

Supplementary material
for the paper
*Fast Globally Optimal Surface Normal Estimation from an
Affine Correspondence*

ICCV2023 submission with ID 10102

This document gives all elements that are missing in the original submission due to the page limit. The main message of this document is that only elementary operations are used in the formulas, This fact explains why the algorithm is rapid.

The missing terms are as follows:

$$\begin{aligned} \text{numer}_1 := & a_{11}w_{11,x}w_{c,y}^2n_y^2 + a_{12}w_{12,x}w_{c,y}^2n_y^2 + a_{21}w_{21,x}w_{c,y}^2n_y^2 + a_{22}w_{22,x}w_{c,y}^2n_y^2 + w_{11,y}^2w_{c,x}n_y^2 + \\ & w_{12,y}^2w_{c,x}n_y^2 + w_{21,y}^2w_{c,x}n_y^2 + w_{22,y}^2w_{c,x}n_y^2 - w_{11,x}w_{11,y}w_{c,y}n_y^2 - w_{12,x}w_{12,y}w_{c,y}n_y^2 - w_{21,x}w_{21,y}w_{c,y}n_y^2 - \\ & w_{22,x}w_{22,y}w_{c,y}n_y^2 - a_{11}w_{11,y}w_{c,x}w_{c,y}n_y^2 - a_{12}w_{12,y}w_{c,x}w_{c,y}n_y^2 - a_{21}w_{21,y}w_{c,x}w_{c,y}n_y^2 - a_{22}w_{22,y}w_{c,x}w_{c,y}n_y^2 + \\ & 2w_{11,y}w_{11,z}w_{c,x}n_y + 2w_{12,y}w_{12,z}w_{c,x}n_y + 2w_{21,y}w_{21,z}w_{c,x}n_y + 2w_{22,y}w_{22,z}w_{c,x}n_y - w_{11,x}w_{11,z}w_{c,y}n_y - \\ & w_{12,x}w_{12,z}w_{c,y}n_y - w_{21,x}w_{21,z}w_{c,y}n_y - w_{22,x}w_{22,z}w_{c,y}n_y - a_{11}w_{11,z}w_{c,x}w_{c,y}n_y - a_{12}w_{12,z}w_{c,x}w_{c,y}n_y - \\ & a_{21}w_{21,z}w_{c,x}w_{c,y}n_y - a_{22}w_{22,z}w_{c,x}w_{c,y}n_y - w_{11,x}w_{11,y}w_{c,z}n_y - w_{12,x}w_{12,y}w_{c,z}n_y - w_{21,x}w_{21,y}w_{c,z}n_y - \\ & w_{22,x}w_{22,y}w_{c,z}n_y - a_{11}w_{11,y}w_{c,x}w_{c,z}n_y - a_{12}w_{12,y}w_{c,x}w_{c,z}n_y - a_{21}w_{21,y}w_{c,x}w_{c,z}n_y - a_{22}w_{22,y}w_{c,x}w_{c,z}n_y + \\ & 2a_{11}w_{11,x}w_{c,y}w_{c,z}n_y + 2a_{12}w_{12,x}w_{c,y}w_{c,z}n_y + 2a_{21}w_{21,x}w_{c,y}w_{c,z}n_y + 2a_{22}w_{22,x}w_{c,y}w_{c,z}n_y + a_{11}w_{11,x}w_{c,z}^2 + \\ & a_{12}w_{12,x}w_{c,z}^2 + a_{21}w_{21,x}w_{c,z}^2 + a_{22}w_{22,x}w_{c,z}^2 + w_{11,z}^2w_{c,x} + w_{12,z}^2w_{c,x} + w_{21,z}^2w_{c,x} + w_{22,z}^2w_{c,x} - w_{11,x}w_{11,z}w_{c,z} - \\ & w_{12,x}w_{12,z}w_{c,z} - w_{21,x}w_{21,z}w_{c,z} - w_{22,x}w_{22,z}w_{c,z} - a_{11}w_{11,z}w_{c,x}w_{c,z} - a_{12}w_{12,z}w_{c,x}w_{c,z} - a_{21}w_{21,z}w_{c,x}w_{c,z} - \\ & a_{22}w_{22,z}w_{c,x}w_{c,z} \end{aligned}$$

$$\begin{aligned} \text{denom}_1 := & -a_{11}n_yw_{11,x}w_{c,x}w_{c,y} + a_{11}n_yw_{11,y}w_{c,x}^2 + a_{12}n_yw_{12,y}w_{c,x}^2 + a_{21}n_yw_{21,y}w_{c,x}^2 + \\ & a_{22}n_yw_{22,y}w_{c,x}^2 - a_{12}n_yw_{12,x}w_{c,x}w_{c,y} - a_{21}n_yw_{21,x}w_{c,x}w_{c,y} - a_{22}n_yw_{22,x}w_{c,x}w_{c,y} - a_{11}w_{11,x}w_{c,x}w_{c,z} + \\ & a_{11}w_{11,z}w_{c,x}^2 + a_{12}w_{12,z}w_{c,x}^2 + a_{21}w_{21,z}w_{c,x}^2 + a_{22}w_{22,z}w_{c,x}^2 - a_{12}w_{12,x}w_{c,x}w_{c,z} - a_{21}w_{21,x}w_{c,x}w_{c,z} - \\ & a_{22}w_{22,x}w_{c,x}w_{c,z} + n_yw_{11,x}w_{c,y} - n_yw_{11,y}w_{c,x} - n_yw_{12,x}w_{c,y} - n_yw_{12,y}w_{c,x} - n_yw_{21,x}w_{c,y} - n_yw_{21,y}w_{c,x} - \\ & n_yw_{22,x}w_{c,y} - n_yw_{22,y}w_{c,x} + n_yw_{12,x}^2w_{c,y} + n_yw_{12,y}^2w_{c,x} + w_{11,x}^2w_{c,z} - w_{11,x}w_{11,z}w_{c,x} - \\ & w_{12,x}w_{12,z}w_{c,x} - w_{21,x}w_{21,z}w_{c,x} - w_{22,x}w_{22,z}w_{c,x} + w_{12,x}^2w_{c,z} + w_{21,x}^2w_{c,z} + w_{22,x}^2w_{c,z} \end{aligned}$$

$$\begin{aligned}
\text{numer}_2 := & ((a_{12}w_{12,y} + a_{21}w_{21,y} + a_{22}w_{22,y})w_{c,z}^2 - (w_{12,y}w_{12,z} + a_{12}w_{c,y}w_{12,z} + w_{21,y}w_{21,z} + \\
& w_{22,y}w_{22,z} + a_{21}w_{21,z}w_{c,y} + a_{22}w_{22,z}w_{c,y})w_{c,z} + (w_{12,z}^2 + w_{21,z}^2 + w_{22,z}^2)w_{c,y})w_{11,x}^2 + (-((a_{12}w_{11,y}w_{12,x} + \\
& a_{11}w_{12,y}w_{12,x} + a_{21}w_{11,y}w_{21,x} + a_{11}w_{21,x}w_{21,y} + a_{22}w_{11,y}w_{22,x} + a_{11}w_{22,x}w_{22,y})w_{c,z}^2) + \\
& w_{11,y}(w_{12,x}w_{12,z} + a_{12}w_{c,x}w_{12,z} + w_{21,x}w_{21,z} + w_{22,x}w_{22,z} + a_{21}w_{21,z}w_{c,x} + a_{22}w_{22,z}w_{c,x})w_{c,z} + \\
& a_{11}(w_{12,y}w_{12,z}w_{c,x} + w_{21,y}w_{21,z}w_{c,x} + w_{22,y}w_{22,z}w_{c,x} + w_{12,x}w_{12,z}w_{c,y} + w_{21,x}w_{21,z}w_{c,y} + \\
& w_{22,x}w_{22,z}w_{c,y})w_{c,z} - (w_{12,z}^2 + w_{21,z}^2 + w_{22,z}^2)w_{c,x}(w_{11,y} + a_{11}w_{c,y}))w_{11,x} + a_{11}w_{11,y}w_{12,z}^2w_{c,x}^2 + \\
& a_{11}w_{11,y}w_{21,z}^2w_{c,x}^2 + a_{12}w_{12,y}w_{21,z}^2w_{c,x}^2 + a_{11}w_{11,y}w_{22,z}^2w_{c,x}^2 + a_{12}w_{12,y}w_{22,z}^2w_{c,x}^2 + a_{21}w_{21,y}w_{22,z}^2w_{c,x}^2 + \\
& a_{21}w_{12,z}^2w_{21,y}w_{c,x}^2 - a_{21}w_{12,y}w_{12,z}w_{21,z}w_{c,x}^2 - a_{12}w_{12,z}w_{21,y}w_{21,z}w_{c,x}^2 + a_{22}w_{12,z}^2w_{22,y}w_{c,x}^2 + \\
& a_{22}w_{21,z}^2w_{22,y}w_{c,x}^2 - a_{22}w_{12,y}w_{12,z}w_{22,z}w_{c,x}^2 - a_{22}w_{21,y}w_{21,z}w_{22,z}w_{c,x}^2 - a_{12}w_{12,z}w_{22,y}w_{22,z}w_{c,x}^2 - \\
& a_{21}w_{21,z}w_{22,y}w_{22,z}w_{c,x}^2 + (a_{21}w_{21,y}w_{12,x}^2 - a_{21}w_{12,y}w_{21,x}w_{12,x} - a_{12}w_{21,x}w_{21,y}w_{12,x} - a_{22}w_{12,y}w_{22,x}w_{12,x} + \\
& a_{12}w_{12,y}w_{21,x}^2 + a_{12}w_{12,y}w_{22,x}^2 + a_{21}w_{21,y}w_{22,x}^2 - a_{22}w_{21,x}w_{21,y}w_{22,x} + a_{11}w_{11,y}(w_{12,x}^2 + w_{21,x}^2 + w_{22,x}^2) + \\
& a_{22}(w_{12,x}^2 + w_{21,x}^2)w_{22,y} - (a_{12}w_{12,x} + a_{21}w_{21,x})w_{22,x}w_{22,y})w_{c,z}^2 - w_{12,x}w_{12,y}w_{21,x}^2w_{c,x} - \\
& w_{12,x}w_{12,y}w_{22,x}^2w_{c,x} - w_{21,x}w_{21,y}w_{22,x}^2w_{c,x} - w_{12,z}^2w_{21,x}w_{21,y}w_{c,x} + w_{12,y}w_{12,z}w_{21,x}w_{21,z}w_{c,x} + \\
& w_{12,x}w_{12,z}w_{21,y}w_{21,z}w_{c,x} - w_{12,z}^2w_{22,x}w_{22,y}w_{c,x} - w_{21,z}^2w_{22,x}w_{22,y}w_{c,x} + w_{12,y}w_{12,z}w_{22,x}w_{22,z}w_{c,x} + \\
& w_{21,y}w_{21,z}w_{22,x}w_{22,z}w_{c,x} + w_{12,x}w_{12,z}w_{22,y}w_{22,z}w_{c,x} + w_{21,x}w_{21,z}w_{22,y}w_{22,z}w_{c,x} + w_{12,z}^2w_{21,x}^2w_{c,y} + \\
& w_{12,x}^2w_{21,z}^2w_{c,y} + w_{12,z}^2w_{22,x}^2w_{c,y} + w_{21,z}^2w_{22,x}^2w_{c,y} + w_{12,x}^2w_{22,z}^2w_{c,y} + w_{21,x}^2w_{22,z}^2w_{c,y} - \\
& 2w_{12,x}w_{12,z}w_{21,x}w_{21,z}w_{c,y} - 2w_{12,x}w_{12,z}w_{22,x}w_{22,z}w_{c,y} - 2w_{21,x}w_{21,z}w_{22,x}w_{22,z}w_{c,y} - \\
& a_{12}w_{12,x}w_{21,z}^2w_{c,x}w_{c,y} - a_{12}w_{12,x}w_{22,z}^2w_{c,x}w_{c,y} - a_{21}w_{21,x}w_{22,z}^2w_{c,x}w_{c,y} - a_{21}w_{12,z}^2w_{21,x}w_{c,x}w_{c,y} + \\
& a_{21}w_{12,x}w_{12,z}w_{21,z}w_{c,x}w_{c,y} + a_{12}w_{12,z}w_{21,x}w_{21,z}w_{c,x}w_{c,y} - a_{22}w_{12,z}^2w_{22,x}w_{c,x}w_{c,y} - \\
& a_{22}w_{21,z}^2w_{22,x}w_{c,x}w_{c,y} + a_{22}w_{12,x}w_{12,z}w_{22,z}w_{c,x}w_{c,y} + a_{22}w_{21,x}w_{21,z}w_{22,z}w_{c,x}w_{c,y} + \\
& a_{12}w_{12,z}w_{22,x}w_{22,z}w_{c,x}w_{c,y} + a_{21}w_{21,z}w_{22,x}w_{22,z}w_{c,x}w_{c,y} + w_{11,z}^2((a_{12}w_{12,y} + a_{21}w_{21,y} + a_{22}w_{22,y})w_{c,x}^2 \\
& - (w_{12,x}w_{12,y} + w_{21,x}w_{21,y} + w_{22,x}w_{22,y})w_{c,x} + (w_{12,x}^2 + w_{21,x}^2 + w_{22,x}^2 - (a_{12}w_{12,x} + a_{21}w_{21,x} + \\
& a_{22}w_{22,x})w_{c,y})w_{c,y}) + (-((w_{21,y}w_{21,z} + a_{21}w_{c,y}w_{21,z} + w_{22,y}w_{22,z} + a_{22}w_{22,z}w_{c,y})w_{12,x}^2) + \\
& (a_{12}(w_{21,y}w_{21,z}w_{c,x} + w_{22,y}w_{22,z}w_{c,x} + w_{21,x}w_{21,z}w_{c,y} + w_{22,x}w_{22,z}w_{c,y}) + w_{12,z}(-2(a_{11}w_{11,y} + a_{21}w_{21,y} + \\
& a_{22}w_{22,y})w_{c,x} + w_{21,x}(w_{21,y} + a_{21}w_{c,y}) + w_{22,x}(w_{22,y} + a_{22}w_{c,y})))w_{12,x} - w_{21,y}w_{21,z}w_{22,x}^2 + \\
& w_{21,x}w_{21,z}w_{22,x}w_{22,y} + w_{21,x}w_{21,y}w_{22,x}w_{22,z} - w_{21,x}^2w_{22,y}w_{22,z} + a_{12}w_{12,z}w_{21,x}w_{21,y}w_{c,x} - \\
& 2a_{11}w_{11,y}w_{21,x}w_{21,z}w_{c,x} + a_{22}w_{21,y}w_{21,z}w_{22,x}w_{c,x} - 2a_{22}w_{21,x}w_{21,z}w_{22,y}w_{c,x} + a_{12}w_{12,z}w_{22,x}w_{22,y}w_{c,x} + \\
& a_{21}w_{21,z}w_{22,x}w_{22,y}w_{c,x} + a_{22}w_{21,x}w_{21,y}w_{22,z}w_{c,x} - 2a_{11}w_{11,y}w_{22,x}w_{22,z}w_{c,x} - 2a_{21}w_{21,y}w_{22,x}w_{22,z}w_{c,x} + \\
& a_{21}w_{21,x}w_{22,y}w_{22,z}w_{c,x} + w_{12,y}(-w_{12,z}(w_{21,x}^2 + w_{22,x}^2) + w_{12,z}(a_{21}w_{21,x} + a_{22}w_{22,x})w_{c,x} - \\
& 2a_{12}(w_{21,x}w_{21,z} + w_{22,x}w_{22,z})w_{c,x} + w_{12,x}(w_{21,x}w_{21,z} + a_{21}w_{c,x}w_{21,z} + w_{22,x}w_{22,z} + a_{22}w_{22,z}w_{c,x})) - \\
& (a_{12}w_{12,z}(w_{21,x}^2 + w_{22,x}^2) + (a_{22}w_{21,x} - a_{21}w_{22,x})(w_{21,x}w_{22,z} - w_{21,z}w_{22,x}))w_{c,y})w_{c,z} + \\
& w_{11,z}(w_{11,y}w_{c,x}(w_{12,x}w_{12,z} + w_{21,x}w_{21,z} + w_{22,x}w_{22,z} - (a_{12}w_{12,z} + a_{21}w_{21,z} + a_{22}w_{22,z})w_{c,x}) + \\
& a_{11}w_{c,x}((w_{12,x}w_{12,z} + w_{21,x}w_{21,z} + w_{22,x}w_{22,z})w_{c,y} - (w_{12,y}w_{12,z} + w_{21,y}w_{21,z} + w_{22,y}w_{22,z})w_{c,x}) - \\
& w_{11,y}(w_{12,x}^2 + w_{21,x}^2 + w_{22,x}^2 - (a_{12}w_{12,x} + a_{21}w_{21,x} + a_{22}w_{22,x})w_{c,x})w_{c,z} + a_{11}((w_{12,x}w_{12,y} + \\
& w_{21,x}w_{21,y} + w_{22,x}w_{22,y})w_{c,x} - (w_{12,x}^2 + w_{21,x}^2 + w_{22,x}^2)w_{c,y})w_{c,z} + w_{11,x}(w_{12,y}w_{12,z}w_{c,x} + \\
& w_{21,y}w_{21,z}w_{c,x} + w_{22,y}w_{22,z}w_{c,x} + a_{12}w_{12,z}w_{c,y}w_{c,x} + a_{21}w_{21,z}w_{c,y}w_{c,x} + a_{22}w_{22,z}w_{c,y}w_{c,x} - \\
& 2w_{12,x}w_{12,z}w_{c,y} - 2w_{21,x}w_{21,z}w_{c,y} - 2w_{22,x}w_{22,z}w_{c,y} + (w_{21,x}w_{21,y} + w_{22,x}w_{22,y} - 2(a_{12}w_{12,y} + a_{21}w_{21,y} + \\
& a_{22}w_{22,y})w_{c,x} + (a_{21}w_{21,x} + a_{22}w_{22,x})w_{c,y} + w_{12,x}(w_{12,y} + a_{12}w_{c,y}))w_{c,z}))
\end{aligned}$$

$$\begin{aligned}
\text{denom}_2 := & ((a_{12}w_{12,z} + a_{21}w_{21,z} + a_{22}w_{22,z})w_{c,y}^2 - (w_{12,y}w_{12,z} + w_{21,y}w_{21,z} + w_{22,y}w_{22,z})w_{c,y} + \\
& (w_{12,y}^2 + w_{21,y}^2 + w_{22,y}^2 - (a_{12}w_{12,y} + a_{21}w_{21,y} + a_{22}w_{22,y})w_{c,y})w_{c,z}w_{11,x}^2 + (a_{11}w_{c,y}((w_{12,y}w_{12,z} + \\
& w_{21,y}w_{21,z} + w_{22,y}w_{22,z})w_{c,x} - (w_{12,x}w_{12,z} + w_{21,x}w_{21,z} + w_{22,x}w_{22,z})w_{c,y}) + w_{11,z}(-(a_{12}w_{12,x} + \\
& a_{21}w_{21,x} + a_{22}w_{22,x})w_{c,y}^2) + (w_{12,x}w_{12,y} + a_{12}w_{c,x}w_{12,y} + w_{21,x}w_{21,y} + w_{22,x}w_{22,y} + a_{21}w_{21,y}w_{c,x} + \\
& a_{22}w_{22,y}w_{c,x})w_{c,y} - (w_{12,y}^2 + w_{21,y}^2 + w_{22,y}^2)w_{c,x}) + a_{11}((w_{12,x}w_{12,y} + w_{21,x}w_{21,y} + w_{22,x}w_{22,y})w_{c,y} - \\
& (w_{12,y}^2 + w_{21,y}^2 + w_{22,y}^2)w_{c,x})w_{c,z})w_{11,x} + a_{11}w_{11,z}w_{12,y}^2w_{c,x}^2 + a_{11}w_{11,z}w_{21,y}^2w_{c,x}^2 + a_{12}w_{12,z}w_{21,y}^2w_{c,x}^2 + \\
& a_{11}w_{11,z}w_{22,y}^2w_{c,x}^2 + a_{12}w_{12,z}w_{22,y}^2w_{c,x}^2 + a_{21}w_{21,z}w_{22,y}^2w_{c,x}^2 - a_{21}w_{12,y}w_{12,z}w_{21,y}w_{c,x}^2 + \\
& a_{21}w_{12,y}^2w_{21,z}w_{c,x}^2 - a_{12}w_{12,y}w_{21,y}w_{21,z}w_{c,x}^2 - a_{22}w_{12,y}w_{12,z}w_{22,y}w_{c,x}^2 - a_{22}w_{21,y}w_{21,z}w_{22,y}w_{c,x}^2 + \\
& a_{22}w_{12,y}^2w_{22,z}w_{c,x}^2 + a_{22}w_{21,y}^2w_{22,z}w_{c,x}^2 - a_{12}w_{12,y}w_{22,y}w_{22,z}w_{c,x}^2 - a_{21}w_{21,y}w_{22,y}w_{22,z}w_{c,x}^2 + \\
& a_{11}w_{11,z}w_{12,x}^2w_{c,y}^2 + a_{11}w_{11,z}w_{21,x}^2w_{c,y}^2 + a_{12}w_{12,z}w_{21,x}^2w_{c,y}^2 + a_{11}w_{11,z}w_{22,x}^2w_{c,y}^2 + a_{12}w_{12,z}w_{22,x}^2w_{c,y}^2 + \\
& a_{21}w_{21,z}w_{22,x}^2w_{c,y}^2 - a_{21}w_{12,x}w_{12,z}w_{21,x}w_{c,y}^2 + a_{21}w_{12,x}^2w_{21,z}w_{c,y}^2 - a_{12}w_{12,x}w_{21,x}w_{21,z}w_{c,y}^2 - \\
& a_{22}w_{12,x}w_{12,z}w_{22,x}w_{c,y}^2 - a_{22}w_{21,x}w_{21,z}w_{22,x}w_{c,y}^2 + a_{22}w_{12,x}^2w_{22,z}w_{c,y}^2 + a_{22}w_{21,x}^2w_{22,z}w_{c,y}^2 - \\
& a_{12}w_{12,x}w_{22,z}w_{c,y}^2 - a_{21}w_{21,x}w_{22,z}w_{c,y}^2 - w_{12,x}w_{12,z}w_{21,y}^2w_{c,x} - w_{12,x}w_{12,z}w_{22,y}^2w_{c,x} - \\
& w_{21,x}w_{21,z}w_{22,y}^2w_{c,x} + w_{12,y}w_{12,z}w_{21,x}w_{21,y}w_{c,x} - w_{12,y}^2w_{21,x}w_{21,z}w_{c,x} + w_{12,x}w_{12,y}w_{21,y}w_{21,z}w_{c,x} + \\
& w_{12,y}w_{12,z}w_{22,x}w_{22,y}w_{c,x} + w_{21,y}w_{21,z}w_{22,x}w_{22,y}w_{c,x} - w_{12,y}^2w_{22,x}w_{22,z}w_{c,x} - w_{21,y}^2w_{22,x}w_{22,z}w_{c,x} + \\
& w_{12,x}w_{12,y}w_{22,y}w_{22,z}w_{c,x} + w_{21,x}w_{21,y}w_{22,y}w_{22,z}w_{c,x} - w_{12,y}w_{12,z}w_{21,x}^2w_{c,y} - w_{12,y}w_{12,z}w_{22,x}^2w_{c,y} - \\
& w_{21,y}w_{21,z}w_{22,x}^2w_{c,y} + w_{12,x}w_{12,z}w_{21,x}w_{21,y}w_{c,y} + w_{12,x}w_{12,y}w_{21,x}w_{21,z}w_{c,y} - w_{12,x}^2w_{21,y}w_{21,z}w_{c,y} + \\
& w_{12,x}w_{12,z}w_{22,x}w_{22,y}w_{c,y} + w_{21,x}w_{21,z}w_{22,x}w_{22,y}w_{c,y} + w_{12,x}w_{12,y}w_{22,x}w_{22,z}w_{c,y} + \\
& w_{21,x}w_{21,y}w_{22,x}w_{22,z}w_{c,y} - w_{12,x}^2w_{22,y}w_{22,z}w_{c,y} - w_{21,x}^2w_{22,y}w_{22,z}w_{c,y} - 2a_{11}w_{11,z}w_{12,x}w_{12,y}w_{c,x}w_{c,y} + \\
& a_{21}w_{12,y}w_{12,z}w_{21,x}w_{c,x}w_{c,y} + a_{21}w_{12,x}w_{12,z}w_{21,y}w_{c,x}w_{c,y} - 2a_{11}w_{11,z}w_{21,x}w_{21,y}w_{c,x}w_{c,y} - \\
& 2a_{12}w_{12,z}w_{21,x}w_{21,y}w_{c,x}w_{c,y} - 2a_{21}w_{12,x}w_{12,y}w_{21,z}w_{c,x}w_{c,y} + a_{12}w_{12,y}w_{21,x}w_{21,z}w_{c,x}w_{c,y} + \\
& a_{12}w_{12,x}w_{21,y}w_{21,z}w_{c,x}w_{c,y} + a_{22}w_{12,y}w_{12,z}w_{22,x}w_{c,x}w_{c,y} + a_{22}w_{21,y}w_{21,z}w_{22,x}w_{c,x}w_{c,y} + \\
& a_{22}w_{12,x}w_{12,z}w_{22,y}w_{c,x}w_{c,y} + a_{22}w_{21,x}w_{21,z}w_{22,y}w_{c,x}w_{c,y} - 2a_{11}w_{11,z}w_{22,x}w_{22,y}w_{c,x}w_{c,y} - \\
& 2a_{12}w_{12,z}w_{22,x}w_{22,y}w_{c,x}w_{c,y} - 2a_{21}w_{21,z}w_{22,x}w_{22,y}w_{c,x}w_{c,y} - 2a_{22}w_{12,x}w_{12,y}w_{22,z}w_{c,x}w_{c,y} - \\
& 2a_{22}w_{21,x}w_{21,y}w_{22,z}w_{c,x}w_{c,y} + a_{12}w_{12,y}w_{22,x}w_{22,z}w_{c,x}w_{c,y} + a_{21}w_{21,y}w_{22,x}w_{22,z}w_{c,x}w_{c,y} + \\
& a_{12}w_{12,x}w_{22,y}w_{22,z}w_{c,x}w_{c,y} + a_{21}w_{21,x}w_{22,y}w_{22,z}w_{c,x}w_{c,y} + ((w_{21,y}^2 + w_{22,y}^2 - (a_{21}w_{21,y} + \\
& a_{22}w_{22,y})w_{c,y})w_{12,x}^2 + a_{12}((w_{21,x}w_{21,y} + w_{22,x}w_{22,y})w_{c,y} - (w_{21,y}^2 + w_{22,y}^2)w_{c,x})w_{12,x} + \\
& w_{12,y}^2(w_{21,x}^2 + w_{22,x}^2 - (a_{21}w_{21,x} + a_{22}w_{22,x})w_{c,x}) + (w_{21,y}w_{22,x} - w_{21,x}w_{22,y})(w_{21,y}w_{22,x} - \\
& a_{21}w_{c,y}w_{22,x} - w_{21,x}w_{22,y} - a_{22}w_{21,y}w_{c,x} + a_{21}w_{22,y}w_{c,x} + a_{22}w_{21,x}w_{c,y}) + w_{12,y}(a_{12}(w_{21,x}w_{21,y} + \\
& w_{22,x}w_{22,y})w_{c,x} - a_{12}(w_{21,x}^2 + w_{22,x}^2)w_{c,y} + w_{12,x}(-2w_{21,x}w_{21,y} + a_{21}w_{c,x}w_{21,y} - 2w_{22,x}w_{22,y} + \\
& a_{22}w_{22,y}w_{c,x} + a_{21}w_{21,x}w_{c,y} + a_{22}w_{22,x}w_{c,y}))w_{c,z} + w_{11,y}((a_{12}w_{12,z} + a_{21}w_{21,z} + a_{22}w_{22,z})w_{c,x}^2 - \\
& (w_{12,x}w_{12,z} + w_{21,x}w_{21,z} + w_{22,x}w_{22,z})w_{c,x} + (w_{12,x}^2 + w_{21,x}^2 + w_{22,x}^2 - (a_{12}w_{12,x} + a_{21}w_{21,x} + \\
& a_{22}w_{22,x})w_{c,x})w_{c,z}) + w_{11,y}(w_{11,z}w_{c,x}(w_{12,x}w_{12,y} + w_{21,x}w_{21,y} + w_{22,x}w_{22,y} - (a_{12}w_{12,y} + \\
& a_{21}w_{21,y} + a_{22}w_{22,y})w_{c,x}) - w_{11,z}(w_{12,x}^2 + w_{21,x}^2 + w_{22,x}^2 - (a_{12}w_{12,x} + a_{21}w_{21,x} + a_{22}w_{22,x})w_{c,x})w_{c,y} + \\
& a_{11}w_{c,x}((w_{12,x}w_{12,z} + w_{21,x}w_{21,z} + w_{22,x}w_{22,z})w_{c,y} - (w_{12,y}w_{12,z} + w_{21,y}w_{21,z} + w_{22,y}w_{22,z})w_{c,x}) + \\
& a_{11}((w_{12,x}w_{12,y} + w_{21,x}w_{21,y} + w_{22,x}w_{22,y})w_{c,x} - (w_{12,x}^2 + w_{21,x}^2 + w_{22,x}^2)w_{c,y})w_{c,z} + \\
& w_{11,x}(w_{12,y}w_{12,z}w_{c,x} + w_{21,y}w_{21,z}w_{c,x} + w_{22,y}w_{22,z}w_{c,x} - 2a_{12}w_{12,z}w_{c,y}w_{c,x} - 2a_{21}w_{21,z}w_{c,y}w_{c,x} - \\
& 2a_{22}w_{22,z}w_{c,y}w_{c,x} + a_{12}w_{12,y}w_{c,z}w_{c,x} + a_{21}w_{21,y}w_{c,z}w_{c,x} + a_{22}w_{22,y}w_{c,z}w_{c,x} + w_{12,x}w_{12,z}w_{c,y} + \\
& w_{21,x}w_{21,z}w_{c,y} + w_{22,x}w_{22,z}w_{c,y} - 2w_{12,x}w_{12,y}w_{c,z} - 2w_{21,x}w_{21,y}w_{c,z} - 2w_{22,x}w_{22,y}w_{c,z} + \\
& a_{12}w_{12,x}w_{c,y}w_{c,z} + a_{21}w_{21,x}w_{c,y}w_{c,z} + a_{22}w_{22,x}w_{c,y}w_{c,z}))
\end{aligned}$$

$$\begin{aligned} \text{numer}_3 := & n_y((a_{11}w_{11,x} + a_{12}w_{12,x} + a_{21}w_{21,x} + a_{22}w_{22,x})w_{c,y}^2 - \\ & w_{c,y}((a_{11}w_{11,y} + a_{12}w_{12,y} + a_{21}w_{21,y} + a_{22}w_{22,y})w_{c,x} + w_{11,x}w_{11,y} + w_{12,x}w_{12,y} + w_{21,x}w_{21,y} + \\ & w_{22,x}w_{22,y}) + (w_{11,y}^2 + w_{12,y}^2 + w_{21,y}^2 + w_{22,y}^2)w_{c,x}) \end{aligned}$$

$$\begin{aligned} \text{denom}_3 := & (a_{11}w_{11,y} + a_{12}w_{12,y} + a_{21}w_{21,y} + a_{22}w_{22,y})w_{c,x}^2 + \\ & w_{c,y}(-(a_{11}w_{11,x} + a_{12}w_{12,x} + a_{21}w_{21,x} + a_{22}w_{22,x})w_{c,x} + w_{11,x}^2 + w_{12,x}^2 + w_{21,x}^2 + w_{22,x}^2) - \\ & w_{c,x}(w_{11,x}w_{11,y} + w_{12,x}w_{12,y} + w_{21,x}w_{21,y} + w_{22,x}w_{22,y}) \end{aligned}$$

$$\text{numer}_4 :=$$

$$\begin{aligned} & (w_{12,y}^2 + w_{21,y}^2 + w_{22,y}^2 + (a_{12}^2 + a_{21}^2 + a_{22}^2)w_{c,y}^2 - 2(a_{12}w_{12,y} + a_{21}w_{21,y} + a_{22}w_{22,y})w_{c,y})w_{11,x}^2 + \\ & 2a_{11}(-(a_{12}w_{12,x} + a_{21}w_{21,x} + a_{22}w_{22,x})w_{c,y}^2 + (w_{12,x}w_{12,y} + a_{12}w_{c,x}w_{12,y} + w_{21,x}w_{21,y} + w_{22,x}w_{22,y} \\ & + a_{21}w_{21,y}w_{c,x} + a_{22}w_{22,y}w_{c,x})w_{c,y} - (w_{12,y}^2 + w_{21,y}^2 + w_{22,y}^2)w_{c,x})w_{11,x} + w_{12,y}^2w_{21,x}^2 + w_{12,x}^2w_{21,y}^2 + \\ & w_{12,y}^2w_{22,x}^2 + w_{21,y}^2w_{22,x}^2 + w_{12,x}^2w_{22,y}^2 + w_{21,x}^2w_{22,y}^2 + a_{11}^2w_{12,y}^2w_{c,x}^2 + a_{21}^2w_{12,y}^2w_{c,x}^2 + a_{22}^2w_{12,y}^2w_{c,x}^2 + \\ & a_{11}^2w_{21,y}^2w_{c,x}^2 + a_{12}^2w_{21,y}^2w_{c,x}^2 + a_{22}^2w_{21,y}^2w_{c,x}^2 + a_{11}^2w_{22,y}^2w_{c,x}^2 + a_{12}^2w_{22,y}^2w_{c,x}^2 + a_{21}^2w_{22,y}^2w_{c,x}^2 - \\ & 2a_{12}a_{21}w_{12,y}w_{21,y}w_{c,x}^2 - 2a_{12}a_{22}w_{12,y}w_{22,y}w_{c,x}^2 - 2a_{21}a_{22}w_{21,y}w_{22,y}w_{c,x}^2 + ((w_{12,x}^2 + w_{21,x}^2 + w_{22,x}^2)a_{11}^2 + \\ & a_{12}^2w_{21,x}^2 + a_{12}^2w_{22,x}^2 + a_{22}^2(w_{12,x}^2 + w_{21,x}^2) - 2a_{12}a_{22}w_{12,x}w_{22,x} - 2a_{21}w_{21,x}(a_{12}w_{12,x} + a_{22}w_{22,x}) + \\ & a_{21}^2(w_{12,x}^2 + w_{22,x}^2))w_{c,y}^2 - 2w_{12,x}w_{12,y}w_{21,x}w_{21,y} - 2w_{12,x}w_{12,y}w_{22,x}w_{22,y} - 2w_{21,x}w_{21,y}w_{22,x}w_{22,y} - \\ & 2a_{12}w_{12,x}w_{21,y}w_{c,x} - 2a_{12}w_{12,x}w_{22,y}w_{c,x} - 2a_{21}w_{21,x}w_{22,y}w_{c,x} - 2a_{21}w_{12,y}w_{21,x}w_{c,x} + \\ & 2a_{21}w_{12,x}w_{12,y}w_{21,y}w_{c,x} + 2a_{12}w_{12,y}w_{21,x}w_{21,y}w_{c,x} - 2a_{22}w_{12,y}w_{22,x}w_{c,x} - 2a_{22}w_{21,y}w_{22,x}w_{c,x} + \\ & 2a_{22}w_{12,x}w_{12,y}w_{22,y}w_{c,x} + 2a_{22}w_{21,x}w_{21,y}w_{22,y}w_{c,x} + 2a_{12}w_{12,y}w_{22,x}w_{22,y}w_{c,x} + \\ & 2a_{21}w_{21,y}w_{22,x}w_{22,y}w_{c,x} + w_{11,y}^2(w_{12,x}^2 + w_{21,x}^2 + w_{22,x}^2 + (a_{12}^2 + a_{21}^2 + a_{22}^2)w_{c,x}^2 - 2(a_{12}w_{12,x} + \\ & a_{21}w_{21,x} + a_{22}w_{22,x})w_{c,x}) - 2(w_{22,x}w_{22,y}w_{c,x}a_{11}^2 - a_{22}(w_{12,x}w_{12,y} + w_{21,x}w_{21,y})w_{22,x} + \\ & a_{22}(w_{12,x}^2 + w_{21,x}^2)w_{22,y} + (a_{11}^2 + a_{22}^2)(w_{12,x}w_{12,y} + w_{21,x}w_{21,y})w_{c,x} + a_{21}^2(w_{12,x}w_{12,y} + \\ & w_{22,x}w_{22,y})w_{c,x} + a_{12}^2(w_{21,x}w_{21,y} + w_{22,x}w_{22,y})w_{c,x} + a_{21}(w_{21,y}w_{12,x}^2 - w_{12,y}w_{21,x}w_{12,x} + \\ & w_{21,y}w_{22,x}^2 - w_{21,x}w_{22,x}w_{22,y} - (a_{12}w_{12,y}w_{21,x} + a_{22}w_{22,y}w_{21,x} + a_{12}w_{12,x}w_{21,y} + a_{22}w_{21,y}w_{22,x})w_{c,x}) + \\ & a_{12}w_{12,y}(w_{21,x}^2 + w_{22,x}(w_{22,x} - a_{22}w_{c,x})) - a_{12}w_{12,x}(w_{21,x}w_{21,y} + w_{22,y}(w_{22,x} + a_{22}w_{c,x})))w_{c,y} + \\ & 2w_{11,y}((w_{11,x} - a_{11}w_{c,x})(-w_{12,x}w_{12,y} + a_{12}w_{c,x}w_{12,y} - w_{21,x}w_{21,y} - w_{22,x}w_{22,y} + a_{21}w_{21,y}w_{c,x} + \\ & a_{22}w_{22,y}w_{c,x}) + (-w_{11,x}w_{c,x}a_{12}^2 + w_{12,x}(w_{11,x} + a_{11}w_{c,x})a_{12} - a_{11}(w_{12,x}^2 + w_{21,x}^2 + w_{22,x}^2) + a_{11}(a_{21}w_{21,x} + \\ & a_{22}w_{22,x})w_{c,x} + w_{11,x}(a_{21}w_{21,x} + a_{22}w_{22,x} - (a_{21}^2 + a_{22}^2)w_{c,x}))w_{c,y}) \end{aligned}$$

$$\begin{aligned} \text{denom}_4 := & (w_{11,y}^2 + w_{12,y}^2 + w_{21,y}^2 + w_{22,y}^2)w_{c,x}^2 - \\ & 2w_{c,x}w_{c,y}(w_{11,x}w_{11,y} + w_{12,x}w_{12,y} + w_{21,x}w_{21,y} + w_{22,x}w_{22,y}) + (w_{11,x}^2 + w_{12,x}^2 + w_{21,x}^2 + w_{22,x}^2)w_{c,y}^2 \end{aligned}$$