrm{a21\ w33\ w51\ w52} + \mathrm{a22\ w43\ w51\ w52} + \mathrm{w31\ w32\ w53} + \mathrm{w41\ w42\ w53} + \mathrm{a21\ w32\ w51\ w53} +$

$\qquad \mathrm{a22\ w42\ w51\ w53} + \mathrm{w12\ (w11 + a11\ w51)\ w53} + \mathrm{w22\ (w21 + a12\ w51)\ w53} -$

$\qquad 2\ \mathrm{a11\ w11\ w52\ w53} - 2\ \mathrm{a12\ w21\ w52\ w53} - 2\ \mathrm{a21\ w31\ w52\ w53} - 2\ \mathrm{a22\ w41\ w52\ w53}\big) +$

$\quad \mathrm{n1}\ \big(-\mathrm{w21\ w23\ w51} - \mathrm{w31\ w33\ w51} - \mathrm{w41\ w43\ w51} + \mathrm{a11\ w13\ w51}^2 +$

$\qquad \mathrm{a12\ w23\ w51}^2 + \mathrm{a21\ w33\ w51}^2 + \mathrm{a22\ w43\ w51}^2 +$

$\qquad \mathrm{n2}\ \big(-\mathrm{w31\ w32\ w51} - \mathrm{w41\ w42\ w51} + \mathrm{a11\ w12\ w51}^2 + \mathrm{a12\ w22\ w51}^2 + \mathrm{a21\ w32\ w51}^2 +$

$\qquad\quad \mathrm{a22\ w42\ w51}^2 + \mathrm{w11}^2\ \mathrm{w52} + \mathrm{w21}^2\ \mathrm{w52} + \mathrm{w31}^2\ \mathrm{w52} + \mathrm{w41}^2\ \mathrm{w52} - \mathrm{a21\ w31\ w51\ w52} -$

$\qquad\quad \mathrm{a22\ w41\ w51\ w52} - \mathrm{w11\ w51\ (w12 + a11\ w52)} - \mathrm{w21\ w51\ (w22 + a12\ w52)}\big) +$

$\qquad \mathrm{w11}^2\ \mathrm{w53} + \mathrm{w21}^2\ \mathrm{w53} + \mathrm{w31}^2\ \mathrm{w53} + \mathrm{w41}^2\ \mathrm{w53} - \mathrm{a12\ w21\ w51\ w53} -$

$\qquad \mathrm{a21\ w31\ w51\ w53} - \mathrm{a22\ w41\ w51\ w53} - \mathrm{w11\ w51\ (w13 + a11\ w53)}\big)\big)$

Now we express the first component to obtain formula (6):

In[ ]:= `Solve[2 (-w13^2 w51 - w23^2 w51 - w33^2 w51 - w43^2 w51 +`

$$n2^2 \left(-w12^2\,w51 - w22^2\,w51 - w32^2\,w51 - w42^2\,w51 + w31\,w32\,w52 + w41\,w42\,w52 + \right.$$
$$a21\,w32\,w51\,w52 + a22\,w42\,w51\,w52 + w12\,(w11 + a11\,w51)\,w52 + w22\,(w21 + a12\,w51)$$
$$\left. w52 - a11\,w11\,w52^2 - a12\,w21\,w52^2 - a21\,w31\,w52^2 - a22\,w41\,w52^2\right) +$$
$$w11\,w13\,w53 + w21\,w23\,w53 + w31\,w33\,w53 + w41\,w43\,w53 + a11\,w13\,w51\,w53 +$$
$$a12\,w23\,w51\,w53 + a21\,w33\,w51\,w53 + a22\,w43\,w51\,w53 -$$
$$a11\,w11\,w53^2 - a12\,w21\,w53^2 - a21\,w31\,w53^2 - a22\,w41\,w53^2 +$$
$$n2\,(-2\,w12\,w13\,w51 - 2\,w22\,w23\,w51 - 2\,w32\,w33\,w51 - 2\,w42\,w43\,w51 + w11\,w13\,w52 +$$
$$w21\,w23\,w52 + w31\,w33\,w52 + w41\,w43\,w52 + a11\,w13\,w51\,w52 + a12\,w23\,w51\,w52 +$$
$$a21\,w33\,w51\,w52 + a22\,w43\,w51\,w52 + w31\,w32\,w53 + w41\,w42\,w53 + a21\,w32\,w51\,w53 +$$
$$a22\,w42\,w51\,w53 + w12\,(w11 + a11\,w51)\,w53 + w22\,(w21 + a12\,w51)\,w53 -$$
$$2\,a11\,w11\,w52\,w53 - 2\,a12\,w21\,w52\,w53 - 2\,a21\,w31\,w52\,w53 - 2\,a22\,w41\,w52\,w53) +$$
$$n1\,\left(-w21\,w23\,w51 - w31\,w33\,w51 - w41\,w43\,w51 + a11\,w13\,w51^2 + \right.$$
$$a12\,w23\,w51^2 + a21\,w33\,w51^2 + a22\,w43\,w51^2 +$$
$$n2\,\left(-w31\,w32\,w51 - w41\,w42\,w51 + a11\,w12\,w51^2 + a12\,w22\,w51^2 + a21\,w32\,w51^2 + \right.$$
$$a22\,w42\,w51^2 + w11^2\,w52 + w21^2\,w52 + w31^2\,w52 + w41^2\,w52 - a21\,w31\,w51\,w52 -$$
$$\left. a22\,w41\,w51\,w52 - w11\,w51\,(w12 + a11\,w52) - w21\,w51\,(w22 + a12\,w52)\right) +$$
$$w11^2\,w53 + w21^2\,w53 + w31^2\,w53 + w41^2\,w53 - a12\,w21\,w51\,w53 - a21\,w31\,w51\,w53 -$$
$$\left.\left. a22\,w41\,w51\,w53 - w11\,w51\,(w13 + a11\,w53)\right)\right) == 0, n1]$ // Simplify$$

Out[ ]=

$$\{\{n1 \to$$
$$-\Big(\big(\big(-w13^2\,w51 - w23^2\,w51 - w33^2\,w51 - w43^2\,w51 + n2^2\,\big(-w12^2\,w51 - w22^2\,w51 - w32^2\,w51 - $$
$$w42^2\,w51 + w31\,w32\,w52 + w41\,w42\,w52 + a21\,w32\,w51\,w52 +$$
$$a22\,w42\,w51\,w52 + w12\,(w11 + a11\,w51)\,w52 + w22\,(w21 + a12\,w51)\,w52 -$$
$$a11\,w11\,w52^2 - a12\,w21\,w52^2 - a21\,w31\,w52^2 - a22\,w41\,w52^2\big) +$$
$$w21\,w23\,w53 + w31\,w33\,w53 + w41\,w43\,w53 + a12\,w23\,w51\,w53 +$$
$$a21\,w33\,w51\,w53 + a22\,w43\,w51\,w53 + w13\,(w11 + a11\,w51)\,w53 -$$
$$a11\,w11\,w53^2 - a12\,w21\,w53^2 - a21\,w31\,w53^2 - a22\,w41\,w53^2 +$$
$$n2\,\big(-2\,w12\,w13\,w51 - 2\,w22\,w23\,w51 - 2\,w32\,w33\,w51 - 2\,w42\,w43\,w51 +$$
$$w11\,w13\,w52 + w21\,w23\,w52 + w31\,w33\,w52 + w41\,w43\,w52 + a11\,w13\,w51\,w52 +$$
$$a12\,w23\,w51\,w52 + a21\,w33\,w51\,w52 + a22\,w43\,w51\,w52 + w31\,w32\,w53 +$$
$$w41\,w42\,w53 + a21\,w32\,w51\,w53 + a22\,w42\,w51\,w53 + w12\,(w11 + a11\,w51)\,w53 +$$
$$w22\,(w21 + a12\,w51)\,w53 - 2\,a11\,w11\,w52\,w53 - 2\,a12\,w21\,w52\,w53 -$$
$$2\,a21\,w31\,w52\,w53 - 2\,a22\,w41\,w52\,w53\big)\big) \Big/$$
$$\big(-w21\,w23\,w51 - w31\,w33\,w51 - w41\,w43\,w51 + a11\,w13\,w51^2 +$$
$$a12\,w23\,w51^2 + a21\,w33\,w51^2 + a22\,w43\,w51^2 +$$
$$n2\,\big(-w31\,w32\,w51 - w41\,w42\,w51 + a11\,w12\,w51^2 + a12\,w22\,w51^2 + a21\,w32\,w51^2 +$$
$$a22\,w42\,w51^2 + w11^2\,w52 + w21^2\,w52 + w31^2\,w52 + w41^2\,w52 - a21\,w31\,w51\,w52 -$$
$$a22\,w41\,w51\,w52 - w11\,w51\,(w12 + a11\,w52) - w21\,w51\,(w22 + a12\,w52)\big) +$$
$$w11^2\,w53 + w21^2\,w53 + w31^2\,w53 + w41^2\,w53 - a12\,w21\,w51\,w53 -$$
$$a21\,w31\,w51\,w53 - a22\,w41\,w51\,w53 - w11\,w51\,(w13 + a11\,w53)\big)\Big)\}\}$$

We compute the partial derivative w.r.t. the second variable (that is, $\partial_2 g(n_x, n_y)$):

$In[\bullet]:=$ `D[`$\left(a11 - \dfrac{n1\ w11 + n2\ w12 + w13}{n1\ w51 + n2\ w52 + w53}\right)^2 +$

$\qquad\left(a12 - \dfrac{n1\ w21 + n2\ w22 + w23}{n1\ w51 + n2\ w52 + w53}\right)^2 + \left(a21 - \dfrac{n1\ w31 + n2\ w32 + w33}{n1\ w51 + n2\ w52 + w53}\right)^2 +$

$\qquad\left(a22 - \dfrac{n1\ w41 + n2\ w42 + w43}{n1\ w51 + n2\ w52 + w53}\right)^2$, `n2]` `// Together // Simplify`

$Out[\bullet]=$

$\dfrac{1}{(n1\ w51 + n2\ w52 + w53)^3}$ $2\ \big(-w13^2\ w52 - w23^2\ w52 - w33^2\ w52 - w43^2\ w52\ +$

$\qquad n1^2\ \big(w31\ w32\ w51 + w41\ w42\ w51 - a11\ w12\ w51^2 - a12\ w22\ w51^2 - a21\ w32\ w51^2\ -$

$\qquad\qquad a22\ w42\ w51^2 - w11^2\ w52 - w21^2\ w52 - w31^2\ w52 - w41^2\ w52 + a21\ w31\ w51\ w52\ +$

$\qquad\qquad a22\ w41\ w51\ w52 + w11\ w51\ (w12 + a11\ w52) + w21\ w51\ (w22 + a12\ w52)\big)\ +$

$\qquad w12\ w13\ w53 + w22\ w23\ w53 + w32\ w33\ w53 + w42\ w43\ w53 + a11\ w13\ w52\ w53\ +$

$\qquad a12\ w23\ w52\ w53 + a21\ w33\ w52\ w53 + a22\ w43\ w52\ w53\ -$

$\qquad a11\ w12\ w53^2 - a12\ w22\ w53^2 - a21\ w32\ w53^2 - a22\ w42\ w53^2\ +$

$\qquad n1\ \big(w12\ w13\ w51 + w22\ w23\ w51 + w32\ w33\ w51 + w42\ w43\ w51\ -$

$\qquad\qquad 2\ w11\ w13\ w52 - 2\ w21\ w23\ w52 - 2\ w31\ w33\ w52 - 2\ w41\ w43\ w52\ +$

$\qquad\qquad a11\ w13\ w51\ w52 + a12\ w23\ w51\ w52 + a21\ w33\ w51\ w52 + a22\ w43\ w51\ w52\ +$

$\qquad\qquad n2\ \big(w12^2\ w51 + w22^2\ w51 + w32^2\ w51 + w42^2\ w51 - w31\ w32\ w52 - w41\ w42\ w52 - a21\ w32$

$\qquad\qquad\qquad w51\ w52 - a22\ w42\ w51\ w52 - w12\ (w11 + a11\ w51)\ w52 - w22\ (w21 + a12\ w51)$

$\qquad\qquad\qquad w52 + a11\ w11\ w52^2 + a12\ w21\ w52^2 + a21\ w31\ w52^2 + a22\ w41\ w52^2\big)\ +$

$\qquad\qquad w11\ w12\ w53 + w21\ w22\ w53 + w31\ w32\ w53 + w41\ w42\ w53 - 2\ a11\ w12\ w51\ w53\ -$

$\qquad\qquad 2\ a12\ w22\ w51\ w53 - 2\ a21\ w32\ w51\ w53 - 2\ a22\ w42\ w51\ w53\ +$

$\qquad\qquad a11\ w11\ w52\ w53 + a12\ w21\ w52\ w53 + a21\ w31\ w52\ w53 + a22\ w41\ w52\ w53\big)\ +$

$\qquad n2\ \big(-w32\ w33\ w52 - w42\ w43\ w52 + a11\ w13\ w52^2 + a12\ w23\ w52^2 + a21\ w33\ w52^2\ +$

$\qquad\qquad a22\ w43\ w52^2 + w12^2\ w53 + w22^2\ w53 + w32^2\ w53 + w42^2\ w53 - a21\ w32\ w52\ w53\ -$

$\qquad\qquad a22\ w42\ w52\ w53 - w12\ w52\ (w13 + a11\ w53) - w22\ w52\ (w23 + a12\ w53)\big)\big)$

We then substitute the "red" expression (that is, n1 expressed with n2) into the "blue" one (that is, into $\partial_2 g(n_x, n_y)$) to define the expression "factorize"

$$\text{factorize} = \frac{1}{(n1\,w51 + n2\,w52 + w53)^3}\,2\,\big(-w13^2\,w52 - w23^2\,w52 - w33^2\,w52 - w43^2\,w52 +$$

$$n1^2\,\big(w31\,w32\,w51 + w41\,w42\,w51 - a11\,w12\,w51^2 - a12\,w22\,w51^2 - a21\,w32\,w51^2 -$$

$$a22\,w42\,w51^2 - w11^2\,w52 - w21^2\,w52 - w31^2\,w52 - w41^2\,w52 + a21\,w31\,w51\,w52 +$$

$$a22\,w41\,w51\,w52 + w11\,w51\,(w12 + a11\,w52) + w21\,w51\,(w22 + a12\,w52)\big) +$$

$$w12\,w13\,w53 + w22\,w23\,w53 + w32\,w33\,w53 + w42\,w43\,w53 + a11\,w13\,w52\,w53 +$$

$$a12\,w23\,w52\,w53 + a21\,w33\,w52\,w53 + a22\,w43\,w52\,w53 -$$

$$a11\,w12\,w53^2 - a12\,w22\,w53^2 - a21\,w32\,w53^2 - a22\,w42\,w53^2 +$$

$$n1\,\big(w12\,w13\,w51 + w22\,w23\,w51 + w32\,w33\,w51 + w42\,w43\,w51 -$$

$$2\,w11\,w13\,w52 - 2\,w21\,w23\,w52 - 2\,w31\,w33\,w52 - 2\,w41\,w43\,w52 +$$

$$a11\,w13\,w51\,w52 + a12\,w23\,w51\,w52 + a21\,w33\,w51\,w52 + a22\,w43\,w51\,w52 +$$

$$n2\,\big(w12^2\,w51 + w22^2\,w51 + w32^2\,w51 + w42^2\,w51 - w31\,w32\,w52 - w41\,w42\,w52 - a21\,w32$$

$$w51\,w52 - a22\,w42\,w51\,w52 - w12\,(w11 + a11\,w51)\,w52 - w22\,(w21 + a12\,w51)$$

$$w52 + a11\,w11\,w52^2 + a12\,w21\,w52^2 + a21\,w31\,w52^2 + a22\,w41\,w52^2\big) +$$

$$w11\,w12\,w53 + w21\,w22\,w53 + w31\,w32\,w53 + w41\,w42\,w53 - 2\,a11\,w12\,w51\,w53 -$$

$$2\,a12\,w22\,w51\,w53 - 2\,a21\,w32\,w51\,w53 - 2\,a22\,w42\,w51\,w53 +$$

$$a11\,w11\,w52\,w53 + a12\,w21\,w52\,w53 + a21\,w31\,w52\,w53 + a22\,w41\,w52\,w53\big) +$$

$$n2\,\big(-w32\,w33\,w52 - w42\,w43\,w52 + a11\,w13\,w52^2 + a12\,w23\,w52^2 + a21\,w33\,w52^2 +$$

$$a22\,w43\,w52^2 + w12^2\,w53 + w22^2\,w53 + w32^2\,w53 + w42^2\,w53 - a21\,w32\,w52\,w53 -$$

$$a22\,w42\,w52\,w53 - w12\,w52\,(w13 + a11\,w53) - w22\,w52\,(w23 + a12\,w53)\big)\big) \, / .$$

$$n1 \to -\Big(\big(-w13^2\,w51 - w23^2\,w51 - w33^2\,w51 - w43^2\,w51 + n2^2\,\big(-w12^2\,w51 - w22^2\,w51 -$$

$$w32^2\,w51 - w42^2\,w51 + w31\,w32\,w52 + w41\,w42\,w52 + a21\,w32\,w51\,w52 +$$

$$a22\,w42\,w51\,w52 + w12\,(w11 + a11\,w51)\,w52 + w22\,(w21 + a12\,w51)\,w52 -$$

$$a11\,w11\,w52^2 - a12\,w21\,w52^2 - a21\,w31\,w52^2 - a22\,w41\,w52^2\big) +$$

$$w21\,w23\,w53 + w31\,w33\,w53 + w41\,w43\,w53 + a12\,w23\,w51\,w53 +$$

$$a21\,w33\,w51\,w53 + a22\,w43\,w51\,w53 + w13\,(w11 + a11\,w51)\,w53 -$$

$$a11\,w11\,w53^2 - a12\,w21\,w53^2 - a21\,w31\,w53^2 - a22\,w41\,w53^2 +$$

$$n2\,(-2\,w12\,w13\,w51 - 2\,w22\,w23\,w51 - 2\,w32\,w33\,w51 - 2\,w42\,w43\,w51 + w11\,w13\,w52 +$$

$$w21\,w23\,w52 + w31\,w33\,w52 + w41\,w43\,w52 + a11\,w13\,w51\,w52 + a12\,w23\,w51\,w52 +$$

$$a21\,w33\,w51\,w52 + a22\,w43\,w51\,w52 + w31\,w32\,w53 + w41\,w42\,w53 + a21\,w32\,w51$$

$$w53 + a22\,w42\,w51\,w53 + w12\,(w11 + a11\,w51)\,w53 + w22\,(w21 + a12\,w51)\,w53 - 2$$

$$a11\,w11\,w52\,w53 - 2\,a12\,w21\,w52\,w53 - 2\,a21\,w31\,w52\,w53 - 2\,a22\,w41\,w52\,w53)\big)\Big) \Big/$$

$$\big(-w21\,w23\,w51 - w31\,w33\,w51 - w41\,w43\,w51 + a11\,w13\,w51^2 + a12\,w23\,w51^2 +$$

$$a21\,w33\,w51^2 + a22\,w43\,w51^2 +$$

$$n2\,\big(-w31\,w32\,w51 - w41\,w42\,w51 + a11\,w12\,w51^2 + a12\,w22\,w51^2 + a21\,w32\,w51^2 +$$

$$a22\,w42\,w51^2 + w11^2\,w52 + w21^2\,w52 + w31^2\,w52 + w41^2\,w52 - a21\,w31\,w51\,w52 -$$

$$a22\,w41\,w51\,w52 - w11\,w51\,(w12 + a11\,w52) - w21\,w51\,(w22 + a12\,w52)\big) +$$

$$w11^2\,w53 + w21^2\,w53 + w31^2\,w53 + w41^2\,w53 - a12\,w21\,w51\,w53 -$$

$$a21\,w31\,w51\,w53 - a22\,w41\,w51\,w53 - w11\,w51\,(w13 + a11\,w53)\big)\Big)$$

The expression "factorize" to be factorized appears below, depending only on n2:

Out[•]=

$$\Big(2\,\big(-w13^2\,w52 - w23^2\,w52 - w33^2\,w52 - w43^2\,w52 + w12\,w13\,w53 + w22\,w23\,w53 +$$

$$w32\,w33\,w53 + w42\,w43\,w53 + a11\,w13\,w52\,w53 + a12\,w23\,w52\,w53 + a21\,w33\,w52\,w53 +$$

$$a22\,w43\,w52\,w53 - a11\,w12\,w53^2 - a12\,w22\,w53^2 - a21\,w32\,w53^2 - a22\,w42\,w53^2 +$$

$$n2 \left(-w32\,w33\,w52 - w42\,w43\,w52 + a11\,w13\,w52^2 + a12\,w23\,w52^2 + a21\,w33\,w52^2 + a22\,w43\,w52^2 + w12^2\,w53 + w22^2\,w53 + w32^2\,w53 + w42^2\,w53 - a21\,w32\,w52\,w53 - a22\,w42\,w52\,w53 - w12\,w52\,(w13 + a11\,w53) - w22\,w52\,(w23 + a12\,w53) \right) -$$

$$\Big(\big(w12\,w13\,w51 + w22\,w23\,w51 + w32\,w33\,w51 + w42\,w43\,w51 - 2\,w11\,w13\,w52 - 2\,w21\,w23\,w52 - 2\,w31\,w33\,w52 - 2\,w41\,w43\,w52 + a11\,w13\,w51\,w52 + a12\,w23\,w51\,w52 + a21\,w33\,w51\,w52 + a22\,w43\,w51\,w52 +$$

$$n2\,\big(w12^2\,w51 + w22^2\,w51 + w32^2\,w51 + w42^2\,w51 - w31\,w32\,w52 - w41\,w42\,w52 - a21\,w32\,w51\,w52 - a22\,w42\,w51\,w52 - w12\,(w11 + a11\,w51)\,w52 - w22\,(w21 + a12\,w51)\,w52 + a11\,w11\,w52^2 + a12\,w21\,w52^2 + a21\,w31\,w52^2 + a22\,w41\,w52^2 \big) +$$

$$w11\,w12\,w53 + w21\,w22\,w53 + w31\,w32\,w53 + w41\,w42\,w53 - 2\,a11\,w12\,w51\,w53 - 2\,a12\,w22\,w51\,w53 - 2\,a21\,w32\,w51\,w53 - 2\,a22\,w42\,w51\,w53 + a11\,w11\,w52\,w53 + a12\,w21\,w52\,w53 + a21\,w31\,w52\,w53 + a22\,w41\,w52\,w53 \big)$$

$$\big(-w13^2\,w51 - w23^2\,w51 - w33^2\,w51 - w43^2\,w51 + n2^2\,\big(-w12^2\,w51 - w22^2\,w51 - w32^2\,w51 - w42^2\,w51 + w31\,w32\,w52 + w41\,w42\,w52 + a21\,w32\,w51\,w52 + a22\,w42\,w51\,w52 + w12\,(w11 + a11\,w51)\,w52 + w22\,(w21 + a12\,w51)\,w52 - a11\,w11\,w52^2 - a12\,w21\,w52^2 - a21\,w31\,w52^2 - a22\,w41\,w52^2 \big) + w21\,w23\,w53 + w31\,w33\,w53 + w41\,w43\,w53 + a12\,w23\,w51\,w53 + a21\,w33\,w51\,w53 + a22\,w43\,w51\,w53 + w13\,(w11 + a11\,w51)\,w53 - a11\,w11\,w53^2 - a12\,w21\,w53^2 - a21\,w31\,w53^2 - a22\,w41\,w53^2 + n2\,\big(-2\,w12\,w13\,w51 - 2\,w22\,w23\,w51 - 2\,w32\,w33\,w51 - 2\,w42\,w43\,w51 + w11\,w13\,w52 + w21\,w23\,w52 + w31\,w33\,w52 + w41\,w43\,w52 + a11\,w13\,w51\,w52 + a12\,w23\,w51\,w52 + a21\,w33\,w51\,w52 + a22\,w43\,w51\,w52 + w31\,w32\,w53 + w41\,w42\,w53 + a21\,w32\,w51\,w53 + a22\,w42\,w51\,w53 + w12\,(w11 + a11\,w51)\,w53 + w22\,(w21 + a12\,w51)\,w53 - 2\,a11\,w11\,w52\,w53 - 2\,a12\,w21\,w52\,w53 - 2\,a21\,w31\,w52\,w53 - 2\,a22\,w41\,w52\,w53 \big) \big) \Big) \Big/$$

$$\big(-w21\,w23\,w51 - w31\,w33\,w51 - w41\,w43\,w51 + a11\,w13\,w51^2 + a12\,w23\,w51^2 + a21\,w33\,w51^2 + a22\,w43\,w51^2 +$$

$$n2\,\big(-w31\,w32\,w51 - w41\,w42\,w51 + a11\,w12\,w51^2 + a12\,w22\,w51^2 + a21\,w32\,w51^2 + a22\,w42\,w51^2 + w11^2\,w52 + w21^2\,w52 + w31^2\,w52 + w41^2\,w52 - a21\,w31\,w51\,w52 - a22\,w41\,w51\,w52 - w11\,w51\,(w12 + a11\,w52) - w21\,w51\,(w22 + a12\,w52) \big) +$$

$$w11^2\,w53 + w21^2\,w53 + w31^2\,w53 + w41^2\,w53 - a12\,w21\,w51\,w53 - a21\,w31\,w51\,w53 - a22\,w41\,w51\,w53 - w11\,w51\,(w13 + a11\,w53) \big) +$$

$$\Big(\big(w31\,w32\,w51 + w41\,w42\,w51 - a11\,w12\,w51^2 - a12\,w22\,w51^2 - a21\,w32\,w51^2 - a22\,w42\,w51^2 - w11^2\,w52 - w21^2\,w52 - w31^2\,w52 - w41^2\,w52 + a21\,w31\,w51\,w52 + a22\,w41\,w51\,w52 + w11\,w51\,(w12 + a11\,w52) + w21\,w51\,(w22 + a12\,w52) \big)$$

$$\big(-w13^2\,w51 - w23^2\,w51 - w33^2\,w51 - w43^2\,w51 + n2^2\,\big(-w12^2\,w51 - w22^2\,w51 - w32^2\,w51 - w42^2\,w51 + w31\,w32\,w52 + w41\,w42\,w52 + a21\,w32\,w51\,w52 + a22\,w42\,w51\,w52 + w12\,(w11 + a11\,w51)\,w52 + w22\,(w21 + a12\,w51)\,w52 - a11\,w11\,w52^2 - a12\,w21\,w52^2 - a21\,w31\,w52^2 - a22\,w41\,w52^2 \big) + w21\,w23\,w53 + w31\,w33\,w53 + w41\,w43\,w53 + a12\,w23\,w51\,w53 + a21\,w33\,w51\,w53 + a22\,w43\,w51\,w53 + w13\,(w11 + a11\,w51)\,w53 - a11\,w11\,w53^2 - a12\,w21\,w53^2 - a21\,w31\,w53^2 - a22\,w41\,w53^2 + n2\,\big(-2\,w12\,w13\,w51 - 2\,w22\,w23\,w51 - 2\,w32\,w33\,w51 - 2\,w42\,w43\,w51 + w11\,w13\,w52 + w21\,w23\,w52 + w31\,w33\,w52 + w41\,w43\,w52 + a11\,w13\,w51\,w52 + a12\,w23\,w51\,w52 + a21\,w33\,w51\,w52 + a22\,w43\,w51\,w52 + w31\,w32\,w53 + w41\,w42\,w53 + a21\,w32\,w51\,w53 + a22\,w42\,w51\,w53 + w12\,(w11 + a11\,w51)\,w53 + w22\,(w21 + a12\,w51)\,w53 - 2\,a11\,w11\,w52$$

$$\text{w53} - 2\,\text{a12}\,\text{w21}\,\text{w52}\,\text{w53} - 2\,\text{a21}\,\text{w31}\,\text{w52}\,\text{w53} - 2\,\text{a22}\,\text{w41}\,\text{w52}\,\text{w53})\Big)^2\Big) \Big/$$

$$\Big(-\text{w21}\,\text{w23}\,\text{w51} - \text{w31}\,\text{w33}\,\text{w51} - \text{w41}\,\text{w43}\,\text{w51} + \text{a11}\,\text{w13}\,\text{w51}^2 + \text{a12}\,\text{w23}\,\text{w51}^2 +$$

$$\text{a21}\,\text{w33}\,\text{w51}^2 + \text{a22}\,\text{w43}\,\text{w51}^2 +$$

$$\text{n2}\,\Big(-\text{w31}\,\text{w32}\,\text{w51} - \text{w41}\,\text{w42}\,\text{w51} + \text{a11}\,\text{w12}\,\text{w51}^2 + \text{a12}\,\text{w22}\,\text{w51}^2 + \text{a21}\,\text{w32}\,\text{w51}^2 +$$

$$\text{a22}\,\text{w42}\,\text{w51}^2 + \text{w11}^2\,\text{w52} + \text{w21}^2\,\text{w52} + \text{w31}^2\,\text{w52} + \text{w41}^2\,\text{w52} - \text{a21}\,\text{w31}\,\text{w51}\,\text{w52} -$$

$$\text{a22}\,\text{w41}\,\text{w51}\,\text{w52} - \text{w11}\,\text{w51}\,(\text{w12} + \text{a11}\,\text{w52}) - \text{w21}\,\text{w51}\,(\text{w22} + \text{a12}\,\text{w52})\Big) +$$

$$\text{w11}^2\,\text{w53} + \text{w21}^2\,\text{w53} + \text{w31}^2\,\text{w53} + \text{w41}^2\,\text{w53} - \text{a12}\,\text{w21}\,\text{w51}\,\text{w53} -$$

$$\text{a21}\,\text{w31}\,\text{w51}\,\text{w53} - \text{a22}\,\text{w41}\,\text{w51}\,\text{w53} - \text{w11}\,\text{w51}\,(\text{w13} + \text{a11}\,\text{w53})\Big)^2\Big)\Big) \Big/$$

$$\Big(\text{n2}\,\text{w52} + \text{w53} - \Big(\text{w51}\,\Big(-\text{w13}^2\,\text{w51} - \text{w23}^2\,\text{w51} - \text{w33}^2\,\text{w51} - \text{w43}^2\,\text{w51} + \text{n2}^2\,\Big(-\text{w12}^2\,\text{w51} -$$

$$\text{w22}^2\,\text{w51} - \text{w32}^2\,\text{w51} - \text{w42}^2\,\text{w51} + \text{w31}\,\text{w32}\,\text{w52} + \text{w41}\,\text{w42}\,\text{w52} + \text{a21}\,\text{w32}\,\text{w51}\,\text{w52} +$$

$$\text{a22}\,\text{w42}\,\text{w51}\,\text{w52} + \text{w12}\,(\text{w11} + \text{a11}\,\text{w51})\,\text{w52} + \text{w22}\,(\text{w21} + \text{a12}\,\text{w51})\,\text{w52} -$$

$$\text{a11}\,\text{w11}\,\text{w52}^2 - \text{a12}\,\text{w21}\,\text{w52}^2 - \text{a21}\,\text{w31}\,\text{w52}^2 - \text{a22}\,\text{w41}\,\text{w52}^2\Big) + \text{w21}\,\text{w23}\,\text{w53} +$$

$$\text{w31}\,\text{w33}\,\text{w53} + \text{w41}\,\text{w43}\,\text{w53} + \text{a12}\,\text{w23}\,\text{w51}\,\text{w53} + \text{a21}\,\text{w33}\,\text{w51}\,\text{w53} +$$

$$\text{a22}\,\text{w43}\,\text{w51}\,\text{w53} + \text{w13}\,(\text{w11} + \text{a11}\,\text{w51})\,\text{w53} - \text{a11}\,\text{w11}\,\text{w53}^2 - \text{a12}\,\text{w21}\,\text{w53}^2 -$$

$$\text{a21}\,\text{w31}\,\text{w53}^2 - \text{a22}\,\text{w41}\,\text{w53}^2 + \text{n2}\,(-2\,\text{w12}\,\text{w13}\,\text{w51} - 2\,\text{w22}\,\text{w23}\,\text{w51} - 2\,\text{w32}\,\text{w33}\,\text{w51} -$$

$$2\,\text{w42}\,\text{w43}\,\text{w51} + \text{w11}\,\text{w13}\,\text{w52} + \text{w21}\,\text{w23}\,\text{w52} + \text{w31}\,\text{w33}\,\text{w52} + \text{w41}\,\text{w43}\,\text{w52} +$$

$$\text{a11}\,\text{w13}\,\text{w51}\,\text{w52} + \text{a12}\,\text{w23}\,\text{w51}\,\text{w52} + \text{a21}\,\text{w33}\,\text{w51}\,\text{w52} + \text{a22}\,\text{w43}\,\text{w51}\,\text{w52} +$$

$$\text{w31}\,\text{w32}\,\text{w53} + \text{w41}\,\text{w42}\,\text{w53} + \text{a21}\,\text{w32}\,\text{w51}\,\text{w53} + \text{a22}\,\text{w42}\,\text{w51}\,\text{w53} +$$

$$\text{w12}\,(\text{w11} + \text{a11}\,\text{w51})\,\text{w53} + \text{w22}\,(\text{w21} + \text{a12}\,\text{w51})\,\text{w53} - 2\,\text{a11}\,\text{w11}\,\text{w52}\,\text{w53} -$$

$$2\,\text{a12}\,\text{w21}\,\text{w52}\,\text{w53} - 2\,\text{a21}\,\text{w31}\,\text{w52}\,\text{w53} - 2\,\text{a22}\,\text{w41}\,\text{w52}\,\text{w53})\Big)\Big) \Big/$$

$$\Big(-\text{w21}\,\text{w23}\,\text{w51} - \text{w31}\,\text{w33}\,\text{w51} - \text{w41}\,\text{w43}\,\text{w51} + \text{a11}\,\text{w13}\,\text{w51}^2 + \text{a12}\,\text{w23}\,\text{w51}^2 +$$

$$\text{a21}\,\text{w33}\,\text{w51}^2 + \text{a22}\,\text{w43}\,\text{w51}^2 +$$

$$\text{n2}\,\Big(-\text{w31}\,\text{w32}\,\text{w51} - \text{w41}\,\text{w42}\,\text{w51} + \text{a11}\,\text{w12}\,\text{w51}^2 + \text{a12}\,\text{w22}\,\text{w51}^2 + \text{a21}\,\text{w32}\,\text{w51}^2 +$$

$$\text{a22}\,\text{w42}\,\text{w51}^2 + \text{w11}^2\,\text{w52} + \text{w21}^2\,\text{w52} + \text{w31}^2\,\text{w52} + \text{w41}^2\,\text{w52} - \text{a21}\,\text{w31}\,\text{w51}\,\text{w52} -$$

$$\text{a22}\,\text{w41}\,\text{w51}\,\text{w52} - \text{w11}\,\text{w51}\,(\text{w12} + \text{a11}\,\text{w52}) - \text{w21}\,\text{w51}\,(\text{w22} + \text{a12}\,\text{w52})\Big) +$$

$$\text{w11}^2\,\text{w53} + \text{w21}^2\,\text{w53} + \text{w31}^2\,\text{w53} + \text{w41}^2\,\text{w53} - \text{a12}\,\text{w21}\,\text{w51}\,\text{w53} -$$

$$\text{a21}\,\text{w31}\,\text{w51}\,\text{w53} - \text{a22}\,\text{w41}\,\text{w51}\,\text{w53} - \text{w11}\,\text{w51}\,(\text{w13} + \text{a11}\,\text{w53})\Big)\Big)^3$$

The actual factorization takes place below, and we obtain formula (7):

Factor[factorize]

Out[•]=

$$-\Big(\Big(2\,(\text{n2}\,\text{w11}\,\text{w12}\,\text{w51} + \text{w11}\,\text{w13}\,\text{w51} + \text{n2}\,\text{w21}\,\text{w22}\,\text{w51} + \text{w21}\,\text{w23}\,\text{w51} + \text{n2}\,\text{w31}\,\text{w32}\,\text{w51} +$$

$$\text{w31}\,\text{w33}\,\text{w51} + \text{n2}\,\text{w41}\,\text{w42}\,\text{w51} + \text{w41}\,\text{w43}\,\text{w51} - \text{a11}\,\text{n2}\,\text{w12}\,\text{w51}^2 - \text{a11}\,\text{w13}\,\text{w51}^2 -$$

$$\text{a12}\,\text{n2}\,\text{w22}\,\text{w51}^2 - \text{a12}\,\text{w23}\,\text{w51}^2 - \text{a21}\,\text{n2}\,\text{w32}\,\text{w51}^2 - \text{a21}\,\text{w33}\,\text{w51}^2 -$$

$$\text{a22}\,\text{n2}\,\text{w42}\,\text{w51}^2 - \text{a22}\,\text{w43}\,\text{w51}^2 - \text{n2}\,\text{w11}^2\,\text{w52} - \text{n2}\,\text{w21}^2\,\text{w52} - \text{n2}\,\text{w31}^2\,\text{w52} -$$

$$\text{n2}\,\text{w41}^2\,\text{w52} + \text{a11}\,\text{n2}\,\text{w11}\,\text{w51}\,\text{w52} + \text{a12}\,\text{n2}\,\text{w21}\,\text{w51}\,\text{w52} + \text{a21}\,\text{n2}\,\text{w31}\,\text{w51}\,\text{w52} +$$

$$\text{a22}\,\text{n2}\,\text{w41}\,\text{w51}\,\text{w52} - \text{w11}^2\,\text{w53} - \text{w21}^2\,\text{w53} - \text{w31}^2\,\text{w53} - \text{w41}^2\,\text{w53} +$$

$$\text{a11}\,\text{w11}\,\text{w51}\,\text{w53} + \text{a12}\,\text{w21}\,\text{w51}\,\text{w53} + \text{a21}\,\text{w31}\,\text{w51}\,\text{w53} + \text{a22}\,\text{w41}\,\text{w51}\,\text{w53})$$

$$(\text{n2}\,\text{w12}\,\text{w13}\,\text{w21}\,\text{w22}\,\text{w51} + \text{w13}^2\,\text{w21}\,\text{w22}\,\text{w51} - \text{n2}\,\text{w11}\,\text{w13}\,\text{w22}^2\,\text{w51} -$$

$$\text{n2}\,\text{w12}^2\,\text{w21}\,\text{w23}\,\text{w51} - \text{w12}\,\text{w13}\,\text{w21}\,\text{w23}\,\text{w51} + \text{n2}\,\text{w11}\,\text{w12}\,\text{w22}\,\text{w23}\,\text{w51} -$$

$$\text{w11}\,\text{w13}\,\text{w22}\,\text{w23}\,\text{w51} + \text{w11}\,\text{w12}\,\text{w23}^2\,\text{w51} + \text{n2}\,\text{w12}\,\text{w13}\,\text{w31}\,\text{w32}\,\text{w51} +$$

$$\text{w13}^2\,\text{w31}\,\text{w32}\,\text{w51} + \text{n2}\,\text{w22}\,\text{w23}\,\text{w31}\,\text{w32}\,\text{w51} + \text{w23}^2\,\text{w31}\,\text{w32}\,\text{w51} - \text{n2}\,\text{w11}\,\text{w13}\,\text{w32}^2\,\text{w51} -$$

$$\text{n2}\,\text{w21}\,\text{w23}\,\text{w32}^2\,\text{w51} - \text{n2}\,\text{w12}^2\,\text{w31}\,\text{w33}\,\text{w51} - \text{w12}\,\text{w13}\,\text{w31}\,\text{w33}\,\text{w51} -$$

$$\text{n2}\,\text{w22}^2\,\text{w31}\,\text{w33}\,\text{w51} - \text{w22}\,\text{w23}\,\text{w31}\,\text{w33}\,\text{w51} + \text{n2}\,\text{w11}\,\text{w12}\,\text{w32}\,\text{w33}\,\text{w51} -$$

$$\text{w11}\,\text{w13}\,\text{w32}\,\text{w33}\,\text{w51} + \text{n2}\,\text{w21}\,\text{w22}\,\text{w32}\,\text{w33}\,\text{w51} - \text{w21}\,\text{w23}\,\text{w32}\,\text{w33}\,\text{w51} +$$

$$w11\ w12\ w33^2\ w51 + w21\ w22\ w33^2\ w51 + n2\ w12\ w13\ w41\ w42\ w51 + w13^2\ w41\ w42\ w51 +$$
$$n2\ w22\ w23\ w41\ w42\ w51 + w23^2\ w41\ w42\ w51 + n2\ w32\ w33\ w41\ w42\ w51 +$$
$$w33^2\ w41\ w42\ w51 - n2\ w11\ w13\ w42^2\ w51 - n2\ w21\ w23\ w42^2\ w51 - n2\ w31\ w33\ w42^2\ w51 -$$
$$n2\ w12^2\ w41\ w43\ w51 - w12\ w13\ w41\ w43\ w51 - n2\ w22^2\ w41\ w43\ w51 -$$
$$w22\ w23\ w41\ w43\ w51 - n2\ w32^2\ w41\ w43\ w51 - w32\ w33\ w41\ w43\ w51 +$$
$$n2\ w11\ w12\ w42\ w43\ w51 - w11\ w13\ w42\ w43\ w51 + n2\ w21\ w22\ w42\ w43\ w51 -$$
$$w21\ w23\ w42\ w43\ w51 + n2\ w31\ w32\ w42\ w43\ w51 - w31\ w33\ w42\ w43\ w51 +$$
$$w11\ w12\ w43^2\ w51 + w21\ w22\ w43^2\ w51 + w31\ w32\ w43^2\ w51 - a12\ n2\ w12\ w13\ w22\ w51^2 -$$
$$a12\ w13^2\ w22\ w51^2 + a11\ n2\ w13\ w22^2\ w51^2 + a12\ n2\ w12^2\ w23\ w51^2 +$$
$$a12\ w12\ w13\ w23\ w51^2 - a11\ n2\ w12\ w22\ w23\ w51^2 + a11\ w13\ w22\ w23\ w51^2 -$$
$$a11\ w12\ w23^2\ w51^2 - a21\ n2\ w12\ w13\ w32\ w51^2 - a21\ w13^2\ w32\ w51^2 -$$
$$a21\ n2\ w22\ w23\ w32\ w51^2 - a21\ w23^2\ w32\ w51^2 + a11\ n2\ w13\ w32^2\ w51^2 +$$
$$a12\ n2\ w23\ w32^2\ w51^2 + a21\ n2\ w12^2\ w33\ w51^2 + a21\ w12\ w13\ w33\ w51^2 +$$
$$a21\ n2\ w22^2\ w33\ w51^2 + a21\ w22\ w23\ w33\ w51^2 - a11\ n2\ w12\ w32\ w33\ w51^2 +$$
$$a11\ w13\ w32\ w33\ w51^2 - a12\ n2\ w22\ w32\ w33\ w51^2 + a12\ w23\ w32\ w33\ w51^2 -$$
$$a11\ w12\ w33^2\ w51^2 - a12\ w22\ w33^2\ w51^2 - a22\ n2\ w12\ w13\ w42\ w51^2 - a22\ w13^2\ w42\ w51^2 -$$
$$a22\ n2\ w22\ w23\ w42\ w51^2 - a22\ w23^2\ w42\ w51^2 - a22\ n2\ w32\ w33\ w42\ w51^2 -$$
$$a22\ w33^2\ w42\ w51^2 + a11\ n2\ w13\ w42^2\ w51^2 + a12\ n2\ w23\ w42^2\ w51^2 +$$
$$a21\ n2\ w33\ w42^2\ w51^2 + a22\ n2\ w12^2\ w43\ w51^2 + a22\ w12\ w13\ w43\ w51^2 +$$
$$a22\ n2\ w22^2\ w43\ w51^2 + a22\ w22\ w23\ w43\ w51^2 + a22\ n2\ w32^2\ w43\ w51^2 +$$
$$a22\ w32\ w33\ w43\ w51^2 - a11\ n2\ w12\ w42\ w43\ w51^2 + a11\ w13\ w42\ w43\ w51^2 -$$
$$a12\ n2\ w22\ w42\ w43\ w51^2 + a12\ w23\ w42\ w43\ w51^2 - a21\ n2\ w32\ w42\ w43\ w51^2 +$$
$$a21\ w33\ w42\ w43\ w51^2 - a11\ w12\ w43^2\ w51^2 - a12\ w22\ w43^2\ w51^2 -$$
$$a21\ w32\ w43^2\ w51^2 - n2\ w12\ w13\ w21^2\ w52 - w13^2\ w21^2\ w52 + n2\ w11\ w13\ w21\ w22\ w52 +$$
$$n2\ w11\ w12\ w21\ w23\ w52 + 2\ w11\ w13\ w21\ w23\ w52 - n2\ w11^2\ w22\ w23\ w52 -$$
$$w11^2\ w23^2\ w52 - n2\ w12\ w13\ w31^2\ w52 - w13^2\ w31^2\ w52 - n2\ w22\ w23\ w31^2\ w52 -$$
$$w23^2\ w31^2\ w52 + n2\ w11\ w13\ w31\ w32\ w52 + n2\ w21\ w23\ w31\ w32\ w52 +$$
$$n2\ w11\ w12\ w31\ w33\ w52 + 2\ w11\ w13\ w31\ w33\ w52 + n2\ w21\ w22\ w31\ w33\ w52 +$$
$$2\ w21\ w23\ w31\ w33\ w52 - n2\ w11^2\ w32\ w33\ w52 - n2\ w21^2\ w32\ w33\ w52 -$$
$$w11^2\ w33^2\ w52 - w21^2\ w33^2\ w52 - n2\ w12\ w13\ w41^2\ w52 - w13^2\ w41^2\ w52 -$$
$$n2\ w22\ w23\ w41^2\ w52 - w23^2\ w41^2\ w52 - n2\ w32\ w33\ w41^2\ w52 - w33^2\ w41^2\ w52 +$$
$$n2\ w11\ w13\ w41\ w42\ w52 + n2\ w21\ w23\ w41\ w42\ w52 + n2\ w31\ w33\ w41\ w42\ w52 +$$
$$n2\ w11\ w12\ w41\ w43\ w52 + 2\ w11\ w13\ w41\ w43\ w52 + n2\ w21\ w22\ w41\ w43\ w52 +$$
$$2\ w21\ w23\ w41\ w43\ w52 + n2\ w31\ w32\ w41\ w43\ w52 + 2\ w31\ w33\ w41\ w43\ w52 -$$
$$n2\ w11^2\ w42\ w43\ w52 - n2\ w21^2\ w42\ w43\ w52 - n2\ w31^2\ w42\ w43\ w52 -$$
$$w11^2\ w43^2\ w52 - w21^2\ w43^2\ w52 - w31^2\ w43^2\ w52 + a12\ n2\ w12\ w13\ w21\ w51\ w52 +$$
$$a12\ w13^2\ w21\ w51\ w52 + a12\ n2\ w11\ w13\ w22\ w51\ w52 - 2\ a11\ n2\ w13\ w21\ w22\ w51\ w52 -$$
$$2\ a12\ n2\ w11\ w12\ w23\ w51\ w52 - a12\ w11\ w13\ w23\ w51\ w52 +$$
$$a11\ n2\ w12\ w21\ w23\ w51\ w52 - a11\ w13\ w21\ w23\ w51\ w52 + a11\ n2\ w11\ w22\ w23\ w51\ w52 +$$
$$a11\ w11\ w23^2\ w51\ w52 + a21\ n2\ w12\ w13\ w31\ w51\ w52 + a21\ w13^2\ w31\ w51\ w52 +$$
$$a21\ n2\ w22\ w23\ w31\ w51\ w52 + a21\ w23^2\ w31\ w51\ w52 + a21\ n2\ w11\ w13\ w32\ w51\ w52 +$$
$$a21\ n2\ w21\ w23\ w32\ w51\ w52 - 2\ a11\ n2\ w13\ w31\ w32\ w51\ w52 -$$
$$2\ a12\ n2\ w23\ w31\ w32\ w51\ w52 - 2\ a21\ n2\ w11\ w12\ w33\ w51\ w52 -$$
$$a21\ w11\ w13\ w33\ w51\ w52 - 2\ a21\ n2\ w21\ w22\ w33\ w51\ w52 - a21\ w21\ w23\ w33\ w51\ w52 +$$
$$a11\ n2\ w12\ w31\ w33\ w51\ w52 - a11\ w13\ w31\ w33\ w51\ w52 + a12\ n2\ w22\ w31\ w33\ w51\ w52 -$$
$$a12\ w23\ w31\ w33\ w51\ w52 + a11\ n2\ w11\ w32\ w33\ w51\ w52 + a12\ n2\ w21\ w32\ w33\ w51\ w52 +$$
$$a11\ w11\ w33^2\ w51\ w52 + a12\ w21\ w33^2\ w51\ w52 + a22\ n2\ w12\ w13\ w41\ w51\ w52 +$$

$a22\ w13^2\ w41\ w51\ w52 + a22\ n2\ w22\ w23\ w41\ w51\ w52 + a22\ w23^2\ w41\ w51\ w52 +$

$a22\ n2\ w32\ w33\ w41\ w51\ w52 + a22\ w33^2\ w41\ w51\ w52 + a22\ n2\ w11\ w13\ w42\ w51\ w52 +$

$a22\ n2\ w21\ w23\ w42\ w51\ w52 + a22\ n2\ w31\ w33\ w42\ w51\ w52 -$

$2\ a11\ n2\ w13\ w41\ w42\ w51\ w52 - 2\ a12\ n2\ w23\ w41\ w42\ w51\ w52 -$

$2\ a21\ n2\ w33\ w41\ w42\ w51\ w52 - 2\ a22\ n2\ w11\ w12\ w43\ w51\ w52 -$

$a22\ w11\ w13\ w43\ w51\ w52 - 2\ a22\ n2\ w21\ w22\ w43\ w51\ w52 - a22\ w21\ w23\ w43\ w51\ w52 -$

$2\ a22\ n2\ w31\ w32\ w43\ w51\ w52 - a22\ w31\ w33\ w43\ w51\ w52 + a11\ n2\ w12\ w41\ w43\ w51\ w52 -$

$a11\ w13\ w41\ w43\ w51\ w52 + a12\ n2\ w22\ w41\ w43\ w51\ w52 - a12\ w23\ w41\ w43\ w51\ w52 +$

$a21\ n2\ w32\ w41\ w43\ w51\ w52 - a21\ w33\ w41\ w43\ w51\ w52 + a11\ n2\ w11\ w42\ w43\ w51\ w52 +$

$a12\ n2\ w21\ w42\ w43\ w51\ w52 + a21\ n2\ w31\ w42\ w43\ w51\ w52 + a11\ w11\ w43^2\ w51\ w52 +$

$a12\ w21\ w43^2\ w51\ w52 + a21\ w31\ w43^2\ w51\ w52 - a12\ n2\ w11\ w13\ w21\ w52^2 +$

$a11\ n2\ w13\ w21^2\ w52^2 + a12\ n2\ w11^2\ w23\ w52^2 - a11\ n2\ w11\ w21\ w23\ w52^2 -$

$a21\ n2\ w11\ w13\ w31\ w52^2 - a21\ n2\ w21\ w23\ w31\ w52^2 + a11\ n2\ w13\ w31^2\ w52^2 +$

$a12\ n2\ w23\ w31^2\ w52^2 + a21\ n2\ w11^2\ w33\ w52^2 + a21\ n2\ w21^2\ w33\ w52^2 -$

$a11\ n2\ w11\ w31\ w33\ w52^2 - a12\ n2\ w21\ w31\ w33\ w52^2 - a22\ n2\ w11\ w13\ w41\ w52^2 -$

$a22\ n2\ w21\ w23\ w41\ w52^2 - a22\ n2\ w31\ w33\ w41\ w52^2 + a11\ n2\ w13\ w41^2\ w52^2 +$

$a12\ n2\ w23\ w41^2\ w52^2 + a21\ n2\ w33\ w41^2\ w52^2 + a22\ n2\ w11^2\ w43\ w52^2 +$

$a22\ n2\ w21^2\ w43\ w52^2 + a22\ n2\ w31^2\ w43\ w52^2 - a11\ n2\ w11\ w41\ w43\ w52^2 -$

$a12\ n2\ w21\ w41\ w43\ w52^2 - a21\ n2\ w31\ w41\ w43\ w52^2 + n2\ w12^2\ w21^2\ w53 +$

$w12\ w13\ w21^2\ w53 - 2\ n2\ w11\ w12\ w21\ w22\ w53 - w11\ w13\ w21\ w22\ w53 +$

$n2\ w11^2\ w22^2\ w53 - w11\ w12\ w21\ w23\ w53 + w11^2\ w22\ w23\ w53 + n2\ w12^2\ w31^2\ w53 +$

$w12\ w13\ w31^2\ w53 + n2\ w22^2\ w31^2\ w53 + w22\ w23\ w31^2\ w53 - 2\ n2\ w11\ w12\ w31\ w32\ w53 -$

$w11\ w13\ w31\ w32\ w53 - 2\ n2\ w21\ w22\ w31\ w32\ w53 - w21\ w23\ w31\ w32\ w53 +$

$n2\ w11^2\ w32^2\ w53 + n2\ w21^2\ w32^2\ w53 - w11\ w12\ w31\ w33\ w53 - w21\ w22\ w31\ w33\ w53 +$

$w11^2\ w32\ w33\ w53 + w21^2\ w32\ w33\ w53 + n2\ w12^2\ w41^2\ w53 + w12\ w13\ w41^2\ w53 +$

$n2\ w22^2\ w41^2\ w53 + w22\ w23\ w41^2\ w53 + n2\ w32^2\ w41^2\ w53 + w32\ w33\ w41^2\ w53 -$

$2\ n2\ w11\ w12\ w41\ w42\ w53 - w11\ w13\ w41\ w42\ w53 - 2\ n2\ w21\ w22\ w41\ w42\ w53 -$

$w21\ w23\ w41\ w42\ w53 - 2\ n2\ w31\ w32\ w41\ w42\ w53 - w31\ w33\ w41\ w42\ w53 +$

$n2\ w11^2\ w42^2\ w53 + n2\ w21^2\ w42^2\ w53 + n2\ w31^2\ w42^2\ w53 - w11\ w12\ w41\ w43\ w53 -$

$w21\ w22\ w41\ w43\ w53 - w31\ w32\ w41\ w43\ w53 + w11^2\ w42\ w43\ w53 +$

$w21^2\ w42\ w43\ w53 + w31^2\ w42\ w43\ w53 - a12\ n2\ w12^2\ w21\ w51\ w53 -$

$a12\ w12\ w13\ w21\ w51\ w53 + a12\ n2\ w11\ w12\ w22\ w51\ w53 + 2\ a12\ w11\ w13\ w22\ w51\ w53 +$

$a11\ n2\ w12\ w21\ w22\ w51\ w53 - a11\ w13\ w21\ w22\ w51\ w53 - a11\ n2\ w11\ w22^2\ w51\ w53 -$

$a12\ w11\ w12\ w23\ w51\ w53 + 2\ a11\ w12\ w21\ w23\ w51\ w53 - a11\ w11\ w22\ w23\ w51\ w53 -$

$a21\ n2\ w12^2\ w31\ w51\ w53 - a21\ w12\ w13\ w31\ w51\ w53 - a21\ n2\ w22^2\ w31\ w51\ w53 -$

$a21\ w22\ w23\ w31\ w51\ w53 + a21\ n2\ w11\ w12\ w32\ w51\ w53 + 2\ a21\ w11\ w13\ w32\ w51\ w53 +$

$a21\ n2\ w21\ w22\ w32\ w51\ w53 + 2\ a21\ w21\ w23\ w32\ w51\ w53 + a11\ n2\ w12\ w31\ w32\ w51\ w53 -$

$a11\ w13\ w31\ w32\ w51\ w53 + a12\ n2\ w22\ w31\ w32\ w51\ w53 - a12\ w23\ w31\ w32\ w51\ w53 -$

$a11\ n2\ w11\ w32^2\ w51\ w53 - a12\ n2\ w21\ w32^2\ w51\ w53 - a21\ w11\ w12\ w33\ w51\ w53 -$

$a21\ w21\ w22\ w33\ w51\ w53 + 2\ a11\ w12\ w31\ w33\ w51\ w53 + 2\ a12\ w22\ w31\ w33\ w51\ w53 -$

$a11\ w11\ w32\ w33\ w51\ w53 - a12\ w21\ w32\ w33\ w51\ w53 - a22\ n2\ w12^2\ w41\ w51\ w53 -$

$a22\ w12\ w13\ w41\ w51\ w53 - a22\ n2\ w22^2\ w41\ w51\ w53 - a22\ w22\ w23\ w41\ w51\ w53 -$

$a22\ n2\ w32^2\ w41\ w51\ w53 - a22\ w32\ w33\ w41\ w51\ w53 + a22\ n2\ w11\ w12\ w42\ w51\ w53 +$

$2\ a22\ w11\ w13\ w42\ w51\ w53 + a22\ n2\ w21\ w22\ w42\ w51\ w53 + 2\ a22\ w21\ w23\ w42\ w51\ w53 +$

$a22\ n2\ w31\ w32\ w42\ w51\ w53 + 2\ a22\ w31\ w33\ w42\ w51\ w53 + a11\ n2\ w12\ w41\ w42\ w51\ w53 -$

$a11\ w13\ w41\ w42\ w51\ w53 + a12\ n2\ w22\ w41\ w42\ w51\ w53 - a12\ w23\ w41\ w42\ w51\ w53 +$

$a21\ n2\ w32\ w41\ w42\ w51\ w53 - a21\ w33\ w41\ w42\ w51\ w53 - a11\ n2\ w11\ w42^2\ w51\ w53 -$

$a12\ n2\ w21\ w42^2\ w51\ w53 - a21\ n2\ w31\ w42^2\ w51\ w53 - a22\ w11\ w12\ w43\ w51\ w53 -$

$a22\ w21\ w22\ w43\ w51\ w53 - a22\ w31\ w32\ w43\ w51\ w53 + 2\ a11\ w12\ w41\ w43\ w51\ w53 +$

$2\ a12\ w22\ w41\ w43\ w51\ w53 + 2\ a21\ w32\ w41\ w43\ w51\ w53 - a11\ w11\ w42\ w43\ w51\ w53 -$

$a12\ w21\ w42\ w43\ w51\ w53 - a21\ w31\ w42\ w43\ w51\ w53 + a12\ n2\ w11\ w12\ w21\ w52\ w53 -$

$a12\ w11\ w13\ w21\ w52\ w53 - a11\ n2\ w12\ w21^2\ w52\ w53 + a11\ w13\ w21^2\ w52\ w53 -$

$a12\ n2\ w11^2\ w22\ w52\ w53 + a11\ n2\ w11\ w21\ w22\ w52\ w53 + a12\ w11^2\ w23\ w52\ w53 -$

$a11\ w11\ w21\ w23\ w52\ w53 + a21\ n2\ w11\ w12\ w31\ w52\ w53 - a21\ w11\ w13\ w31\ w52\ w53 +$

$a21\ n2\ w21\ w22\ w31\ w52\ w53 - a21\ w21\ w23\ w31\ w52\ w53 - a11\ n2\ w12\ w31^2\ w52\ w53 +$

$a11\ w13\ w31^2\ w52\ w53 - a12\ n2\ w22\ w31^2\ w52\ w53 + a12\ w23\ w31^2\ w52\ w53 -$

$a21\ n2\ w11^2\ w32\ w52\ w53 - a21\ n2\ w21^2\ w32\ w52\ w53 + a11\ n2\ w11\ w31\ w32\ w52\ w53 +$

$a12\ n2\ w21\ w31\ w32\ w52\ w53 + a21\ w11^2\ w33\ w52\ w53 + a21\ w21^2\ w33\ w52\ w53 -$

$a11\ w11\ w31\ w33\ w52\ w53 - a12\ w21\ w31\ w33\ w52\ w53 + a22\ n2\ w11\ w12\ w41\ w52\ w53 -$

$a22\ w11\ w13\ w41\ w52\ w53 + a22\ n2\ w21\ w22\ w41\ w52\ w53 - a22\ w21\ w23\ w41\ w52\ w53 +$

$a22\ n2\ w31\ w32\ w41\ w52\ w53 - a22\ w31\ w33\ w41\ w52\ w53 - a11\ n2\ w12\ w41^2\ w52\ w53 +$

$a11\ w13\ w41^2\ w52\ w53 - a12\ n2\ w22\ w41^2\ w52\ w53 + a12\ w23\ w41^2\ w52\ w53 -$

$a21\ n2\ w32\ w41^2\ w52\ w53 + a21\ w33\ w41^2\ w52\ w53 - a22\ n2\ w11^2\ w42\ w52\ w53 -$

$a22\ n2\ w21^2\ w42\ w52\ w53 - a22\ n2\ w31^2\ w42\ w52\ w53 + a11\ n2\ w11\ w41\ w42\ w52\ w53 +$

$a12\ n2\ w21\ w41\ w42\ w52\ w53 + a21\ n2\ w31\ w41\ w42\ w52\ w53 + a22\ w11^2\ w43\ w52\ w53 +$

$a22\ w21^2\ w43\ w52\ w53 + a22\ w31^2\ w43\ w52\ w53 - a11\ w11\ w41\ w43\ w52\ w53 -$

$a12\ w21\ w41\ w43\ w52\ w53 - a21\ w31\ w41\ w43\ w52\ w53 + a12\ w11\ w12\ w21\ w53^2 -$

$a11\ w12\ w21^2\ w53^2 - a12\ w11^2\ w22\ w53^2 + a11\ w11\ w21\ w22\ w53^2 +$

$a21\ w11\ w12\ w31\ w53^2 + a21\ w21\ w22\ w31\ w53^2 - a11\ w12\ w31^2\ w53^2 -$

$a12\ w22\ w31^2\ w53^2 - a21\ w11^2\ w32\ w53^2 - a21\ w21^2\ w32\ w53^2 + a11\ w11\ w31\ w32\ w53^2 +$

$a12\ w21\ w31\ w32\ w53^2 + a22\ w11\ w12\ w41\ w53^2 + a22\ w21\ w22\ w41\ w53^2 +$

$a22\ w31\ w32\ w41\ w53^2 - a11\ w12\ w41^2\ w53^2 - a12\ w22\ w41^2\ w53^2 -$

$a21\ w32\ w41^2\ w53^2 - a22\ w11^2\ w42\ w53^2 - a22\ w21^2\ w42\ w53^2 - a22\ w31^2\ w42\ w53^2 +$

$a11\ w11\ w41\ w42\ w53^2 + a12\ w21\ w41\ w42\ w53^2 + a21\ w31\ w41\ w42\ w53^2 \big) \big) \Big/$

$\big(n2^2\ w12^2\ w51^2 + 2\ n2\ w12\ w13\ w51^2 + w13^2\ w51^2 + n2^2\ w22^2\ w51^2 + 2\ n2\ w22\ w23\ w51^2 +$

$w23^2\ w51^2 + n2^2\ w32^2\ w51^2 + 2\ n2\ w32\ w33\ w51^2 + w33^2\ w51^2 + n2^2\ w42^2\ w51^2 +$

$2\ n2\ w42\ w43\ w51^2 + w43^2\ w51^2 - 2\ n2^2\ w11\ w12\ w51\ w52 - 2\ n2\ w11\ w13\ w51\ w52 -$

$2\ n2^2\ w21\ w22\ w51\ w52 - 2\ n2\ w21\ w23\ w51\ w52 - 2\ n2^2\ w31\ w32\ w51\ w52 -$

$2\ n2\ w31\ w33\ w51\ w52 - 2\ n2^2\ w41\ w42\ w51\ w52 - 2\ n2\ w41\ w43\ w51\ w52 + n2^2\ w11^2\ w52^2 +$

$n2^2\ w21^2\ w52^2 + n2^2\ w31^2\ w52^2 + n2^2\ w41^2\ w52^2 - 2\ n2\ w11\ w12\ w51\ w53 -$

$2\ w11\ w13\ w51\ w53 - 2\ n2\ w21\ w22\ w51\ w53 - 2\ w21\ w23\ w51\ w53 -$

$2\ n2\ w31\ w32\ w51\ w53 - 2\ w31\ w33\ w51\ w53 - 2\ n2\ w41\ w42\ w51\ w53 -$

$2\ w41\ w43\ w51\ w53 + 2\ n2\ w11^2\ w52\ w53 + 2\ n2\ w21^2\ w52\ w53 + 2\ n2\ w31^2\ w52\ w53 +$

$2\ n2\ w41^2\ w52\ w53 + w11^2\ w53^2 + w21^2\ w53^2 + w31^2\ w53^2 + w41^2\ w53^2 \big)^2 \big)$

It is seen that its structure is $\frac{\text{linear}_1\ \text{linear}_2}{\text{quadratic}^2}$.

We need to solve $\frac{\text{linear}_1\ \text{linear}_2}{\text{quadratic}^2} = 0$ for n2. Clearly, we can ignore the denominator, so we consider only the two factors in the numerator as two separate equations.

linear1 = n2 w11 w12 w51 + w11 w13 w51 + n2 w21 w22 w51 + w21 w23 w51 + n2 w31 w32 w51 + w31 w33 w51 + n2 w41 w42 w51 + w41 w43 w51 – a11 n2 w12 w51^2 – a11 w13 w51^2 – a12 n2 w22 w51^2 – a12 w23 w51^2 – a21 n2 w32 w51^2 – a21 w33 w51^2 – a22 n2 w42 w51^2 – a22 w43 w51^2 – n2 w11^2 w52 – n2 w21^2 w52 – n2 w31^2 w52 – n2 w41^2 w52 + a11 n2 w11 w51 w52 + a12 n2 w21 w51 w52 + a21 n2 w31 w51 w52 + a22 n2 w41 w51 w52 – w11^2 w53 – w21^2 w53 – w31^2 w53 – w41^2 w53 + a11 w11 w51 w53 + a12 w21 w51 w53 + a21 w31 w51 w53 + a22 w41 w51 w53

linear2 = n2 w12 w13 w21 w22 w51 + w13^2 w21 w22 w51 – n2 w11 w13 w22^2 w51 – n2 w12^2 w21 w23 w51 – w12 w13 w21 w23 w51 + n2 w11 w12 w22 w23 w51 – w11 w13 w22 w23 w51 + w11 w12 w23^2 w51 + n2 w12 w13 w31 w32 w51 + w13^2 w31 w32 w51 + n2 w22 w23 w31 w32 w51 + w23^2 w31 w32 w51 – n2 w11 w13 w32^2 w51 – n2 w21 w23 w32^2 w51 – n2 w12^2 w31 w33 w51 – w12 w13 w31 w33 w51 – n2 w22^2 w31 w33 w51 – w22 w23 w31 w33 w51 + n2 w11 w12 w32 w33 w51 – w11 w13 w32 w33 w51 + n2 w21 w22 w32 w33 w51 – w21 w23 w32 w33 w51 + w11 w12 w33^2 w51 + w21 w22 w33^2 w51 + n2 w12 w13 w41 w42 w51 + w13^2 w41 w42 w51 + n2 w22 w23 w41 w42 w51 + w23^2 w41 w42 w51 + n2 w32 w33 w41 w42 w51 + w33^2 w41 w42 w51 – n2 w11 w13 w42^2 w51 – n2 w21 w23 w42^2 w51 – n2 w31 w33 w42^2 w51 – n2 w12^2 w41 w43 w51 – w12 w13 w41 w43 w51 – n2 w22^2 w41 w43 w51 – w22 w23 w41 w43 w51 – n2 w32^2 w41 w43 w51 – w32 w33 w41 w43 w51 + n2 w11 w12 w42 w43 w51 – w11 w13 w42 w43 w51 + n2 w21 w22 w42 w43 w51 – w21 w23 w42 w43 w51 + n2 w31 w32 w42 w43 w51 – w31 w33 w42 w43 w51 + w11 w12 w43^2 w51 + w21 w22 w43^2 w51 + w31 w32 w43^2 w51 – a12 n2 w12 w13 w22 w51^2 – a12 w13^2 w22 w51^2 + a11 n2 w13 w22^2 w51^2 + a12 n2 w12^2 w23 w51^2 + a12 w12 w13 w23 w51^2 – a11 n2 w12 w22 w23 w51^2 + a11 w13 w22 w23 w51^2 – a11 w12 w23^2 w51^2 – a21 n2 w12 w13 w32 w51^2 – a21 w13^2 w32 w51^2 – a21 n2 w22 w23 w32 w51^2 – a21 w23^2 w32 w51^2 + a11 n2 w13 w32^2 w51^2 + a12 n2 w23 w32^2 w51^2 + a21 n2 w12^2 w33 w51^2 + a21 w12 w13 w33 w51^2 + a21 n2 w22^2 w33 w51^2 + a21 w22 w23 w33 w51^2 – a11 n2 w12 w32 w33 w51^2 + a11 w13 w32 w33 w51^2 – a12 n2 w22 w32 w33 w51^2 + a12 w23 w32 w33 w51^2 – a11 w12 w33^2 w51^2 – a12 w22 w33^2 w51^2 – a22 n2 w12 w13 w42 w51^2 – a22 w13^2 w42 w51^2 – a22 n2 w22 w23 w42 w51^2 – a22 w23^2 w42 w51^2 – a22 n2 w32 w33 w42 w51^2 – a22 w33^2 w42 w51^2 + a11 n2 w13 w42^2 w51^2 + a12 n2 w23 w42^2 w51^2 + a21 n2 w33 w42^2 w51^2 + a22 n2 w12^2 w43 w51^2 + a22 w12 w13 w43 w51^2 + a22 n2 w22^2 w43 w51^2 + a22 w22 w23 w43 w51^2 + a22 n2 w32^2 w43 w51^2 + a22 w32 w33 w43 w51^2 – a11 n2 w12 w42 w43 w51^2 + a11 w13 w42 w43 w51^2 – a12 n2 w22 w42 w43 w51^2 + a12 w23 w42 w43 w51^2 – a21 n2 w32 w42 w43 w51^2 + a21 w33 w42 w43 w51^2 – a11 w12 w43^2 w51^2 – a12 w22 w43^2 w51^2 – a21 w32 w43^2 w51^2 – n2 w12 w13 w21^2 w52 – w13^2 w21^2 w52 + n2 w11 w13 w21 w22 w52 + n2 w11 w12 w21 w23 w52 + 2 w11 w13 w21 w23 w52 – n2 w11^2 w22 w23 w52 – w11^2 w23^2 w52 – n2 w12 w13 w31^2 w52 – w13^2 w31^2 w52 – n2 w22 w23 w31^2 w52 – w23^2 w31^2 w52 + n2 w11 w13 w31 w32 w52 + n2 w21 w23 w31 w32 w52 + n2 w11 w12 w31 w33 w52 + 2 w11 w13 w31 w33 w52 + n2 w21 w22 w31 w33 w52 + 2 w21 w23 w31 w33 w52 – n2 w11^2 w32 w33 w52 – n2 w21^2 w32 w33 w52 – w11^2 w33^2 w52 – w21^2 w33^2 w52 – n2 w12 w13 w41^2 w52 – w13^2 w41^2 w52 – n2 w22 w23 w41^2 w52 – w23^2 w41^2 w52 – n2 w32 w33 w41^2 w52 – w33^2 w41^2 w52 + n2 w11 w13 w41 w42 w52 + n2 w21 w23 w41 w42 w52 + n2 w31 w33 w41 w42 w52 + n2 w11 w12 w41 w43 w52 + 2 w11 w13 w41 w43 w52 + n2 w21 w22 w41 w43 w52 + 2 w21 w23 w41 w43 w52 + n2 w31 w32 w41 w43 w52 + 2 w31 w33 w41 w43 w52 – n2 w11^2 w42 w43 w52 – n2 w21^2 w42 w43 w52 – n2 w31^2 w42 w43 w52 – w11^2 w43^2 w52 – w21^2 w43^2 w52 – w31^2 w43^2 w52 + a12 n2 w12 w13 w21 w51 w52 + a12 w13^2 w21 w51 w52 + a12 n2 w11 w13 w22 w51 w52 – 2 a11 n2 w13 w21 w22 w51 w52 – 2 a12 n2 w11 w12 w23 w51 w52 – a12 w11 w13 w23 w51 w52 + a11 n2 w12 w21 w23 w51 w52 – a11 w13 w21 w23 w51 w52 +

$a_{11} n_2 w_{11} w_{22} w_{23} w_{51} w_{52} + a_{11} w_{11} w_{23}^2 w_{51} w_{52} + a_{21} n_2 w_{12} w_{13} w_{31} w_{51} w_{52} +$

$a_{21} w_{13}^2 w_{31} w_{51} w_{52} + a_{21} n_2 w_{22} w_{23} w_{31} w_{51} w_{52} + a_{21} w_{23}^2 w_{31} w_{51} w_{52} +$

$a_{21} n_2 w_{11} w_{13} w_{32} w_{51} w_{52} + a_{21} n_2 w_{21} w_{23} w_{32} w_{51} w_{52} - 2 a_{11} n_2 w_{13} w_{31} w_{32} w_{51} w_{52} -$

$2 a_{12} n_2 w_{23} w_{31} w_{32} w_{51} w_{52} - 2 a_{21} n_2 w_{11} w_{12} w_{33} w_{51} w_{52} - a_{21} w_{11} w_{13} w_{33} w_{51} w_{52} -$

$2 a_{21} n_2 w_{21} w_{22} w_{33} w_{51} w_{52} - a_{21} w_{21} w_{23} w_{33} w_{51} w_{52} + a_{11} n_2 w_{12} w_{31} w_{33} w_{51} w_{52} -$

$a_{11} w_{13} w_{31} w_{33} w_{51} w_{52} + a_{12} n_2 w_{22} w_{31} w_{33} w_{51} w_{52} - a_{12} w_{23} w_{31} w_{33} w_{51} w_{52} +$

$a_{11} n_2 w_{11} w_{32} w_{33} w_{51} w_{52} + a_{12} n_2 w_{21} w_{32} w_{33} w_{51} w_{52} + a_{11} w_{11} w_{33}^2 w_{51} w_{52} +$

$a_{12} w_{21} w_{33}^2 w_{51} w_{52} + a_{22} n_2 w_{12} w_{13} w_{41} w_{51} w_{52} + a_{22} w_{13}^2 w_{41} w_{51} w_{52} +$

$a_{22} n_2 w_{22} w_{23} w_{41} w_{51} w_{52} + a_{22} w_{23}^2 w_{41} w_{51} w_{52} + a_{22} n_2 w_{32} w_{33} w_{41} w_{51} w_{52} +$

$a_{22} w_{33}^2 w_{41} w_{51} w_{52} + a_{22} n_2 w_{11} w_{13} w_{42} w_{51} w_{52} + a_{22} n_2 w_{21} w_{23} w_{42} w_{51} w_{52} +$

$a_{22} n_2 w_{31} w_{33} w_{42} w_{51} w_{52} - 2 a_{11} n_2 w_{13} w_{41} w_{42} w_{51} w_{52} - 2 a_{12} n_2 w_{23} w_{41} w_{42} w_{51} w_{52} -$

$2 a_{21} n_2 w_{33} w_{41} w_{42} w_{51} w_{52} - 2 a_{22} n_2 w_{11} w_{12} w_{43} w_{51} w_{52} - a_{22} w_{11} w_{13} w_{43} w_{51} w_{52} -$

$2 a_{22} n_2 w_{21} w_{22} w_{43} w_{51} w_{52} - a_{22} w_{21} w_{23} w_{43} w_{51} w_{52} - 2 a_{22} n_2 w_{31} w_{32} w_{43} w_{51} w_{52} -$

$a_{22} w_{31} w_{33} w_{43} w_{51} w_{52} + a_{11} n_2 w_{12} w_{41} w_{43} w_{51} w_{52} - a_{11} w_{13} w_{41} w_{43} w_{51} w_{52} +$

$a_{12} n_2 w_{22} w_{41} w_{43} w_{51} w_{52} - a_{12} w_{23} w_{41} w_{43} w_{51} w_{52} + a_{21} n_2 w_{32} w_{41} w_{43} w_{51} w_{52} -$

$a_{21} w_{33} w_{41} w_{43} w_{51} w_{52} + a_{11} n_2 w_{11} w_{42} w_{43} w_{51} w_{52} + a_{12} n_2 w_{21} w_{42} w_{43} w_{51} w_{52} +$

$a_{21} n_2 w_{31} w_{42} w_{43} w_{51} w_{52} + a_{11} w_{11} w_{43}^2 w_{51} w_{52} + a_{12} w_{21} w_{43}^2 w_{51} w_{52} +$

$a_{21} w_{31} w_{43}^2 w_{51} w_{52} - a_{12} n_2 w_{11} w_{13} w_{21} w_{52}^2 + a_{11} n_2 w_{13} w_{21}^2 w_{52}^2 +$

$a_{12} n_2 w_{11}^2 w_{23} w_{52}^2 - a_{11} n_2 w_{11} w_{21} w_{23} w_{52}^2 - a_{21} n_2 w_{11} w_{13} w_{31} w_{52}^2 -$

$a_{21} n_2 w_{21} w_{23} w_{31} w_{52}^2 + a_{11} n_2 w_{13} w_{31}^2 w_{52}^2 + a_{12} n_2 w_{23} w_{31}^2 w_{52}^2 +$

$a_{21} n_2 w_{11}^2 w_{33} w_{52}^2 + a_{21} n_2 w_{21}^2 w_{33} w_{52}^2 - a_{11} n_2 w_{11} w_{31} w_{33} w_{52}^2 -$

$a_{12} n_2 w_{21} w_{31} w_{33} w_{52}^2 - a_{22} n_2 w_{11} w_{13} w_{41} w_{52}^2 - a_{22} n_2 w_{21} w_{23} w_{41} w_{52}^2 -$

$a_{22} n_2 w_{31} w_{33} w_{41} w_{52}^2 + a_{11} n_2 w_{13} w_{41}^2 w_{52}^2 + a_{12} n_2 w_{23} w_{41}^2 w_{52}^2 +$

$a_{21} n_2 w_{33} w_{41}^2 w_{52}^2 + a_{22} n_2 w_{11}^2 w_{43} w_{52}^2 + a_{22} n_2 w_{21}^2 w_{43} w_{52}^2 + a_{22} n_2 w_{31}^2 w_{43} w_{52}^2 -$

$a_{11} n_2 w_{11} w_{41} w_{43} w_{52}^2 - a_{12} n_2 w_{21} w_{41} w_{43} w_{52}^2 - a_{21} n_2 w_{31} w_{41} w_{43} w_{52}^2 +$

$n_2 w_{12}^2 w_{21}^2 w_{53} + w_{12} w_{13} w_{21}^2 w_{53} - 2 n_2 w_{11} w_{12} w_{21} w_{22} w_{53} - w_{11} w_{13} w_{21} w_{22} w_{53} +$

$n_2 w_{11}^2 w_{22}^2 w_{53} - w_{11} w_{12} w_{21} w_{23} w_{53} + w_{11}^2 w_{22} w_{23} w_{53} + n_2 w_{12}^2 w_{31}^2 w_{53} +$

$w_{12} w_{13} w_{31}^2 w_{53} + n_2 w_{22}^2 w_{31}^2 w_{53} + w_{22} w_{23} w_{31}^2 w_{53} - 2 n_2 w_{11} w_{12} w_{31} w_{32} w_{53} -$

$w_{11} w_{13} w_{31} w_{32} w_{53} - 2 n_2 w_{21} w_{22} w_{31} w_{32} w_{53} - w_{21} w_{23} w_{31} w_{32} w_{53} +$

$n_2 w_{11}^2 w_{32}^2 w_{53} + n_2 w_{21}^2 w_{32}^2 w_{53} - w_{11} w_{12} w_{31} w_{33} w_{53} - w_{21} w_{22} w_{31} w_{33} w_{53} +$

$w_{11}^2 w_{32} w_{33} w_{53} + w_{21}^2 w_{32} w_{33} w_{53} + n_2 w_{12}^2 w_{41}^2 w_{53} + w_{12} w_{13} w_{41}^2 w_{53} +$

$n_2 w_{22}^2 w_{41}^2 w_{53} + w_{22} w_{23} w_{41}^2 w_{53} + n_2 w_{32}^2 w_{41}^2 w_{53} + w_{32} w_{33} w_{41}^2 w_{53} -$

$2 n_2 w_{11} w_{12} w_{41} w_{42} w_{53} - w_{11} w_{13} w_{41} w_{42} w_{53} - 2 n_2 w_{21} w_{22} w_{41} w_{42} w_{53} -$

$w_{21} w_{23} w_{41} w_{42} w_{53} - 2 n_2 w_{31} w_{32} w_{41} w_{42} w_{53} - w_{31} w_{33} w_{41} w_{42} w_{53} + n_2 w_{11}^2 w_{42}^2 w_{53} +$

$n_2 w_{21}^2 w_{42}^2 w_{53} + n_2 w_{31}^2 w_{42}^2 w_{53} - w_{11} w_{12} w_{41} w_{43} w_{53} - w_{21} w_{22} w_{41} w_{43} w_{53} -$

$w_{31} w_{32} w_{41} w_{43} w_{53} + w_{11}^2 w_{42} w_{43} w_{53} + w_{21}^2 w_{42} w_{43} w_{53} + w_{31}^2 w_{42} w_{43} w_{53} -$

$a_{12} n_2 w_{12}^2 w_{21} w_{51} w_{53} - a_{12} w_{12} w_{13} w_{21} w_{51} w_{53} + a_{12} n_2 w_{11} w_{12} w_{22} w_{51} w_{53} +$

$2 a_{12} w_{11} w_{13} w_{22} w_{51} w_{53} + a_{11} n_2 w_{12} w_{21} w_{22} w_{51} w_{53} - a_{11} w_{13} w_{21} w_{22} w_{51} w_{53} -$

$a_{11} n_2 w_{11} w_{22}^2 w_{51} w_{53} - a_{12} w_{11} w_{12} w_{23} w_{51} w_{53} + 2 a_{11} w_{12} w_{21} w_{23} w_{51} w_{53} -$

$a_{11} w_{11} w_{22} w_{23} w_{51} w_{53} - a_{21} n_2 w_{12}^2 w_{31} w_{51} w_{53} - a_{21} w_{12} w_{13} w_{31} w_{51} w_{53} -$

$a_{21} n_2 w_{22}^2 w_{31} w_{51} w_{53} - a_{21} w_{22} w_{23} w_{31} w_{51} w_{53} + a_{21} n_2 w_{11} w_{12} w_{32} w_{51} w_{53} +$

$2 a_{21} w_{11} w_{13} w_{32} w_{51} w_{53} + a_{21} n_2 w_{21} w_{22} w_{32} w_{51} w_{53} + 2 a_{21} w_{21} w_{23} w_{32} w_{51} w_{53} +$

$a_{11} n_2 w_{12} w_{31} w_{32} w_{51} w_{53} - a_{11} w_{13} w_{31} w_{32} w_{51} w_{53} + a_{12} n_2 w_{22} w_{31} w_{32} w_{51} w_{53} -$

$a_{12} w_{23} w_{31} w_{32} w_{51} w_{53} - a_{11} n_2 w_{11} w_{32}^2 w_{51} w_{53} - a_{12} n_2 w_{21} w_{32}^2 w_{51} w_{53} -$

$a_{21} w_{11} w_{12} w_{33} w_{51} w_{53} - a_{21} w_{21} w_{22} w_{33} w_{51} w_{53} + 2 a_{11} w_{12} w_{31} w_{33} w_{51} w_{53} +$

$2 a_{12} w_{22} w_{31} w_{33} w_{51} w_{53} - a_{11} w_{11} w_{32} w_{33} w_{51} w_{53} - a_{12} w_{21} w_{32} w_{33} w_{51} w_{53} -$

$a_{22} n_2 w_{12}^2 w_{41} w_{51} w_{53} - a_{22} w_{12} w_{13} w_{41} w_{51} w_{53} - a_{22} n_2 w_{22}^2 w_{41} w_{51} w_{53} -$

a22 w22 w23 w41 w51 w53 − a22 n2 w32² w41 w51 w53 − a22 w32 w33 w41 w51 w53 +

a22 n2 w11 w12 w42 w51 w53 + 2 a22 w11 w13 w42 w51 w53 + a22 n2 w21 w22 w42 w51 w53 +

2 a22 w21 w23 w42 w51 w53 + a22 n2 w31 w32 w42 w51 w53 + 2 a22 w31 w33 w42 w51 w53 +

a11 n2 w12 w41 w42 w51 w53 − a11 w13 w41 w42 w51 w53 + a12 n2 w22 w41 w42 w51 w53 −

a12 w23 w41 w42 w51 w53 + a21 n2 w32 w41 w42 w51 w53 − a21 w33 w41 w42 w51 w53 −

a11 n2 w11 w42² w51 w53 − a12 n2 w21 w42² w51 w53 − a21 n2 w31 w42² w51 w53 −

a22 w11 w12 w43 w51 w53 − a22 w21 w22 w43 w51 w53 − a22 w31 w32 w43 w51 w53 +

2 a11 w12 w41 w43 w51 w53 + 2 a12 w22 w41 w43 w51 w53 + 2 a21 w32 w41 w43 w51 w53 −

a11 w11 w42 w43 w51 w53 − a12 w21 w42 w43 w51 w53 − a21 w31 w42 w43 w51 w53 +

a12 n2 w11 w12 w21 w52 w53 − a12 w11 w13 w21 w52 w53 − a11 n2 w12 w21² w52 w53 +

a11 w13 w21² w52 w53 − a12 n2 w11² w22 w52 w53 + a11 n2 w11 w21 w22 w52 w53 +

a12 w11² w23 w52 w53 − a11 w11 w21 w23 w52 w53 + a21 n2 w11 w12 w31 w52 w53 −

a21 w11 w13 w31 w52 w53 + a21 n2 w21 w22 w31 w52 w53 − a21 w21 w23 w31 w52 w53 −

a11 n2 w12 w31² w52 w53 + a11 w13 w31² w52 w53 − a12 n2 w22 w31² w52 w53 +

a12 w23 w31² w52 w53 − a21 n2 w11² w32 w52 w53 − a21 n2 w21² w32 w52 w53 +

a11 n2 w11 w31 w32 w52 w53 + a12 n2 w21 w31 w32 w52 w53 + a21 w11² w33 w52 w53 +

a21 w21² w33 w52 w53 − a11 w11 w31 w33 w52 w53 − a12 w21 w31 w33 w52 w53 +

a22 n2 w11 w12 w41 w52 w53 − a22 w11 w13 w41 w52 w53 + a22 n2 w21 w22 w41 w52 w53 −

a22 w21 w23 w41 w52 w53 + a22 n2 w31 w32 w41 w52 w53 − a22 w31 w33 w41 w52 w53 −

a11 n2 w12 w41² w52 w53 + a11 w13 w41² w52 w53 − a12 n2 w22 w41² w52 w53 +

a12 w23 w41² w52 w53 − a21 n2 w32 w41² w52 w53 + a21 w33 w41² w52 w53 −

a22 n2 w11² w42 w52 w53 − a22 n2 w21² w42 w52 w53 − a22 n2 w31² w42 w52 w53 +

a11 n2 w11 w41 w42 w52 w53 + a12 n2 w21 w41 w42 w52 w53 + a21 n2 w31 w41 w42 w52 w53 +

a22 w11² w43 w52 w53 + a22 w21² w43 w52 w53 + a22 w31² w43 w52 w53 −

a11 w11 w41 w43 w52 w53 − a12 w21 w41 w43 w52 w53 − a21 w31 w41 w43 w52 w53 +

a12 w11 w12 w21 w53² − a11 w12 w21² w53² − a12 w11² w22 w53² + a11 w11 w21 w22 w53² +

a21 w11 w12 w31 w53² + a21 w21 w22 w31 w53² − a11 w12 w31² w53² − a12 w22 w31² w53² −

a21 w11² w32 w53² − a21 w21² w32 w53² + a11 w11 w31 w32 w53² + a12 w21 w31 w32 w53² +

a22 w11 w12 w41 w53² + a22 w21 w22 w41 w53² + a22 w31 w32 w41 w53² − a11 w12 w41² w53² −

a12 w22 w41² w53² − a21 w32 w41² w53² − a22 w11² w42 w53² − a22 w21² w42 w53² −

a22 w31² w42 w53² + a11 w11 w41 w42 w53² + a12 w21 w41 w42 w53² + a21 w31 w41 w42 w53²

First we solve the equation linear1 = 0 for n2 and substitute its solution into the "red" equation:

```
Solve[linear1 == 0, n2]
```

Out[•]=

$\{\{$n2 → $($w11 w13 w51 + w21 w23 w51 + w31 w33 w51 + w41 w43 w51 − a11 w13 w51² − a12 w23 w51² −

a21 w33 w51² − a22 w43 w51² − w11² w53 − w21² w53 − w31² w53 − w41² w53 +

a11 w11 w51 w53 + a12 w21 w51 w53 + a21 w31 w51 w53 + a22 w41 w51 w53$)$ $/$

$($ −w11 w12 w51 − w21 w22 w51 − w31 w32 w51 − w41 w42 w51 + a11 w12 w51² +

a12 w22 w51² + a21 w32 w51² + a22 w42 w51² + w11² w52 + w21² w52 + w31² w52 +

w41² w52 − a11 w11 w51 w52 − a12 w21 w51 w52 − a21 w31 w51 w52 − a22 w41 w51 w52$)$ $\}\}$

$\text{In[\circ]:=}$ **n1** $\to$ $-$ $\Big(\big(-$w13^2 w51 $-$ w23^2 w51 $-$ w33^2 w51 $-$ w43^2 w51 $+$

n2^2 $\big(-$w12^2 w51 $-$ w22^2 w51 $-$ w32^2 w51 $-$ w42^2 w51 $+$ w31 w32 w52 $+$ w41 w42 w52 $+$ a21

w32 w51 w52 $+$ a22 w42 w51 w52 $+$ w12 (w11 $+$ a11 w51) w52 $+$ w22 (w21 $+$ a12 w51)

w52 $-$ a11 w11 w52^2 $-$ a12 w21 w52^2 $-$ a21 w31 w52^2 $-$ a22 w41 w52$^2\big)$ $+$

w21 w23 w53 $+$ w31 w33 w53 $+$ w41 w43 w53 $+$ a12 w23 w51 w53 $+$

a21 w33 w51 w53 $+$ a22 w43 w51 w53 $+$ w13 (w11 $+$ a11 w51) w53 $-$

a11 w11 w53^2 $-$ a12 w21 w53^2 $-$ a21 w31 w53^2 $-$ a22 w41 w53^2 $+$

n2 ($-$2 w12 w13 w51 $-$ 2 w22 w23 w51 $-$ 2 w32 w33 w51 $-$ 2 w42 w43 w51 $+$

w11 w13 w52 $+$ w21 w23 w52 $+$ w31 w33 w52 $+$ w41 w43 w52 $+$ a11 w13 w51 w52 $+$

a12 w23 w51 w52 $+$ a21 w33 w51 w52 $+$ a22 w43 w51 w52 $+$

w31 w32 w53 $+$ w41 w42 w53 $+$ a21 w32 w51 w53 $+$ a22 w42 w51 w53 $+$

w12 (w11 $+$ a11 w51) w53 $+$ w22 (w21 $+$ a12 w51) w53 $-$ 2 a11 w11 w52 w53 $-$

2 a12 w21 w52 w53 $-$ 2 a21 w31 w52 w53 $-$ 2 a22 w41 w52 w53)$\big)$ $\Big/$

$\big(-$w21 w23 w51 $-$ w31 w33 w51 $-$ w41 w43 w51 $+$ a11 w13 w51^2 $+$ a12 w23 w51^2 $+$

a21 w33 w51^2 $+$ a22 w43 w51^2 $+$

n2 $\big(-$w31 w32 w51 $-$ w41 w42 w51 $+$ a11 w12 w51^2 $+$ a12 w22 w51^2 $+$ a21 w32 w51^2 $+$

a22 w42 w51^2 $+$ w11^2 w52 $+$ w21^2 w52 $+$ w31^2 w52 $+$ w41^2 w52 $-$ a21 w31 w51 w52 $-$

a22 w41 w51 w52 $-$ w11 w51 (w12 $+$ a11 w52) $-$ w21 w51 (w22 $+$ a12 w52)$\big)$ $+$

w11^2 w53 $+$ w21^2 w53 $+$ w31^2 w53 $+$ w41^2 w53 $-$ a12 w21 w51 w53 $-$

a21 w31 w51 w53 $-$ a22 w41 w51 w53 $-$ w11 w51 (w13 $+$ a11 w53)$\big)\Big)$ /.

n2 $\to$ $\big($w11 w13 w51 $+$ w21 w23 w51 $+$ w31 w33 w51 $+$ w41 w43 w51 $-$ a11 w13 w51^2 $-$

a12 w23 w51^2 $-$ a21 w33 w51^2 $-$ a22 w43 w51^2 $-$ w11^2 w53 $-$

w21^2 w53 $-$ w31^2 w53 $-$ w41^2 w53 $+$ a11 w11 w51 w53 $+$

a12 w21 w51 w53 $+$ a21 w31 w51 w53 $+$ a22 w41 w51 w53$\big)$ $\Big/$

$\big(-$w11 w12 w51 $-$ w21 w22 w51 $-$ w31 w32 w51 $-$ w41 w42 w51 $+$ a11 w12 w51^2 $+$

a12 w22 w51^2 $+$ a21 w32 w51^2 $+$ a22 w42 w51^2 $+$ w11^2 w52 $+$ w21^2 w52 $+$

w31^2 w52 $+$ w41^2 w52 $-$ a11 w11 w51 w52 $-$ a12 w21 w51 w52 $-$

a21 w31 w51 w52 $-$ a22 w41 w51 w52$\big)$ // Simplify

⚫⚫⚫ **Simplify** : Expression

$-\Bigg(\bigg(\langle\!\langle 22\rangle\!\rangle + \dfrac{(\langle\!\langle 1\rangle\!\rangle)\,\langle\!\langle 1\rangle\!\rangle^2}{(\langle\!\langle 1\rangle\!\rangle)^2} + ((\text{w11 w13 w51} + \text{w21 w23 w51} + \text{w31 w33 w51} + \langle\!\langle 17\rangle\!\rangle + \text{a11 w11 w51 w53} +$

$\text{a12 w21 w51 w53} + \text{a21 w31 w51 w53} + \text{a22 w41 w51 w53})(\langle\!\langle 1\rangle\!\rangle))\big/(-\text{w11 w12 w51} - \text{w21}$

$\text{w22 w51} - \text{w31 w32 w51} - \text{w41 w42 w51} + \text{a11 w12 w51}^2 + \langle\!\langle 6\rangle\!\rangle + \text{w41}^2\,\text{w52} - \text{a11 w11 w51}$

$\text{w52} - \text{a12 w21 w51 w52} - \text{a21 w31 w51 w52} - \text{a22 w41 w51 w52})\bigg)\Big/\big(\langle\!\langle 21\rangle\!\rangle + ((-\text{w31 w32}$

$\text{w51} - \text{w41 w42 w51} + \text{a11 w12 w51}^2 + \text{a12 w22 w51}^2 + \langle\!\langle 14\rangle\!\rangle)(\text{w11 w13 w51} + \text{w21 w23}$

$\text{w51} + \text{w31 w33 w51} + \langle\!\langle 19\rangle\!\rangle + \text{a21 w31 w51 w53} + \text{a22 w41 w51 w53})\big/(-\text{w11 w12}$

$\text{w51} - \text{w21 w22 w51} - \text{w31 w32 w51} - \text{w41 w42 w51} + \text{a11 w12 Power}\,[\langle\!\langle 2\rangle\!\rangle] + \langle\!\langle 6\rangle\!\rangle + \langle\!\langle 1$

$\rangle\!\rangle - \text{a11 w11 w51 w52} - \text{a12 w21 w51 w52} - \text{a21 w31 w51 w52} - \text{a22 w41 w51 w52})\big)\Bigg)$

simplified to ComplexInfinity. ⓘ

$\text{Out[\circ]=}$

n1 $\to$ ComplexInfinity

This means that the solution of linear1 = 0 would yield a formula that makes a denominator identically equal to 0, hence the equation linear1 = 0 cannot yield the optimal solution.

We thus have to deal with the only remaining candidate, linear2 = 0.
We solve this equation for n2 and get formula (8):

```
Solve[linear2 == 0, n2]
```

Out[◦]=

$\{\{n2 \rightarrow \big(-w13^2\,w21\,w22\,w51 + w12\,w13\,w21\,w23\,w51 + w11\,w13\,w22\,w23\,w51 - w11\,w12\,w23^2\,w51 -$

$w13^2\,w31\,w32\,w51 - w23^2\,w31\,w32\,w51 + w12\,w13\,w31\,w33\,w51 + w22\,w23\,w31\,w33\,w51 +$

$w11\,w13\,w32\,w33\,w51 + w21\,w23\,w32\,w33\,w51 - w11\,w12\,w33^2\,w51 - w21\,w22\,w33^2\,w51 -$

$w13^2\,w41\,w42\,w51 - w23^2\,w41\,w42\,w51 - w33^2\,w41\,w42\,w51 + w12\,w13\,w41\,w43\,w51 +$

$w22\,w23\,w41\,w43\,w51 + w32\,w33\,w41\,w43\,w51 + w11\,w13\,w42\,w43\,w51 +$

$w21\,w23\,w42\,w43\,w51 + w31\,w33\,w42\,w43\,w51 - w11\,w12\,w43^2\,w51 -$

$w21\,w22\,w43^2\,w51 - w31\,w32\,w43^2\,w51 + a12\,w13^2\,w22\,w51^2 - a12\,w12\,w13\,w23\,w51^2 -$

$a11\,w13\,w22\,w23\,w51^2 + a11\,w12\,w23^2\,w51^2 + a21\,w13^2\,w32\,w51^2 + a21\,w23^2\,w32\,w51^2 -$

$a21\,w12\,w13\,w33\,w51^2 - a21\,w22\,w23\,w33\,w51^2 - a11\,w13\,w32\,w33\,w51^2 -$

$a12\,w23\,w32\,w33\,w51^2 + a11\,w12\,w33^2\,w51^2 + a12\,w22\,w33^2\,w51^2 + a22\,w13^2\,w42\,w51^2 +$

$a22\,w23^2\,w42\,w51^2 + a22\,w33^2\,w42\,w51^2 - a22\,w12\,w13\,w43\,w51^2 -$

$a22\,w22\,w23\,w43\,w51^2 - a22\,w32\,w33\,w43\,w51^2 - a11\,w13\,w42\,w43\,w51^2 -$

$a12\,w23\,w42\,w43\,w51^2 - a21\,w33\,w42\,w43\,w51^2 + a11\,w12\,w43^2\,w51^2 +$

$a12\,w22\,w43^2\,w51^2 + a21\,w32\,w43^2\,w51^2 + w13^2\,w21^2\,w52 - 2\,w11\,w13\,w21\,w23\,w52 +$

$w11^2\,w23^2\,w52 + w13^2\,w31^2\,w52 + w23^2\,w31^2\,w52 - 2\,w11\,w13\,w31\,w33\,w52 -$

$2\,w21\,w23\,w31\,w33\,w52 + w11^2\,w33^2\,w52 + w21^2\,w33^2\,w52 + w13^2\,w41^2\,w52 +$

$w23^2\,w41^2\,w52 + w33^2\,w41^2\,w52 - 2\,w11\,w13\,w41\,w43\,w52 - 2\,w21\,w23\,w41\,w43\,w52 -$

$2\,w31\,w33\,w41\,w43\,w52 + w11^2\,w43^2\,w52 + w21^2\,w43^2\,w52 + w31^2\,w43^2\,w52 -$

$a12\,w13^2\,w21\,w51\,w52 + a12\,w11\,w13\,w23\,w51\,w52 + a11\,w13\,w21\,w23\,w51\,w52 -$

$a11\,w11\,w23^2\,w51\,w52 - a21\,w13^2\,w31\,w51\,w52 - a21\,w23^2\,w31\,w51\,w52 +$

$a21\,w11\,w13\,w33\,w51\,w52 + a21\,w21\,w23\,w33\,w51\,w52 + a11\,w13\,w31\,w33\,w51\,w52 +$

$a12\,w23\,w31\,w33\,w51\,w52 - a11\,w11\,w33^2\,w51\,w52 - a12\,w21\,w33^2\,w51\,w52 -$

$a22\,w13^2\,w41\,w51\,w52 - a22\,w23^2\,w41\,w51\,w52 - a22\,w33^2\,w41\,w51\,w52 +$

$a22\,w11\,w13\,w43\,w51\,w52 + a22\,w21\,w23\,w43\,w51\,w52 + a22\,w31\,w33\,w43\,w51\,w52 +$

$a11\,w13\,w41\,w43\,w51\,w52 + a12\,w23\,w41\,w43\,w51\,w52 + a21\,w33\,w41\,w43\,w51\,w52 -$

$a11\,w11\,w43^2\,w51\,w52 - a12\,w21\,w43^2\,w51\,w52 - a21\,w31\,w43^2\,w51\,w52 -$

$w12\,w13\,w21^2\,w53 + w11\,w13\,w21\,w22\,w53 + w11\,w12\,w21\,w23\,w53 - w11^2\,w22\,w23\,w53 -$

$w12\,w13\,w31^2\,w53 - w22\,w23\,w31^2\,w53 + w11\,w13\,w31\,w32\,w53 + w21\,w23\,w31\,w32\,w53 +$

$w11\,w12\,w31\,w33\,w53 + w21\,w22\,w31\,w33\,w53 - w11^2\,w32\,w33\,w53 - w21^2\,w32\,w33\,w53 -$

$w12\,w13\,w41^2\,w53 - w22\,w23\,w41^2\,w53 - w32\,w33\,w41^2\,w53 + w11\,w13\,w41\,w42\,w53 +$

$w21\,w23\,w41\,w42\,w53 + w31\,w33\,w41\,w42\,w53 + w11\,w12\,w41\,w43\,w53 +$

$w21\,w22\,w41\,w43\,w53 + w31\,w32\,w41\,w43\,w53 - w11^2\,w42\,w43\,w53 - w21^2\,w42\,w43\,w53 -$

$w31^2\,w42\,w43\,w53 + a12\,w12\,w13\,w21\,w51\,w53 - 2\,a12\,w11\,w13\,w22\,w51\,w53 +$

$a11\,w13\,w21\,w22\,w51\,w53 + a12\,w11\,w12\,w23\,w51\,w53 - 2\,a11\,w12\,w21\,w23\,w51\,w53 +$

$a11\,w11\,w22\,w23\,w51\,w53 + a21\,w12\,w13\,w31\,w51\,w53 + a21\,w22\,w23\,w31\,w51\,w53 -$

$2\,a21\,w11\,w13\,w32\,w51\,w53 - 2\,a21\,w21\,w23\,w32\,w51\,w53 + a11\,w13\,w31\,w32\,w51\,w53 +$

$a12\,w23\,w31\,w32\,w51\,w53 + a21\,w11\,w12\,w33\,w51\,w53 + a21\,w21\,w22\,w33\,w51\,w53 -$

$2\,a11\,w12\,w31\,w33\,w51\,w53 - 2\,a12\,w22\,w31\,w33\,w51\,w53 + a11\,w11\,w32\,w33\,w51\,w53 +$

$a12\,w21\,w32\,w33\,w51\,w53 + a22\,w12\,w13\,w41\,w51\,w53 + a22\,w22\,w23\,w41\,w51\,w53 +$

$a22\,w32\,w33\,w41\,w51\,w53 - 2\,a22\,w11\,w13\,w42\,w51\,w53 - 2\,a22\,w21\,w23\,w42\,w51\,w53 -$

$2\,a22\,w31\,w33\,w42\,w51\,w53 + a11\,w13\,w41\,w42\,w51\,w53 + a12\,w23\,w41\,w42\,w51\,w53 +$

$$
\begin{aligned}
& a21\,w33\,w41\,w42\,w51\,w53 + a22\,w11\,w12\,w43\,w51\,w53 + a22\,w21\,w22\,w43\,w51\,w53 + \\
& a22\,w31\,w32\,w43\,w51\,w53 - 2\,a11\,w12\,w41\,w43\,w51\,w53 - 2\,a12\,w22\,w41\,w43\,w51\,w53 - \\
& 2\,a21\,w32\,w41\,w43\,w51\,w53 + a11\,w11\,w42\,w43\,w51\,w53 + a12\,w21\,w42\,w43\,w51\,w53 + \\
& a21\,w31\,w42\,w43\,w51\,w53 + a12\,w11\,w13\,w21\,w52\,w53 - a11\,w13\,w21^2\,w52\,w53 - \\
& a12\,w11^2\,w23\,w52\,w53 + a11\,w11\,w21\,w23\,w52\,w53 + a21\,w11\,w13\,w31\,w52\,w53 + \\
& a21\,w21\,w23\,w31\,w52\,w53 - a11\,w13\,w31^2\,w52\,w53 - a12\,w23\,w31^2\,w52\,w53 - \\
& a21\,w11^2\,w33\,w52\,w53 - a21\,w21^2\,w33\,w52\,w53 + a11\,w11\,w31\,w33\,w52\,w53 + \\
& a12\,w21\,w31\,w33\,w52\,w53 + a22\,w11\,w13\,w41\,w52\,w53 + a22\,w21\,w23\,w41\,w52\,w53 + \\
& a22\,w31\,w33\,w41\,w52\,w53 - a11\,w13\,w41^2\,w52\,w53 - a12\,w23\,w41^2\,w52\,w53 - \\
& a21\,w33\,w41^2\,w52\,w53 - a22\,w11^2\,w43\,w52\,w53 - a22\,w21^2\,w43\,w52\,w53 - \\
& a22\,w31^2\,w43\,w52\,w53 + a11\,w11\,w41\,w43\,w52\,w53 + a12\,w21\,w41\,w43\,w52\,w53 + \\
& a21\,w31\,w41\,w43\,w52\,w53 - a12\,w11\,w12\,w21\,w53^2 + a11\,w12\,w21^2\,w53^2 + \\
& a12\,w11^2\,w22\,w53^2 - a11\,w11\,w21\,w22\,w53^2 - a21\,w11\,w12\,w31\,w53^2 - \\
& a21\,w21\,w22\,w31\,w53^2 + a11\,w12\,w31^2\,w53^2 + a12\,w22\,w31^2\,w53^2 + \\
& a21\,w11^2\,w32\,w53^2 + a21\,w21^2\,w32\,w53^2 - a11\,w11\,w31\,w32\,w53^2 - \\
& a12\,w21\,w31\,w32\,w53^2 - a22\,w11\,w12\,w41\,w53^2 - a22\,w21\,w22\,w41\,w53^2 - \\
& a22\,w31\,w32\,w41\,w53^2 + a11\,w12\,w41^2\,w53^2 + a12\,w22\,w41^2\,w53^2 + \\
& a21\,w32\,w41^2\,w53^2 + a22\,w11^2\,w42\,w53^2 + a22\,w21^2\,w42\,w53^2 + a22\,w31^2\,w42\,w53^2 - \\
& a11\,w11\,w41\,w42\,w53^2 - a12\,w21\,w41\,w42\,w53^2 - a21\,w31\,w41\,w42\,w53^2 \Big) \,\Big/ \\[4pt]
& \Big(w12\,w13\,w21\,w22\,w51 - w11\,w13\,w22^2\,w51 - w12^2\,w21\,w23\,w51 + w11\,w12\,w22\,w23\,w51 + \\
& w12\,w13\,w31\,w32\,w51 + w22\,w23\,w31\,w32\,w51 - w11\,w13\,w32^2\,w51 - w21\,w23\,w32^2\,w51 - \\
& w12^2\,w31\,w33\,w51 - w22^2\,w31\,w33\,w51 + w11\,w12\,w32\,w33\,w51 + w21\,w22\,w32\,w33\,w51 + \\
& w12\,w13\,w41\,w42\,w51 + w22\,w23\,w41\,w42\,w51 + w32\,w33\,w41\,w42\,w51 - \\
& w11\,w13\,w42^2\,w51 - w21\,w23\,w42^2\,w51 - w31\,w33\,w42^2\,w51 - w12^2\,w41\,w43\,w51 - \\
& w22^2\,w41\,w43\,w51 - w32^2\,w41\,w43\,w51 + w11\,w12\,w42\,w43\,w51 + w21\,w22\,w42\,w43\,w51 + \\
& w31\,w32\,w42\,w43\,w51 - a12\,w12\,w13\,w22\,w51^2 + a11\,w13\,w22^2\,w51^2 + a12\,w12^2\,w23\,w51^2 - \\
& a11\,w12\,w22\,w23\,w51^2 - a21\,w12\,w13\,w32\,w51^2 - a21\,w22\,w23\,w32\,w51^2 + \\
& a11\,w13\,w32^2\,w51^2 + a12\,w23\,w32^2\,w51^2 + a21\,w12^2\,w33\,w51^2 + a21\,w22^2\,w33\,w51^2 - \\
& a11\,w12\,w32\,w33\,w51^2 - a12\,w22\,w32\,w33\,w51^2 - a22\,w12\,w13\,w42\,w51^2 - \\
& a22\,w22\,w23\,w42\,w51^2 - a22\,w32\,w33\,w42\,w51^2 + a11\,w13\,w42^2\,w51^2 + \\
& a12\,w23\,w42^2\,w51^2 + a21\,w33\,w42^2\,w51^2 + a22\,w12^2\,w43\,w51^2 + a22\,w22^2\,w43\,w51^2 + \\
& a22\,w32^2\,w43\,w51^2 - a11\,w12\,w42\,w43\,w51^2 - a12\,w22\,w42\,w43\,w51^2 - \\
& a21\,w32\,w42\,w43\,w51^2 - w12\,w13\,w21^2\,w52 + w11\,w13\,w21\,w22\,w52 + \\
& w11\,w12\,w21\,w23\,w52 - w11^2\,w22\,w23\,w52 - w12\,w13\,w31^2\,w52 - w22\,w23\,w31^2\,w52 + \\
& w11\,w13\,w31\,w32\,w52 + w21\,w23\,w31\,w32\,w52 + w11\,w12\,w31\,w33\,w52 + \\
& w21\,w22\,w31\,w33\,w52 - w11^2\,w32\,w33\,w52 - w21^2\,w32\,w33\,w52 - w12\,w13\,w41^2\,w52 - \\
& w22\,w23\,w41^2\,w52 - w32\,w33\,w41^2\,w52 + w11\,w13\,w41\,w42\,w52 + w21\,w23\,w41\,w42\,w52 + \\
& w31\,w33\,w41\,w42\,w52 + w11\,w12\,w41\,w43\,w52 + w21\,w22\,w41\,w43\,w52 + \\
& w31\,w32\,w41\,w43\,w52 - w11^2\,w42\,w43\,w52 - w21^2\,w42\,w43\,w52 - w31^2\,w42\,w43\,w52 + \\
& a12\,w12\,w13\,w21\,w51\,w52 + a12\,w11\,w13\,w22\,w51\,w52 - 2\,a11\,w13\,w21\,w22\,w51\,w52 - \\
& 2\,a12\,w11\,w12\,w23\,w51\,w52 + a11\,w12\,w21\,w23\,w51\,w52 + a11\,w11\,w22\,w23\,w51\,w52 + \\
& a21\,w12\,w13\,w31\,w51\,w52 + a21\,w22\,w23\,w31\,w51\,w52 + a21\,w11\,w13\,w32\,w51\,w52 + \\
& a21\,w21\,w23\,w32\,w51\,w52 - 2\,a11\,w13\,w31\,w32\,w51\,w52 - 2\,a12\,w23\,w31\,w32\,w51\,w52 - \\
& 2\,a21\,w11\,w12\,w33\,w51\,w52 - 2\,a21\,w21\,w22\,w33\,w51\,w52 + a11\,w12\,w31\,w33\,w51\,w52 + \\
& a12\,w22\,w31\,w33\,w51\,w52 + a11\,w11\,w32\,w33\,w51\,w52 + a12\,w21\,w32\,w33\,w51\,w52 + \\
& a22\,w12\,w13\,w41\,w51\,w52 + a22\,w22\,w23\,w41\,w51\,w52 + a22\,w32\,w33\,w41\,w51\,w52 + \\
& a22\,w11\,w13\,w42\,w51\,w52 + a22\,w21\,w23\,w42\,w51\,w52 + a22\,w31\,w33\,w42\,w51\,w52 -
\end{aligned}
$$

$2\,a11\,w13\,w41\,w42\,w51\,w52 - 2\,a12\,w23\,w41\,w42\,w51\,w52 - 2\,a21\,w33\,w41\,w42\,w51\,w52 -$
$2\,a22\,w11\,w12\,w43\,w51\,w52 - 2\,a22\,w21\,w22\,w43\,w51\,w52 - 2\,a22\,w31\,w32\,w43\,w51\,w52 +$
$a11\,w12\,w41\,w43\,w51\,w52 + a12\,w22\,w41\,w43\,w51\,w52 + a21\,w32\,w41\,w43\,w51\,w52 +$
$a11\,w11\,w42\,w43\,w51\,w52 + a12\,w21\,w42\,w43\,w51\,w52 + a21\,w31\,w42\,w43\,w51\,w52 -$
$a12\,w11\,w13\,w21\,w52^2 + a11\,w13\,w21^2\,w52^2 + a12\,w11^2\,w23\,w52^2 -$
$a11\,w11\,w21\,w23\,w52^2 - a21\,w11\,w13\,w31\,w52^2 - a21\,w21\,w23\,w31\,w52^2 +$
$a11\,w13\,w31^2\,w52^2 + a12\,w23\,w31^2\,w52^2 + a21\,w11^2\,w33\,w52^2 + a21\,w21^2\,w33\,w52^2 -$
$a11\,w11\,w31\,w33\,w52^2 - a12\,w21\,w31\,w33\,w52^2 - a22\,w11\,w13\,w41\,w52^2 -$
$a22\,w21\,w23\,w41\,w52^2 - a22\,w31\,w33\,w41\,w52^2 + a11\,w13\,w41^2\,w52^2 +$
$a12\,w23\,w41^2\,w52^2 + a21\,w33\,w41^2\,w52^2 + a22\,w11^2\,w43\,w52^2 + a22\,w21^2\,w43\,w52^2 +$
$a22\,w31^2\,w43\,w52^2 - a11\,w11\,w41\,w43\,w52^2 - a12\,w21\,w41\,w43\,w52^2 -$
$a21\,w31\,w41\,w43\,w52^2 + w12^2\,w21^2\,w53 - 2\,w11\,w12\,w21\,w22\,w53 +$
$w11^2\,w22^2\,w53 + w12^2\,w31^2\,w53 + w22^2\,w31^2\,w53 - 2\,w11\,w12\,w31\,w32\,w53 -$
$2\,w21\,w22\,w31\,w32\,w53 + w11^2\,w32^2\,w53 + w21^2\,w32^2\,w53 + w12^2\,w41^2\,w53 +$
$w22^2\,w41^2\,w53 + w32^2\,w41^2\,w53 - 2\,w11\,w12\,w41\,w42\,w53 - 2\,w21\,w22\,w41\,w42\,w53 -$
$2\,w31\,w32\,w41\,w42\,w53 + w11^2\,w42^2\,w53 + w21^2\,w42^2\,w53 + w31^2\,w42^2\,w53 -$
$a12\,w12^2\,w21\,w51\,w53 + a12\,w11\,w12\,w22\,w51\,w53 + a11\,w12\,w21\,w22\,w51\,w53 -$
$a11\,w11\,w22^2\,w51\,w53 - a21\,w12^2\,w31\,w51\,w53 - a21\,w22^2\,w31\,w51\,w53 +$
$a21\,w11\,w12\,w32\,w51\,w53 + a21\,w21\,w22\,w32\,w51\,w53 + a11\,w12\,w31\,w32\,w51\,w53 +$
$a12\,w22\,w31\,w32\,w51\,w53 - a11\,w11\,w32^2\,w51\,w53 - a12\,w21\,w32^2\,w51\,w53 -$
$a22\,w12^2\,w41\,w51\,w53 - a22\,w22^2\,w41\,w51\,w53 - a22\,w32^2\,w41\,w51\,w53 +$
$a22\,w11\,w12\,w42\,w51\,w53 + a22\,w21\,w22\,w42\,w51\,w53 + a22\,w31\,w32\,w42\,w51\,w53 +$
$a11\,w12\,w41\,w42\,w51\,w53 + a12\,w22\,w41\,w42\,w51\,w53 + a21\,w32\,w41\,w42\,w51\,w53 -$
$a11\,w11\,w42^2\,w51\,w53 - a12\,w21\,w42^2\,w51\,w53 - a21\,w31\,w42^2\,w51\,w53 +$
$a12\,w11\,w12\,w21\,w52\,w53 - a11\,w12\,w21^2\,w52\,w53 - a12\,w11^2\,w22\,w52\,w53 +$
$a11\,w11\,w21\,w22\,w52\,w53 + a21\,w11\,w12\,w31\,w52\,w53 + a21\,w21\,w22\,w31\,w52\,w53 -$
$a11\,w12\,w31^2\,w52\,w53 - a12\,w22\,w31^2\,w52\,w53 - a21\,w11^2\,w32\,w52\,w53 -$
$a21\,w21^2\,w32\,w52\,w53 + a11\,w11\,w31\,w32\,w52\,w53 + a12\,w21\,w31\,w32\,w52\,w53 +$
$a22\,w11\,w12\,w41\,w52\,w53 + a22\,w21\,w22\,w41\,w52\,w53 + a22\,w31\,w32\,w41\,w52\,w53 -$
$a11\,w12\,w41^2\,w52\,w53 - a12\,w22\,w41^2\,w52\,w53 - a21\,w32\,w41^2\,w52\,w53 -$
$a22\,w11^2\,w42\,w52\,w53 - a22\,w21^2\,w42\,w52\,w53 - a22\,w31^2\,w42\,w52\,w53 +$
$a11\,w11\,w41\,w42\,w52\,w53 + a12\,w21\,w41\,w42\,w52\,w53 + a21\,w31\,w41\,w42\,w52\,w53 \Big)\Big\}\Big\}$

This yields the second component of the optimal vector $\boldsymbol{n}_g{}^*$ in Case A.

If we now substitute this last "yellow" expression into the "red" expression for n1, we obtain the first component of the optimal vector $\boldsymbol{n}_g{}^*$ in Case A.

Case B, n3=0

We make the substitution n3 = 0:

$In[\circ]:=$
$$\left(a11 - \frac{n1\ w11 + n2\ w12 + n3\ w13}{n1\ w51 + n2\ w52 + n3\ w53}\right)^2 + \left(a12 - \frac{n1\ w21 + n2\ w22 + n3\ w23}{n1\ w51 + n2\ w52 + n3\ w53}\right)^2 +$$
$$\left(a21 - \frac{n1\ w31 + n2\ w32 + n3\ w33}{n1\ w51 + n2\ w52 + n3\ w53}\right)^2 + \left(a22 - \frac{n1\ w41 + n2\ w42 + n3\ w43}{n1\ w51 + n2\ w52 + n3\ w53}\right)^2\ /.\ n3 \to 0$$

$Out[\circ]=$
$$\left(a11 - \frac{n1\ w11 + n2\ w12}{n1\ w51 + n2\ w52}\right)^2 + \left(a12 - \frac{n1\ w21 + n2\ w22}{n1\ w51 + n2\ w52}\right)^2 +$$
$$\left(a21 - \frac{n1\ w31 + n2\ w32}{n1\ w51 + n2\ w52}\right)^2 + \left(a22 - \frac{n1\ w41 + n2\ w42}{n1\ w51 + n2\ w52}\right)^2$$

We compute its partial derivative w.r.t. the first variable (that is, $\partial_1 h(n_x, n_y)$):

$In[\circ]:=$
$$D\left[\left(a11 - \frac{n1\ w11 + n2\ w12}{n1\ w51 + n2\ w52}\right)^2 + \left(a12 - \frac{n1\ w21 + n2\ w22}{n1\ w51 + n2\ w52}\right)^2 +\right.$$
$$\left.\left(a21 - \frac{n1\ w31 + n2\ w32}{n1\ w51 + n2\ w52}\right)^2 + \left(a22 - \frac{n1\ w41 + n2\ w42}{n1\ w51 + n2\ w52}\right)^2,\ n1\right]\ //\ \text{Together}$$

$Out[\circ]=$
$$\frac{1}{(n1\ w51 + n2\ w52)^3}$$
$$2\ \left(- n1\ n2\ w11\ w12\ w51 - n2^2\ w12^2\ w51 - n1\ n2\ w21\ w22\ w51 - n2^2\ w22^2\ w51 - n1\ n2\ w31\ w32\ w51 -\right.$$
$$n2^2\ w32^2\ w51 - n1\ n2\ w41\ w42\ w51 - n2^2\ w42^2\ w51 + a11\ n1\ n2\ w12\ w51^2 +$$
$$a12\ n1\ n2\ w22\ w51^2 + a21\ n1\ n2\ w32\ w51^2 + a22\ n1\ n2\ w42\ w51^2 + n1\ n2\ w11^2\ w52 +$$
$$n2^2\ w11\ w12\ w52 + n1\ n2\ w21^2\ w52 + n2^2\ w21\ w22\ w52 + n1\ n2\ w31^2\ w52 + n2^2\ w31\ w32\ w52 +$$
$$n1\ n2\ w41^2\ w52 + n2^2\ w41\ w42\ w52 - a11\ n1\ n2\ w11\ w51\ w52 + a11\ n2^2\ w12\ w51\ w52 -$$
$$a12\ n1\ n2\ w21\ w51\ w52 + a12\ n2^2\ w22\ w51\ w52 - a21\ n1\ n2\ w31\ w51\ w52 +$$
$$a21\ n2^2\ w32\ w51\ w52 - a22\ n1\ n2\ w41\ w51\ w52 + a22\ n2^2\ w42\ w51\ w52 -$$
$$\left.a11\ n2^2\ w11\ w52^2 - a12\ n2^2\ w21\ w52^2 - a21\ n2^2\ w31\ w52^2 - a22\ n2^2\ w41\ w52^2\right)$$

By solving for n1, we get formula (10):

$In[\circ]:=$
$$\text{Solve}\left[- n1\ n2\ w11\ w12\ w51 - n2^2\ w12^2\ w51 - n1\ n2\ w21\ w22\ w51 - n2^2\ w22^2\ w51 -\right.$$
$$n1\ n2\ w31\ w32\ w51 - n2^2\ w32^2\ w51 - n1\ n2\ w41\ w42\ w51 - n2^2\ w42^2\ w51 + a11\ n1\ n2\ w12\ w51^2 +$$
$$a12\ n1\ n2\ w22\ w51^2 + a21\ n1\ n2\ w32\ w51^2 + a22\ n1\ n2\ w42\ w51^2 + n1\ n2\ w11^2\ w52 +$$
$$n2^2\ w11\ w12\ w52 + n1\ n2\ w21^2\ w52 + n2^2\ w21\ w22\ w52 + n1\ n2\ w31^2\ w52 + n2^2\ w31\ w32\ w52 +$$
$$n1\ n2\ w41^2\ w52 + n2^2\ w41\ w42\ w52 - a11\ n1\ n2\ w11\ w51\ w52 + a11\ n2^2\ w12\ w51\ w52 -$$
$$a12\ n1\ n2\ w21\ w51\ w52 + a12\ n2^2\ w22\ w51\ w52 - a21\ n1\ n2\ w31\ w51\ w52 +$$
$$a21\ n2^2\ w32\ w51\ w52 - a22\ n1\ n2\ w41\ w51\ w52 + a22\ n2^2\ w42\ w51\ w52 -$$
$$\left.a11\ n2^2\ w11\ w52^2 - a12\ n2^2\ w21\ w52^2 - a21\ n2^2\ w31\ w52^2 - a22\ n2^2\ w41\ w52^2 == 0,\ n1\right]$$

$Out[\circ]=$
$$\left\{\left\{n1 \to\right.\right.$$
$$\left(n2\ w12^2\ w51 + n2\ w22^2\ w51 + n2\ w32^2\ w51 + n2\ w42^2\ w51 - n2\ w11\ w12\ w52 - n2\ w21\ w22\ w52 -\right.$$
$$n2\ w31\ w32\ w52 - n2\ w41\ w42\ w52 - a11\ n2\ w12\ w51\ w52 - a12\ n2\ w22\ w51\ w52 -$$
$$a21\ n2\ w32\ w51\ w52 - a22\ n2\ w42\ w51\ w52 + a11\ n2\ w11\ w52^2 +$$
$$\left.a12\ n2\ w21\ w52^2 + a21\ n2\ w31\ w52^2 + a22\ n2\ w41\ w52^2\right)\ \Big/$$
$$\left(- w11\ w12\ w51 - w21\ w22\ w51 - w31\ w32\ w51 - w41\ w42\ w51 + a11\ w12\ w51^2 +\right.$$
$$a12\ w22\ w51^2 + a21\ w32\ w51^2 + a22\ w42\ w51^2 + w11^2\ w52 + w21^2\ w52 + w31^2\ w52 +$$
$$\left.\left.\left.w41^2\ w52 - a11\ w11\ w51\ w52 - a12\ w21\ w51\ w52 - a21\ w31\ w51\ w52 - a22\ w41\ w51\ w52\right)\right\}\right\}$$

By taking the partial derivative w.r.t. the second variable (that is, $\partial_2 h(n_x, n_y)$) and replacing n1 with the above expression, we get an expression which is identically equal to 0:

In[ ]:=
$$\left(D\left[\left(a11 - \frac{n1\ w11 + n2\ w12}{n1\ w51 + n2\ w52}\right)^2 + \left(a12 - \frac{n1\ w21 + n2\ w22}{n1\ w51 + n2\ w52}\right)^2 + \right.\right.$$
$$\left.\left(a21 - \frac{n1\ w31 + n2\ w32}{n1\ w51 + n2\ w52}\right)^2 + \left(a22 - \frac{n1\ w41 + n2\ w42}{n1\ w51 + n2\ w52}\right)^2, \text{n2}\right] \text{ // Together}\right) \text{ /.}$$
$$\text{n1} \rightarrow \left(\text{n2}\ w12^2\ w51 + \text{n2}\ w22^2\ w51 + \text{n2}\ w32^2\ w51 + \text{n2}\ w42^2\ w51 - \text{n2}\ w11\ w12\ w52 -\right.$$
$$\text{n2}\ w21\ w22\ w52 - \text{n2}\ w31\ w32\ w52 - \text{n2}\ w41\ w42\ w52 - a11\ \text{n2}\ w12\ w51\ w52 -$$
$$a12\ \text{n2}\ w22\ w51\ w52 - a21\ \text{n2}\ w32\ w51\ w52 - a22\ \text{n2}\ w42\ w51\ w52 +$$
$$\left.a11\ \text{n2}\ w11\ w52^2 + a12\ \text{n2}\ w21\ w52^2 + a21\ \text{n2}\ w31\ w52^2 + a22\ \text{n2}\ w41\ w52^2\right) \text{ /}$$
$$\left(-w11\ w12\ w51 - w21\ w22\ w51 - w31\ w32\ w51 - w41\ w42\ w51 + a11\ w12\ w51^2 +\right.$$
$$a12\ w22\ w51^2 + a21\ w32\ w51^2 + a22\ w42\ w51^2 + w11^2\ w52 + w21^2\ w52 + w31^2\ w52 + w41^2\ w52 -$$
$$\left.a11\ w11\ w51\ w52 - a12\ w21\ w51\ w52 - a21\ w31\ w51\ w52 - a22\ w41\ w51\ w52\right) \text{ // Simplify}$$

Out[ ]=

0

This means that the second equation is automatically satisfied, hence system (9) has a one-parameter family of candidate optima.

By substituting the expression for n1 into the objective function, we get (14):

In[ ]:=

$$\left(a11 - \frac{n1\,w11 + n2\,w12}{n1\,w51 + n2\,w52}\right)^2 + \left(a12 - \frac{n1\,w21 + n2\,w22}{n1\,w51 + n2\,w52}\right)^2 +$$
$$\left(a21 - \frac{n1\,w31 + n2\,w32}{n1\,w51 + n2\,w52}\right)^2 + \left(a22 - \frac{n1\,w41 + n2\,w42}{n1\,w51 + n2\,w52}\right)^2\ /.$$
$$n1 \to \left(n2\,w12^2\,w51 + n2\,w22^2\,w51 + n2\,w32^2\,w51 + n2\,w42^2\,w51 - n2\,w11\,w12\,w52 - \right.$$
$$n2\,w21\,w22\,w52 - n2\,w31\,w32\,w52 - n2\,w41\,w42\,w52 - a11\,n2\,w12\,w51\,w52 - $$
$$a12\,n2\,w22\,w51\,w52 - a21\,n2\,w32\,w51\,w52 - a22\,n2\,w42\,w51\,w52 + $$
$$\left. a11\,n2\,w11\,w52^2 + a12\,n2\,w21\,w52^2 + a21\,n2\,w31\,w52^2 + a22\,n2\,w41\,w52^2\right) \Big/$$
$$\left(-w11\,w12\,w51 - w21\,w22\,w51 - w31\,w32\,w51 - w41\,w42\,w51 + a11\,w12\,w51^2 + \right.$$
$$a12\,w22\,w51^2 + a21\,w32\,w51^2 + a22\,w42\,w51^2 + w11^2\,w52 + w21^2\,w52 + w31^2\,w52 + w41^2\,w52 - $$
$$\left. a11\,w11\,w51\,w52 - a12\,w21\,w51\,w52 - a21\,w31\,w51\,w52 - a22\,w41\,w51\,w52\right)\ //\ \text{Simplify}$$

Out[ ]=

$$\left(w22^2\,w31^2 - 2\,w21\,w22\,w31\,w32 + w21^2\,w32^2 + w22^2\,w41^2 + w32^2\,w41^2 - 2\,w21\,w22\,w41\,w42 - \right.$$
$$2\,w31\,w32\,w41\,w42 + w21^2\,w42^2 + w31^2\,w42^2 - 2\,a21\,w22^2\,w31\,w51 + 2\,a21\,w21\,w22\,w32\,w51 + $$
$$2\,a12\,w22\,w31\,w32\,w51 - 2\,a12\,w21\,w32^2\,w51 - 2\,a22\,w22^2\,w41\,w51 - 2\,a22\,w32^2\,w41\,w51 + $$
$$2\,a22\,w21\,w22\,w42\,w51 + 2\,a22\,w31\,w32\,w42\,w51 + 2\,a12\,w22\,w41\,w42\,w51 + $$
$$2\,a21\,w32\,w41\,w42\,w51 - 2\,a12\,w21\,w42^2\,w51 - 2\,a21\,w31\,w42^2\,w51 + a11^2\,w22^2\,w51^2 + $$
$$a21^2\,w22^2\,w51^2 + a22^2\,w22^2\,w51^2 - 2\,a12\,a21\,w22\,w32\,w51^2 + a11^2\,w32^2\,w51^2 + $$
$$a12^2\,w32^2\,w51^2 + a22^2\,w32^2\,w51^2 - 2\,a12\,a22\,w22\,w42\,w51^2 - 2\,a21\,a22\,w32\,w42\,w51^2 + $$
$$a11^2\,w42^2\,w51^2 + a12^2\,w42^2\,w51^2 + a21^2\,w42^2\,w51^2 + w12^2\,\left(w21^2 + w31^2 + w41^2 - \right.$$
$$\left. 2\,a12\,w21\,w51 - 2\,a21\,w31\,w51 - 2\,a22\,w41\,w51 + a12^2\,w51^2 + a21^2\,w51^2 + a22^2\,w51^2\right) + $$
$$2\,a21\,w21\,w22\,w31\,w52 - 2\,a12\,w22\,w31^2\,w52 - 2\,a21\,w21^2\,w32\,w52 + $$
$$2\,a12\,w21\,w31\,w32\,w52 + 2\,a22\,w21\,w22\,w41\,w52 + 2\,a22\,w31\,w32\,w41\,w52 - $$
$$2\,a12\,w22\,w41^2\,w52 - 2\,a21\,w32\,w41^2\,w52 - 2\,a22\,w21^2\,w42\,w52 - 2\,a22\,w31^2\,w42\,w52 + $$
$$2\,a12\,w21\,w41\,w42\,w52 + 2\,a21\,w31\,w41\,w42\,w52 - 2\,a11^2\,w21\,w22\,w51\,w52 - $$
$$2\,a21^2\,w21\,w22\,w51\,w52 - 2\,a22^2\,w21\,w22\,w51\,w52 + 2\,a12\,a21\,w22\,w31\,w51\,w52 + $$
$$2\,a12\,a21\,w21\,w32\,w51\,w52 - 2\,a11^2\,w31\,w32\,w51\,w52 - 2\,a12^2\,w31\,w32\,w51\,w52 - $$
$$2\,a22^2\,w31\,w32\,w51\,w52 + 2\,a12\,a22\,w22\,w41\,w51\,w52 + 2\,a21\,a22\,w32\,w41\,w51\,w52 + $$
$$2\,a12\,a22\,w21\,w42\,w51\,w52 + 2\,a21\,a22\,w31\,w42\,w51\,w52 - 2\,a11^2\,w41\,w42\,w51\,w52 - $$
$$2\,a12^2\,w41\,w42\,w51\,w52 - 2\,a21^2\,w41\,w42\,w51\,w52 + a11^2\,w21^2\,w52^2 + a21^2\,w21^2\,w52^2 + $$
$$a22^2\,w21^2\,w52^2 - 2\,a12\,a21\,w21\,w31\,w52^2 + a11^2\,w31^2\,w52^2 + a12^2\,w31^2\,w52^2 + $$
$$a22^2\,w31^2\,w52^2 - 2\,a12\,a22\,w21\,w41\,w52^2 - 2\,a21\,a22\,w31\,w41\,w52^2 + a11^2\,w41^2\,w52^2 + $$
$$a12^2\,w41^2\,w52^2 + a21^2\,w41^2\,w52^2 + w11^2\,\left(w22^2 + w32^2 + w42^2 - 2\,a12\,w22\,w52 - \right.$$
$$\left. 2\,a21\,w32\,w52 - 2\,a22\,w42\,w52 + a12^2\,w52^2 + a21^2\,w52^2 + a22^2\,w52^2\right) - 2\,a11\,w11$$
$$\left(w22^2\,w51 + w32^2\,w51 + w42^2\,w51 - w41\,w42\,w52 - a22\,w42\,w51\,w52 - w22\,(w21 + a12\,w51)\right.$$
$$\left. w52 - w32\,(w31 + a21\,w51)\,w52 + a12\,w21\,w52^2 + a21\,w31\,w52^2 + a22\,w41\,w52^2\right) - 2\,w12$$
$$\left(w11\,\left(w41\,w42 - a12\,w22\,w51 - a21\,w32\,w51 - a22\,w42\,w51 - a22\,w41\,w52 + a12^2\,w51\,w52 + \right.\right.$$
$$\left. a21^2\,w51\,w52 + a22^2\,w51\,w52 + w21\,(w22 - a12\,w52) + w31\,(w32 - a21\,w52)\right) + $$
$$a11\,\left(-w41\,w42\,w51 + a12\,w22\,w51^2 + a21\,w32\,w51^2 + a22\,w42\,w51^2 + \right.$$
$$w21^2\,w52 + w31^2\,w52 + w41^2\,w52 - a22\,w41\,w51\,w52 - $$
$$\left.\left.\left.\left. w21\,w51\,(w22 + a12\,w52) - w31\,w51\,(w32 + a21\,w52)\right)\right)\right) \right/$$
$$\left(w12^2\,w51^2 + w22^2\,w51^2 + w32^2\,w51^2 + w42^2\,w51^2 - 2\,w11\,w12\,w51\,w52 - \right.$$
$$2\,w21\,w22\,w51\,w52 - 2\,w31\,w32\,w51\,w52 - 2\,w41\,w42\,w51\,w52 + $$
$$\left. w11^2\,w52^2 + w21^2\,w52^2 + w31^2\,w52^2 + w41^2\,w52^2\right)$$