Theoretical and Numerical Analysis of 3D Reconstruction Using Point and Line Incidences
Supplementary Material

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Overview

In this Supplementary Material, we prove all the mathematical results from the main body of the paper. For convenience of the reader, we start in Appendix A by explaining the elementary notions of algebraic geometry, that are helpful to understand the rest of the Supplementary Material. Results that appear in the main body of the paper are restated and given the same numbering. Additional results not stated in the main body are numbered independently.

Appendix B deals with Section 1 apart from the Euclidean distance degree. In Appendix C we provide helpful background for the EDD calculations that are carried out in Appendix D. In Appendix E we provide pseudocode for the different reconstruction approaches from Section 2.

A. Algebraic Geometry Preliminaries

The *complex projective space* of dimension n is the set of one-dimensional linear subspaces of $\mathbb{C}^{n+1}$, equivalently $\mathbb{P}^n := (\mathbb{C}^{n+1} \setminus \{0\})/\sim$, where $\sim$ denotes the equivalence relation defined by

$$x \sim y \Leftrightarrow x = \lambda y \quad \text{for some} \quad 0 \neq \lambda \in \mathbb{C}. \tag{1}$$

The ring of polynomials in $n+1$ variables is denoted by $R := \mathbb{C}[x_0, \ldots, x_n]$. A subset $X \subseteq \mathbb{P}^n$ is called a *projective algebraic variety*, when there exists a collection $\{f_1, \ldots, f_k\}$ of homogeneous polynomials such that $X = \{x \in \mathbb{P}^n \mid f_1(x) = \cdots f_k(x) = 0\}$. In other words, X is the vanishing set of the polynomials f_i for $i = 1, \ldots, k$ such that each term of the polynomial has degree d and $f_i(\lambda x_0, \ldots, \lambda x_n) = \lambda^d f_i(x_0, \ldots, x_n)$. Similarly, a subset $X \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$ is an algebraic variety, if X is the vanishing set of multi-homogeneous polynomials $\{f_1, \ldots, f_k\}$.

The Zariski topology on $\mathbb{P}^n$ (or $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$) is the topology whose closed sets are algebraic varieties. Therefore, given a set U, its Zariski closure, denoted $\overline{U}$, is the smallest variety containing U.

Let $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$ be projective varieties. A map $\varphi : X \longrightarrow Y$ is regular if it can be written as

$$\varphi(x) = [\varphi_0(x) : \cdots : \varphi_m(x)] \tag{2}$$

for some polynomials $\varphi_0, \ldots, \varphi_m$ that do not vanish simultaneously. If there is a regular map $\psi : Y \longrightarrow X$ such that $\varphi \circ \psi = \mathrm{Id}_Y$ and $\psi \circ \varphi = \mathrm{Id}_X$, we say that X and Y are *isomorphic*, and we denote it by $X \cong Y$. If $U \subseteq X$ is a Zariski dense open set and $\varphi : U \longrightarrow Y$ is a regular map, we say that φ is a *rational map* from X to Y, and denote it by $\varphi : X \dashrightarrow Y$. See [1, Section 1] for background on the basic properties of rational maps that are used in this section.

Given a variety X, we define its *ideal* as the set

$$I(X) = \{f \in R \mid f(x) = 0 \text{ for every } x \in X\} \tag{3}$$

of homogeneous polynomials that vanish in every element of X. For every ideal I, it is possible to find a (not necessarily unique) finite set of polynomials $\{f_1, \ldots, f_k\} \subseteq I$, such that every element $f \in I$ can be written as

$$f(x) = g_1(x)f_1(x) + \cdots + g_k(x)f_k(x), \tag{4}$$

for some polynomials $g_i(x) \in R$. In this scenario, we say that I is *generated* by $\{f_1, \dots, f_k\}$, and this is denoted as $I = \langle f_1, \dots, f_k\rangle$.

Given a variety X and its ideal $I(X) = \langle f_1, \dots, f_k\rangle$, we say that a point $a \in X$ is *smooth* if the rank of the Jacobian matrix $J(a) := \left[\frac{\partial f_i}{\partial x_i}(a)\right]$ is equal to the codimension of X. This definition is independent of the choice of generators of $I(X)$. For a broader description and results on the smoothness of algebraic varieties, we refer the reader to [5].

We use the notation $\vee$ to denote the *join* of two vectors spaces, meaning $U \vee V = \mathrm{span}\{U, V\}$. Similarly, $\wedge$ denotes the intersection of linear spaces.

B. Anchored Multiview Varieties

For a camera matrix $C : \mathbb{P}^3 \to \mathbb{P}^2$, the *back-projected line* of $x \in \mathbb{P}^2$ is the line in $\mathbb{P}^3$ that contains all points that are by C projected onto x. Similarly, for an image line $\ell \in \mathrm{Gr}(1, \mathbb{P}^2)$, its back-projected plane is the plane in $\mathbb{P}^3$ containing all lines that are by C projected onto ℓ. Under the identification $\mathrm{Gr}(1, \mathbb{P}^2) \cong (\mathbb{P}^2)^\vee \cong \mathbb{P}^2$ we describe the *back-projected plane* of ℓ by its defining linear equation $C^T\ell$. We may parameterize lines in $\mathrm{Gr}(1, \mathbb{P}^3)$ by two points spanning it.

Throughout this work, we assume that any camera arrangement has at least one camera and all centers are pairwise disjoint.

B.1. The linear isomorphisms

Consider an arrangement $\widetilde{\mathcal{C}}$ of full-rank 2×2 matrices, and an arrangement $\widehat{\mathcal{C}}$ of full-rank 2×3 matrices. We define $\mathcal{M}^{1,1}_{\widetilde{\mathcal{C}}}$, and $\mathcal{M}^{2,1}_{\widehat{\mathcal{C}}}$, respectively as the Zariski closure of the image of the joint maps

$$\begin{aligned} \Phi_{\widetilde{\mathcal{C}}} : \mathbb{P}^1 &\longrightarrow (\mathbb{P}^1)^m, \\ X &\longmapsto (\widetilde{C}_1 X, \dots, \widetilde{C}_m X) \end{aligned} \tag{5}$$

and

$$\begin{aligned} \Phi_{\widehat{\mathcal{C}}} : \mathbb{P}^2 &\dashrightarrow (\mathbb{P}^1)^m, \\ X &\longmapsto (\widehat{C}_1 X, \dots \widehat{C}_m X). \end{aligned} \tag{6}$$

Theorem 1.3.

1. *Let $\phi_L : L \to \mathbb{P}^1$ and $\psi_{\mathcal{C},i} : C_i \cdot L \to \mathbb{P}^1$ be any choices of linear isomorphisms. Let $\widetilde{\mathcal{C}}$ denote the arrangement of matrices $\widetilde{C}_i := \psi_{\mathcal{C},i} \circ C_i \circ \phi_L^{-1}$. Then*
$$\psi_{\mathcal{C},L} := (\psi_{\mathcal{C},1}, \dots, \psi_{\mathcal{C},m}) : \mathcal{M}^L_{\mathcal{C}} \to \mathcal{M}^{1,1}_{\widetilde{\mathcal{C}}} \tag{7}$$
is a linear isomorphism.

2. *Let $\phi_X : \Lambda(X) \to \mathbb{P}^2$ and $\psi_{\mathcal{C},i} : \Lambda(C_i X) \to \mathbb{P}^1$ be any choices of linear isomorphisms. Let $\widehat{\mathcal{C}}$ denote the arrangement of matrices $\widehat{C}_i := \psi_{\mathcal{C},i} \circ C_i \circ \phi_X^{-1}$. Then*
$$\psi_{\mathcal{C},X} := (\psi_{\mathcal{C},1}, \dots, \psi_{\mathcal{C},m}) : \mathcal{L}^X_{\mathcal{C}} \to \mathcal{M}^{2,1}_{\widehat{\mathcal{C}}} \tag{8}$$
is a linear isomorphism.

We interpret $C_i \circ \phi_X^{-1} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ as a matrix as follows. Let H be a plane in $\mathbb{P}^3$ disjoint from X. Then the following map $\phi_X^{-1} : \mathbb{P}^2 \to \Lambda(X)$ is defined by a linear mapping $f_X : \mathbb{P}^2 \to H$ such that $\phi_X^{-1}(Y) = \mathrm{span}\{X, f_X(Y)\}$. As a matrix, $C_i \circ \phi_X^{-1}$ is equal to $[C_i X]_\times C_i f_X$, where

$$[a]_\times := \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}. \tag{9}$$

We often work with Zariski closures of images of rational maps. By Chevalley's theorem [14, Theorem 4.19], we may equivalently take Euclidean closures. With this in mind, we can use the following lemma.

Lemma B.1. *Let $\psi : \mathcal{X} \to \mathcal{Y}$ be an isomorphism and $U \subseteq \mathcal{X}, V \subseteq \mathcal{Y}$ sets whose Euclidean closures equals their Zariski closures. If $\psi(U) = V$, then $\psi(\overline{U}) = \overline{V}$.*

Proof. Take a point $v \in \overline{V} \setminus V$. Then there is a sequence $V \ni v^{(n)} \to v$ in Euclidean topology such that $u^{(n)} = \psi^{-1}(v^{(n)}) \in U$ converges in Euclidean topology by continuity of ψ to a point $u \in \overline{U}$ for which $\psi(u) = v$. We have shown $\overline{V} \subseteq \psi(\overline{U})$. Similarly we show $\overline{U} \subseteq \psi^{-1}(\overline{V})$ from which it follows that $\psi(\overline{U}) \subseteq \overline{V}$. □

Proof of Theorem 1.3.

1. It is worth noting that $\Phi_{\mathcal{C}}|_L$ is well-defined everywhere, as L does not contain any center. Additionally, $\Phi_{\widetilde{\mathcal{C}}}$ is defined everywhere. In particular, the images of both maps are Zariski closed.

By construction, $\psi_{\mathcal{C},L}(\Phi_{\mathcal{C}}|L(X)) = \Phi_{\widetilde{\mathcal{C}}}(\phi_L(X))$, which shows that $\psi_{\mathcal{C},L}$ is a well-defined map.

Take a point $x \in \mathcal{M}^{1,1}_{\widetilde{\mathcal{C}}}$, then there is a point $X \in \mathbb{P}^1$ such that $x_i = \widetilde{C}_i X$ for each i. Consider $x' \in \mathcal{M}^L_{\mathcal{C}}$, the image of $X' = \phi_L^{-1}(X)$ such that $x'_i = C_i X'$. By construction, x is the image of x' under $\psi_{\mathcal{C},L}$, which shows surjectivity.

For injectivity, assume that $\psi_{\mathcal{C},L}(X) = \psi_{\mathcal{C},L}(X')$. Then for each i, $C_i X = C_i X'$. However, since the line L does not meet any of the centers, the back-projected lines must meet in exactly one point inside L, meaning that $X = X'$.

2. As we wish to use Lemma B.1, we let

$$\mathcal{X} = \Lambda(C_1 X) \times \cdots \times \Lambda(C_m X), \quad \mathcal{Y} = (\mathbb{P}^1)^m. \tag{10}$$

Note that $\psi_{\mathcal{C},X} : \mathcal{X} \to \mathcal{Y}$ is an isomorphism by construction. Further, let U be the image of $\Upsilon_{\mathcal{C}}|_{\Lambda(X)}$, and $V = \mathrm{Im}\ \Phi_{\widehat{\mathcal{C}}}$. One can show $\psi_{\mathcal{C},X}(U) = V$ via similar calculations to *1*. □

B.2. Irreducibility, dimension, and equations

In the main body of the paper it was claimed that the anchored multiview varieties under natural conditions equal,

$$\mathcal{X}_{\mathcal{C}}^{L} = \{(x_1, \ldots, x_m) \in \mathcal{M}_{\mathcal{C}} : x_i \in C_i \cdot L\}, \tag{11}$$

$$\mathcal{Y}_{\mathcal{C}}^{X} = \{(\ell_1, \ldots, \ell_m) \in \mathcal{L}_{\mathcal{C}} : C_i X \in \ell_i\}. \tag{12}$$

This provides an alternative characterization to the closure of the images of restrictions of $\Phi_{\mathcal{C}}$ and $\Upsilon_{\mathcal{C}}$, which is a useful fact that we formalize in the following lemma.

Proposition 1.2. *Consider an arrangement of m cameras $\mathcal{C} = (C_1, \ldots, C_m)$, a point $X \in \mathbb{P}^3$ and a line L in $\mathbb{P}^3$ satisfying the conditions of Definition 1.1.*

1. *If there are two different camera centers c_i and c_j such that the span of $\{c_i, c_j, L\}$ is $\mathbb{P}^3$, then*

$$\mathcal{M}_{\mathcal{C}}^{L} = \{(x_1, \ldots, x_m) \in \mathcal{M}_{\mathcal{C}} : x_i \in C_i \cdot L\}. \tag{13}$$

2. *If for each camera center c_i, the line spanned by c_i and X does not contain any other camera center, then*

$$\mathcal{L}_{\mathcal{C}}^{X} = \{(\ell_1, \ldots, \ell_m) \in \mathcal{L}_{\mathcal{C}} : C_i X \in \ell_i\}. \tag{14}$$

Proof.

1. We recall the assumption that L contains no center. Then $\Phi_{\mathcal{C}}|_L$ is defined everywhere and the image of this map is closed. Let $x \in \mathrm{Im}\Phi_{\mathcal{C}}|_L$. There is an $X \in L$ such that $x = \Phi_{\mathcal{C}}(X)$. Therefore $x \in \mathcal{M}_{\mathcal{C}}$ and $x_i \in C_i \cdot L$. Conversely, if $x \in \mathcal{M}_{\mathcal{C}}$ and $x_i \in C_i \cdot L$, then since the back-projected line of x_i meet L in unique points $X_i \in L$, we just have to argue that X_i are all the same. This is trivial if there is only one camera. If there are two centers c_i, c_j that together with L span $\mathbb{P}^3$, then the planes $c_i \vee L$ and $c_j \vee L$ meet in exactly the line L. Then the back-projected lines of x_i, x_j must meet inside L, implying $X_i = X_j$. For any other center c_k, we either have that c_i, c_k and L span $\mathbb{P}^3$ or c_j, c_k and L span $\mathbb{P}^3$. This either implies $X_i = X_k$ or $X_j = X_k$ by the above. Either way, repeating this process shows that all X_i are equal and $x = \Phi_{\mathcal{C}}(X)$ for $X = X_i$.

2. For any line $L \in \Lambda(X)$ that does not meet any center, it is clear that $\ell = \Upsilon_{\mathcal{C}}(L)$ satisfies $\ell \in \mathcal{L}_{\mathcal{C}}$ and $C_i X \in \ell_i$. Therefore $\mathcal{L}_{\mathcal{C}}^{X} \subseteq \mathcal{Y}_{\mathcal{C}}^{X}$. For the other inclusion, we take an element $\ell \in \mathcal{Y}_{\mathcal{C}}^{X}$. If the intersection of the back-projected planes H_i of ℓ_i contain a line L through X meeting no center, then $\ell = \Upsilon_{\mathcal{C}}|_{\Lambda(X)}(L)$. This especially happens when H_i intersect in a plane. Note that if the intersection contains a line L that doesn't meet X, then the intersection contains the plane $X \vee L$. We are left to check what happens if H_i intersect in exactly a line L that meets a center, say c_i. By assumption, no other center is contained in this line. Therefore $H_j, j \neq i$ is equal to $c_j \vee L$. Let $L^{(n)} \in \Lambda(X)$ be any sequence of lines in H_i meeting no centers and such that $L^{(n)} \to L$. It is clear that $c_j \vee L^{(n)} \to H_j$ for $j \neq i$ $n \to \infty$ and $c_i \vee L^{(n)} = H_i$ for each n. Then $\Upsilon_{\mathcal{C}}(L^{(n)}) \to \ell$, showing $\ell \in \mathcal{L}_{\mathcal{C}}^{X}$ and we are done. $\square$

If the assumptions of Proposition 1.2 do not hold, then the result doesn't either. In the first statement, let c_1, c_2 be centers that together with L span a plane P. Given any point $X \in P \setminus \{c_1, c_2\}$, the element $x = (C_1 X, C_2 X)$ satisfies that $x \in \mathcal{M}_{\mathcal{C}}$ and $x_i \in C_i \cdot L$. However, generally for a point $X \in P \setminus L$, we have $x \notin \mathrm{Im}\Phi_{\mathcal{C}}|_L$. For the second statement, consider two centers c_1, c_2 that together with X span a line L. Consider two distinct planes H_1, H_2, both containing the line L. They meet therefore exactly in L and the pair of corresponding image lines (ℓ_1, ℓ_2) lies in $\mathcal{Y}_{\mathcal{C}}^{X}$ for the camera arrangement given by these two cameras. However, in the image of $\Upsilon_{\mathcal{C}}|_{\Lambda(X)}$, the back-projected planes H_1 and H_2 are always the same.

Proposition 1.4. *$\mathcal{M}_{\mathcal{C}}^{L}$ and $\mathcal{L}_{\mathcal{C}}^{X}$ are irreducible. Further,*

1. *$\mathcal{M}_{\mathcal{C}}^{L}$ is isomorphic to $\mathbb{P}^1$. In particular, $\dim \mathcal{M}_{\mathcal{C}}^{L} = 1$.*
2. *If the span of the centers c_i and the point X are not collinear, then $\dim \mathcal{L}_{\mathcal{C}}^{X} = 2$.*

Proof. Both varieties are irreducible since the image of any rational map from an irreducible variety is irreducible.

1. Since we assume no center lies in L, $\Phi_{\mathcal{C}}$ restricted to L is defined everywhere, and therefore the image of this restriction equals $\mathcal{M}_{\mathcal{C}}^{L}$. This map is further injective since if $x \in \mathcal{M}_{\mathcal{C}}^{L}$, then the back-projected line of $x_i \in \mathbb{P}^2$ meets L in exactly a point X, which implies that X is the only point on L for which $x = \Phi_{\mathcal{C}}(X)$.

2. Note that since $\dim \Lambda(X) = 2$, we have $\dim \mathcal{L}_{\mathcal{C}}^{X} \leq 2$. Let $U \subseteq \Lambda(X)$ be the subset of lines that meets no center. Without restriction, assume c_1, c_2, and X span a plane. Each line $L \in U$ uniquely defines two planes via $c_1 \vee L, c_2 \vee L$. Since $\dim \Lambda(X) = 2$, projection of $\mathcal{L}_{\mathcal{C}}^{X}$ onto the factors of c_1, c_2 is at least two dimensional, showing the other inequality $\dim \mathcal{L}_{\mathcal{C}}^{X} \geq 2$. $\square$

For the result below, let F^{ij} denote the fundamental matrix of C_i and C_j, see [8, 16].

Proposition 1.5. *For a point $X \in \mathbb{P}^3$ and line L in $\mathbb{P}^3$, let $\mathcal{C}$ be a generic (random) camera arrangement of m cameras.*

1. *$x \in \mathcal{M}_{\mathcal{C}}^{L}$ if and only if $x_1^T F^{1j} x_j = 0$ for every $j = 2, \ldots, m$ and $x_i^T C_i \cdot L = 0$ for every $i = 1, \ldots, m$.*
2. *$\ell \in \mathcal{L}_{\mathcal{C}}^{X}$ if and only if*

$$\begin{aligned} \det \begin{bmatrix} C_1^T \ell_1 & C_2^T \ell_2 & C_i^T \ell_i \end{bmatrix} &= 0, \\ \det \begin{bmatrix} C_1^T \ell_1 & C_3^T \ell_3 & C_i^T \ell_i \end{bmatrix} &= 0 \end{aligned} \tag{15}$$

for $i = 3, \ldots, m$ and $\ell_i^T C_i X = 0$ for every $i = 1, \ldots, m$.

Proof. Note that in the generic case, the conditions of Proposition 1.2 hold.

1. Recall that $x_i^T C_i \cdot L = 0$ is equivalent to $x_i \in C_i \cdot L$. As in the proof of Proposition 1.4, $x \in \mathcal{M}_{\mathcal{C}}^L$ is uniquely determined by the intersection $X \in L$ of its back-projected lines. The back-projected lines L_i of x_i intersect if and only if the pairs (L_1, L_i) intersect for $i \geq 2$, which in turn is equivalent to $x_1^T F^{1i} x_i = 0$ for the fundamental matrix F^{1i}.

2. Recall that $\ell_i^T C_i X = 0$ is equivalent to $C_i X \in \ell_i$. By [1, Theorem 2.5], $\ell \in \mathcal{L}_{\mathcal{C}}$ if and only if the back-projected planes meet in a line (assuming generic centers), and

$$\det \begin{bmatrix} C_i^T \ell_i & C_j^T \ell_j & C_k^T \ell_k \end{bmatrix} = 0, \tag{16}$$

is equivalent to the back-projected planes of ℓ_i, ℓ_j, ℓ_k meeting in at least a line. By Proposition 1.2, we are left to show direction $\Leftarrow$. Let H_i denote the back-projected plane of ℓ_i. For $m = 2$, the two back-projected planes always meet. For $m \geq 3$ we have that c_1, c_2, c_3 with X span $\mathbb{P}^3$ by genericity. Especially, the back-projected planes of ℓ_1, ℓ_2, ℓ_3 meet in a line by setting $i = 1, j = 2, k = 3$ in Equation (16). Also, since c_1, c_2, c_3, X span $\mathbb{P}^3$, they meet exactly in a line. If $m \geq 4$, it suffices to show that H_l for $l \geq 4$ meets H_1, H_2, H_3 in a line. Note that either H_1, H_2 or H_1, H_3 meet exactly in a line. Let $i, j \in \{1, 2, 3\}$ denote indices for which this happens. Then for i, j and $k = 4$, Equation (16) guarantees that H_i, H_j, H_4 meet in exactly a line, which suffices. □

B.3. Smoothness and multidegrees

First, similar to what is done in the proof of the smoothness properties of the multiview variety $\mathcal{M}_{\mathcal{C}}$ in [16], we use that multiview varieties are isomorphic to corresponding varieties of back-projected lines or planes.

Proposition 1.6.

1. *$\mathcal{M}_{\mathcal{C}}^L$ smooth.*
2. *If there are exactly two cameras, or the centers together with the point X span $\mathbb{P}^3$, then $\mathcal{L}_{\mathcal{C}}^X$ is smooth.*

Proof.

1. Since $\mathcal{M}_{\mathcal{C}}^L$ is isomorphic to $\mathbb{P}^1$ by Proposition 1.4, it is smooth.

2. Assume that the line $c_i \vee X$ contains c_j for $j \neq i$. In the image we always have $H_i = H_j$ for the back-projected planes of ℓ_i, ℓ_j. Therefore $\mathcal{L}_{\mathcal{C}}^X$ is isomorphic to $\mathcal{L}_{\mathcal{C}'}^X$, where $\mathcal{C}'$ is equal to $\mathcal{C}$ after having removed the smallest amount of cameras from $\mathcal{C}$ such that each line $c_i \vee X$ contains exactly one center, namely c_i itself. We, therefore, assume now that $\mathcal{C}$ has this property: $c_i \vee X$ contains only the center c_i for each i.

If $m = 1$, then one can check that $\mathcal{L}_{\mathcal{C}}^X$ is isomorphic to $\mathbb{P}^1$ and if $m = 2$, that $\mathcal{L}_{\mathcal{C}}^X$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. The latter is for instance because any choice of ℓ_1, ℓ_2, where $C_i X \in \ell_i$ guarantees that the back-projected planes H_i meet in a line containing X. Therefore we now assume that there are at least three cameras.

First, we note that $\Lambda(X)$ is a smooth variety and the lines $c_i \vee X$ are smooth subvarieties. Up to linear transformation, we may assume that $X = (1 : 0 : 0 : 0)$ without loss of generality. Let $a = (a_0 : a_1 : a_2 : a_3)$ be distinct from X. Then $(0 : 0 : a_0 : 0 : a_1 : a_2)$ are the Plücker coordinates of $a \vee X$. In particular,

$$\Lambda(X) = \{w \in \mathbb{P}^5 : w_0 = w_1 = w_3 = 0\}. \tag{17}$$

In Plücker coordinates, the line $L = a \vee X$ in coordinates w and the fixed point $b = (b_0 : b_1 : b_2 : b_3) \in \mathbb{P}^3$ spann the plane:

$$(0 : w_2 b_1 - w_4 b_0 : w_2 b_2 - w_5 b_0 : w_4 b_2 - w_5 b_1). \tag{18}$$

The three linear non-zero functions in w in Equation (18) vanishes if and only if b lies in the line L. Denote them by $f_{b,1}, f_{b,2}, f_{b,3}$ and $f_b(L) = (0 : f_{b,1}(L) : f_{b,2}(L) : f_{b,3}(L))$ for a line $L \in \Lambda(X)$. Let $c \neq X$. Since the blow-up of a linear space at a linear space is smooth, then

$$\Gamma_C := \overline{\{(L, f_c(L)) : c \vee X \neq L \in \Lambda(X)\}}, \tag{19}$$

is a smooth variety in $\Lambda(X) \times \mathrm{Gr}(1, \mathbb{P}^3)$. Keep in mind that $f_c(L) = c \vee L$. Next we consider the joint blow-up $\Gamma_{\mathcal{C}}$ defined as

$$\overline{\{(L, f_{c_1}(L), \ldots, f_{c_m}(L)) : c_i \vee X \neq L \in \Lambda(X)\}} \tag{20}$$

in $\Lambda(X) \times \mathrm{Gr}(1, \mathbb{P}^3)^m$. Take an element $\ell \in \mathcal{L}_{\mathcal{C}}^X$. By the assumption that there are at least three cameras, and the centers together with the point X span $\mathbb{P}^3$, we have that the back-projected planes meet in exactly a line L containing X. We have also assumed that L contains at most one center. If $c_i \in L$, then fix the index i, otherwise choose any index i. Consider the natural projection,

$$\pi_i : \Gamma_{\mathcal{C}} \to \Gamma_{C_i}. \tag{21}$$

Restricting to the set where L meets none of the other centers $c_j, j \neq i$, this map is an isomorphism. Since Γ_{c_i} is smooth, that means that any element of $\Gamma_{\mathcal{C}}$, where L does not meet $c_j, j \neq i$ is smooth. But since i was arbitrary, all of $\Gamma_{\mathcal{C}}$ is smooth. Finally, since the back-projected planes always meet in exactly a line, the projection onto the last m coordinates

$$\pi : \Gamma_{\mathcal{C}} \to \widetilde{\mathcal{L}}_{\mathcal{C}}^X, \tag{22}$$

is an isomorphism and therefore $\widetilde{\mathcal{L}}_{\mathcal{C}}^X$ is smooth, but this is the variety of the back-projected planes of $\mathcal{L}_{\mathcal{C}}^X$. In particular, they are isomorphic, and therefore $\mathcal{L}_{\mathcal{C}}^X$ is also smooth. □

We denote by $L_d \subseteq \mathbb{P}^h$ a general linear subspace of codimension d, meaning dimension $h-d$. The *multidegree* of a variety $\mathcal{X} \subseteq \mathbb{P}^{h_1} \times \cdots \times \mathbb{P}^{h_m}$ is the function

$$D(d_1, \ldots, d_m) := \#(\mathcal{X} \cap (L_{d_1}^{(1)} \times \cdots \times L_{d_m}^{(m)})), \quad (23)$$

for $(d_1, \ldots, d_m) \in \mathbb{N}^n$ such that $d_1 + \cdots + d_m = \dim \mathcal{X}$. First note that for any multiview variety, the function D is symmetric under generic camera conditions. This implies that for any permutation $\sigma \in S_n$, $D(d_1, \ldots, d_m)$ is equal to $D(d_{\sigma(1)}, \ldots, d_{\sigma(m)})$.

Proposition B.2. *Let $\mathcal{C}$ be a generic arrangement of cameras.*

1. *The multidegree of $\mathcal{M}_{\mathcal{C}}^L$ is given by the single number $D(1, 0, \ldots, 0) = 1$.*
2. *The multidegree of $\mathcal{L}_{\mathcal{C}}^X$ is given by the two numbers $D(2, 0, \ldots, 0) = 0$ and $D(1, 1, 0, \ldots, 0) = 1$.*

Proof.
1. Since $\mathcal{M}_{\mathcal{C}}^L$ is of dimension 1 and due to symmetry, we only need to consider one number, namely $D(1, 0, \ldots, 0)$. A generic linear form intersecting the line $C_1 \cdot L$ leaves one point, say x_1. Its back-projected line meets L in a unique point X. Recall that any point outside the back-projected line is not projected onto x_1 by C_1. Since $\mathcal{M}_{\mathcal{C}}^L$ equals the image of $\Phi_{\mathcal{C}}|_L$, X is therefore the unique point on L such that $x = \Phi_{\mathcal{L}}(X)$.
2. By symmetry and the fact that $\dim \mathcal{L}_{\mathcal{C}}^X = 2$, we only need to determine $D(2, 0, \ldots, 0)$ and $D(1, 1, 0, \ldots, 0)$. Two generic linear forms intersecting $\Lambda(C_1 X) \subseteq \mathbb{P}^2$ leaves an empty set, why $D(2, 0, \ldots, 0) = 0$. Intersecting $\Lambda(C_1 X)$ and $\Lambda(C_2 X)$ each with generic linear forms leaves one point in each copy of $\mathbb{P}^2$, say ℓ_1 and ℓ_2 that intersect $C_1 X$ and $C_2 X$ respectively. Since they are generic such that $C_i X \in \ell_i$ their back-projected planes H_i both contain X. By genericity, H_i meet in a unique line through X that meets no center, showing $D(1, 1, 0, \ldots, 0) = 1$. $\square$

B.4. The Euclidean distance problem

This section will be used for the proof of Theorem 1.8. It also explains in more detail the reduction of parameters mentioned in Section 2.1.1, via Theorem B.4.

It is not always true that linearly isomorphic varieties have the same Euclidean distance degree. Take for instance the circle and ellipse in $\mathbb{R}^2$. The circle has EDD 2 and the ellipse has EDD 4. However, under additional assumptions, the EDD is the same:

Proposition B.3. *Let $X \subseteq \mathbb{C}^n, Y \subseteq \mathbb{C}^m$ (with $n \geq m$) and let $\psi : X \to Y$ sending x to $Ax + b$ be an affine isomorphism given by a real full-rank matrix A such that $AA^T = I$. Fix a generic $u \in \mathbb{C}^n$. Then,* $\mathrm{EDD}(X) = \mathrm{EDD}(Y)$.

In particular, if $y_1^*, \ldots, y_k^*$ are the solutions to the critical equations of the ED problem on Y given $Au + b$, then $x_1^* = A^T(y_1^* - b), \ldots, x_k^* = A^T(y_k^* - b)$ are the solutions to the critical equations of the ED problem on X given u.

We work with the critical equations, as defined in [2]. Before the proof, we recall some linear algebra: We use that $A^T A$ is a projection matrix onto $\mathrm{Im} A^T A$, and that $\mathbb{C}^n = \mathrm{Im} A^T A \oplus \ker A^T A$, by which we mean that any $x \in \mathbb{C}^n$ can be written as a unique sum $x_1 + x_2$ with $x_1 \in \mathrm{Im} A^T A, x_2 \in \ker A^T A$. Over the real numbers, this is an orthogonal decomposition, i.e. for x_1, x_2 as above we have $x_1 \cdot x_2 = 0$ with respect to the standard inner product. Moreover, the rank of $A^T A$ is the rank of A. This implies that A^T is injective on $\mathrm{Im} A$ and $X \subseteq \mathrm{Im} A^T A$.

Proof. It is not hard to see that shifting a variety by a constant does not change the EDD. Therefore, we put $b = 0$ and continue.

Let $I_X = \langle f_1, \ldots, f_r \rangle$ be the defining ideal of X. Let $g_i = f_i \circ A^T$. We claim that $I_Y = \langle g_1, \ldots, g_r \rangle$ is the defining ideal of Y. Indeed, $y \in Y$ if and only if $A^T y \in X$ if and only if $f_i(A^T y) = 0$ for each i. Further, a full-rank linear change of coordinates preserves the radicality of ideals.

Since ψ is an isomorphism, $x \in X$ is smooth if and only if $Ax \in Y$ is smooth. Now for a generic $u \in \mathbb{C}^n$, let $z^* = (z_1, \ldots, z_m) \in Y$ be smooth and a solution to the critical equations given Au. Write $c_Y = \mathrm{codim}_{\mathbb{C}^m} Y$. Then

$$g_i(z^*) = 0 \text{ for all } i \text{ \& rank} \begin{bmatrix} (z^* - Au)^T \\ \nabla g_1(z^*) \\ \vdots \\ \nabla g_k(z^*) \end{bmatrix} = c_Y. \quad (24)$$

We define $w^* = A^T z^*$ and prove that its a solution to the critical equations of X given u. First, note that $f_i(w^*) = 0$ for each i by construction, and $A(w^* - A^T Au) = z^* - Au$. By the chain rule, $\nabla g_i(z^*) = \nabla f_i(A^T z^*) A^T$. Thus we also have

$$\mathrm{rank} \left(\begin{bmatrix} (w^* - A^T Au)^T \\ \nabla f_1(w^*) \\ \vdots \\ \nabla f_k(w^*) \end{bmatrix} A^T \right) = c_Y. \quad (25)$$

Note that the submatrix of the last k rows of the matrix in Equation (25) has rank c_Y. Now we argue that for $c_X =$

$\mathrm{codim}_{\mathbb{C}^n} X$,

$$\mathrm{rank}\begin{bmatrix}(w^*-A^TAu)^T\\ \nabla f_1(w^*)\\ \vdots\\ \nabla f_k(w^*)\end{bmatrix}=c_X. \tag{26}$$

Because w^* is smooth in X, last k rows of the matrix in Equation (26) are of rank c_X. The $(w^*-A^TAu)^T$ lies in the row span of those k rows, and observe that w^*-A^TAu lies in $\mathrm{Im}A^TA$. This is because z^* lies in the image of A. Therefore,

$$A(w^*-A^TAu)\in\mathrm{span}\{A\nabla f_i(w^*)^T\} \tag{27}$$

implies

$$w^*-A^TAu\in\mathrm{span}\{A^TA\nabla f_i(w^*)^T\}. \tag{28}$$

Since $X\subseteq\mathrm{Im}A^TA$, it follows that $\nabla f_i(w^*)^T$ span $\ker A^TA$. This is because f_i generate the (real) linear forms l_j that vanish on this linear space $\mathrm{Im}A^TA$ and their gradients span the (real) orthogonal complement $\ker A^TA$. So let λ_i be such that w^*-A^TAu equals the sum of $\lambda_iA^TA\nabla f_i(w^*)^T$. Then w^*-A^TAu equals $\sum\lambda_i\nabla f_i(w^*)^T-v$, for some $v\in\ker A^TA$, spanned by $\nabla f_i(w^*)^T$. This proves Equation (26).

Finally, we motivate why we can change A^TAu to u in Equation (26). Showing that A^TAu-u is linearly dependent on $\nabla f_i(w^*)^T$ is suffcient due to the fact that $(w^*-A^TAu)+(A^TAu-u)=w^*-u$. However, A^TAu-u lies in $\ker A^TA$, and therefore this follows from the above.

For the other direction, let w^* be a smooth point satisfying

$$f_i(w^*)=0\ \text{ for all } i\ \&\ \mathrm{rank}\begin{bmatrix}(w^*-u)^T\\ \nabla f_1(w^*)\\ \vdots\\ \nabla f_k(w^*)\end{bmatrix}=c_X. \tag{29}$$

Then $(w^*-u)^T$ is a linear combination of the rows $\nabla f_i(w^*)$. Then $(w^*-u)^TA^T$ is a linear combination of $\nabla f_i(w^*)A^T$. Writing $z^*=Aw^*$ and recalling that this is a smooth point of Y, we have that $(z^*-Au)^T$ is a linear combination of $\nabla g_i(z^*)$. Therefore,

$$g_i(z^*)=0\ \text{ for all } i\ \&\ \mathrm{rank}\begin{bmatrix}(z^*-Au)^T\\ \nabla g_1(z^*)\\ \vdots\\ \nabla g_k(z^*)\end{bmatrix}=c_Y, \tag{30}$$

are all satisfied. □

In the theorem below we use the relation between $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ and $\widehat{\mathcal{C}}$ from Theorem 1.3.

Theorem B.4. *Let $U_i\subseteq\mathbb{P}^2$ be affine patches and write $U=U_1\times\cdots\times U_m$. Fix real matrices $A_i:\mathbb{C}^3\to\mathbb{C}^2$ such that $A_iA_i^T=I$. Let $A:(\mathbb{C}^3)^m\to(\mathbb{C}^2)^m$ be the map that sends $(x_1,\dots,x_m)\in(\mathbb{P}^2)^m$ to $(A_1x_1,\dots,A_mx_m)\in(\mathbb{P}^1)^m$. Let $u\in U_1\times\cdots\times U_m$ be generic.*

1. *Assume $U_i\cap(C_i\cdot L)\neq\emptyset$ for each i. Write $V_i=A_i(U_i\cap(C_i\cdot L))\subseteq\mathbb{R}^2$, and let $V=V_1\times\cdots\times V_m$. If y^* is a critical point of the ED problem for $\mathcal{M}^{1,1}_{\widetilde{\mathcal{C}}}\cap V$ given Au, then $x^*=A^Ty^*$ is a critical point of the ED problem for $\mathcal{M}^L_{\mathcal{C}}\cap U$ given u.*
2. *Assume $U_i\cap\Lambda(C_iX)\neq\emptyset$ for each i. Write $V_i=A_i(U_i\cap\Lambda(C_iX))$, and let $V=V_1\times\cdots\times V_m$. If y^* is a critical point of the ED problem for $\mathcal{M}^{2,1}_{\widehat{\mathcal{C}}}\cap V$ given Au, then $x^*=A^Ty^*$ is a critical point of the ED problem for $\mathcal{L}^X_{\mathcal{C}}\cap U$ given u.*

In both cases, this is a bijection of critical points.

Proof. It is a consequence of Theorem 1.3 that A is an isomorphism of affine varieties in both *1.* and *2* (we set $\psi_{\mathcal{C},i}=A_i$). Then we can directly apply Proposition B.3. □

C. Euclidean Distance Degree Preliminaries

The main theorem of this article is:

Theorem 1.7. *Let $\mathcal{C}$ be a generic arrangement of m cameras.*

1. $\mathrm{EDD}(\mathcal{M}^L_{\mathcal{C}})=3m-2$.
2. *If $m\geq 3$, then* $\mathrm{EDD}(\mathcal{L}^X_{\mathcal{C}})=\frac{9}{2}m^2-\frac{19}{2}m+3$.

In order to compute these two Euclidean distance degrees we make use of the following theorem:

Theorem C.1 (Theorem 3.8 of [12])**.** *Let $X\subseteq\mathbb{C}^n$ be a smooth variety and let U_β denote the complement of the hypersurface $\sum_{1\leq i\leq n}(z_i-\beta_i)^2+\beta_0=0$ in $\mathbb{C}^n$ where $z\in\mathbb{C}^n$ and $\beta\in\mathbb{C}^{n+1}$. Then,*

$$\mathrm{EDD}(X)=(-1)^{\dim X}\chi(X\cap U_\beta). \tag{31}$$

Here χ is the topological Euler characteristic. In the next section, we closely follow [12], by specializing their techniques to our setting. First, we provide the reader with helpful preliminaries. We often take this section for granted and do not always refer to specific results from it.

We have verified with numerical evidence that these formulas hold for $m\leq 10$. The code is attached.

C.1. The Euler characteristic

There are different approaches to defining the Euler characteristic of a topological space. References to the broader topic of algebraic topology include [9, 13]. For instance, given a *triangulation* of a topological space, the Euler characteristic is the alternating sum

$$k_0 - k_1 + k_2 - \dots, \tag{32}$$

where k_i is the number of simplices of dimension i. An n-*simplex* is a polytope of dimension n with $n+1$ vertices, and a *triangulation* is essentially a way of writing a space as a union of simplices that intersect in a good way. Importantly, all real and complex algebraic varieties can be triangulated [10] with respect to Euclidean topology.

The Euler characteristic can more generally be defined for CW complexes and any topological space through singular homology. For spaces where all definitions apply, they are the same.

The following is used in [12].

Lemma C.2. *Let N, M be subvarieties of a complex variety.*

1. $\chi(M \cup N) = \chi(M) + \chi(N) - \chi(M \cap N)$.
2. $\chi(M \setminus N) = \chi(M) - \chi(N)$.

The above does not hold over the real numbers. For instance, $\chi(\mathbb{R}) = 1$, while $\chi(\{x\}) = 1$ and $\chi(\mathbb{R} \setminus \{x\}) = 2$.

Lemma C.3 ([9, Section 2.1])**.** *Let $f : X \to Y$ be a homeomorphism, such as an isomorphism between varieties, then*

$$\chi(X) = \chi(Y). \tag{33}$$

Lemma C.4 ([13, Chapter 10, Section 1])**.** *The Euler Characteristic of $\mathbb{P}^n$ is $n+1$.*

C.2. Chow rings

We refer to [3, 4] for a thorough treatment of intersection theory, and [6] for a friendly introduction. Here we recall the basic definitions and results that are needed to understand this material.

Let X be a variety. We denote by $Z(X)$ the free abelian group of formal integral linear combinations of irreducible subvarieties of X. An *effective cycle* is a formal sum $\sum n_i Y_i$ of irreducible subvarieties Y_i with $n_i \geq 0$. A *zero-cycle* is a formal sum of zero-dimensional varieties Y_i. The *degree* of a zero-cycle is the sum of the associated integers n_i as in [4, Definition 1.4]. We say that two irreducible subvarieties $Y_0, Y_\infty \in Z(X)$ are *rationally equivalent*, and write $Y_0 \sim Y_\infty$ or $Y_0 \equiv Y_\infty$ if there exists an irreducible variety $W \subseteq X \times \mathbb{P}^1$, whose projection onto $\mathbb{P}^1$ is dense, such that $W \cap (X \times \{(1:0)\}) = Y_0$ and $W \cap (X \times \{(0:1)\}) = Y_\infty$.

The Chow group of X is

$$\mathrm{CH}(X) = Z(X)/\sim. \tag{34}$$

For a subvariety, $Y \subseteq X$, write $[Y]$ for the class in $\mathrm{CH}(X)$ of its associated effective cycle. We now aim to turn this group into a ring, by giving it a multiplicative structure.

Let X be an irreducible variety and let Y_1, Y_2 be subvarieties. Y_1 and Y_2 *intersect transversely* at $p \in Y_1 \cap Y_2$ if Y_1, Y_2 and X are smooth at p and $T_pY_1 + T_pY_2 = T_pX$. Further, Y_1 and Y_2 are *generically transverse* if they intersect transversely at generic points of every irreducible component of the intersection $Y_1 \cap Y_2$.

Theorem C.5. *Let X be a smooth variety. Then there is a unique product structure on $\mathrm{CH}(X)$ such that whenever A, B are generically transverse subvarieties of X, then $[A][B] = [A \cap B]$. This product makes $\mathrm{CH}(X)$ into a graded ring, where the grading is given by codimension.*

A natural example of Chow rings are those of products of projective space,

$$\mathrm{CH}((\mathbb{P}^n)^s) \cong \mathbb{Z}[[H_1], \dots, [H_s]]/\langle [H_1]^{n+1}, \dots, [H_s]^{n+1} \rangle. \tag{35}$$

In the above ring isomorphism, $[H_i]$ represent the class of a hyperplane in $\mathrm{CH}(\mathbb{P}^n)$ in factor i.

To a morphism of smooth varieties $f : X \to Y$, we can associate, the *pushforward* $f_* : \mathrm{CH}(X) \to \mathrm{CH}(Y)$ and the *pullback* $f^* : \mathrm{CH}(Y) \to \mathrm{CH}(X)$, two Chow ring maps.

We define the pushforward on irreducible subvarieties $A \subseteq X$ by setting

$$f_*(A) := \begin{cases} 0 & \text{if the generic fiber of } f|_A \\ & \text{is infinite,} \\ d[f(A)] & \text{if the generic fiber of } f|_A \\ & \text{has cardinality } d. \end{cases} \tag{36}$$

By *generic fiber* we mean $f|_A^{-1}(y)$ for generic $y \in f(A)$.

We say that $A \subseteq Y$ is *generically transverse* to f if $f^{-1}(A)$ is generically reduced and the codimension of $f^{-1}(A)$ in X equals the codimension of A in Y. The pullback f_* is defined as the unique map $\mathrm{CH}(Y) \to \mathrm{CH}(X)$ such that, if $A \subseteq Y$ is generically transverse to f, then $f^*[A] := [f^{-1}(A)]$; see [3, Theorem 1.23].

C.3. Chern classes

In intersection theory, Chern classes are algebraic invariants of a variety that lie in its Chow ring. General references again include [3, 4]. Here we only state the properties of them that we use.

Chern classes $c(E)$ are in general defined for vector bundles E, but when the vector bundle is the tangent bundle of a smooth variety X, then we write $c(X)$ for the total Chern class.

Lemma C.6 (Whitney Sum Formula [4, Theorem 3.2])**.** *For a short exact sequence $0 \to E' \to E \to E'' \to 0$ of vector* *bundles on a variety X, we have for the total Chern classes that*

$$c(E) = c(E')c(E''). \tag{37}$$

By a *divisor* of X we mean a subvariety of codimension one. Let $i : A \hookrightarrow X$ be the inclusion map for a subvariety $A \subseteq X$. For an element $[V] \in \mathrm{CH}(X)$, the restriction $[V]|_A$ denotes the pullback $i^*[V]$. If V is generically transverse to i, then $[V]|_A = [V \cap A]$. We observe that $[V]|_A[U]|_A$ equals $i^*[V]i^*[U]$ and since i^* is a ring homorphism, this equals $i^*([V][U]) = ([V][U])_A$.

Lemma C.7 (Adjunction Formula [4, Example 3.2.11], [3, Theorem 5.3])**.** *If X is smooth variety and D a smooth divisor on X, then*

$$c(D) = \frac{c(X)|_D}{(1+[D])|_D}. \tag{38}$$

Lemma C.8 (Functoriality [3, Theorem 5.3])**.** *Let $f : X \to Y$ be morphism of smooth varieties, then*

$$f^*c(E) = c(f^*E), \tag{39}$$

for vector bundles E on Y.

By putting E to be the tangent bundle of Y, its pullback equals X if f is an isomorphism, which we make precise below.

Lemma C.9. *Let $f : X \to Y$ be an isomorphism, then*

$$c(X) = f^*c(Y). \tag{40}$$

Lemma C.10 ([4, Example 3.2.11])**.** *Let $[H]$ be the class of a hyperplane of $\mathbb{P}^n$. Then we have*

$$c(\mathbb{P}^n) = (1+[H])^{n+1}. \tag{41}$$

An important property we use is the next result. The *top* Chern class $c_{\mathrm{top}}(X)$ of X is the zero-cycle part written of $c(X) \in \mathrm{CH}(X)$.

Theorem C.11 (Chern-Gauss-Bonnet [7, Section 3.3])**.** *For a smooth variety X, we have*

$$\chi(X) = \deg(c_{\mathrm{top}}(X)). \tag{42}$$

It happens that authors use the integration symbol for the degree of a zero-cycle $[Z]$ in X the follows sense,

$$\int_X [Z] := \deg(Z). \tag{43}$$

More generally, let $[Z]$ be a formal sum of irreducible subvarieties of X of codimension k. Consider the inclusion $i : A \hookrightarrow X$ for a k-dimensional variety A. We get the following,

$$\int_A [Z]|_A = \int_A i^*[Z] = \int_X [Z][A], \tag{44}$$

where we assume that $i^{-1}(Z)$ is a 0-dimensional.

Next, we consider Chern classes of blow-ups. First some notation. Let $X \subseteq Y$ be an inclusion of smooth varieties. Let $\tilde{Y}$ be the blow-up of Y at X. Let $\tilde{X}$ be the exceptional locus. Let π, ρ be the projection maps and j, i the inclusion maps. The following diagram commutes:

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{j} & \tilde{Y} \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\pi} \\ X & \xrightarrow{i} & Y \end{array}. \tag{45}$$

Porteus' formula [4, Theorem 15.4] gives an expression for the Chern class of $\tilde{Y}$ in terms of the Chern classes of X and Y. For our purposes, we only need the following special case of this theorem, which follows from [4, Example 15.4.2] as stated in [12].

Theorem C.12. *In Equation (45), let X be a set of m distinct points X_i. Then*

$$c(\tilde{Y}) = \pi^*c(Y) + \sum_{i=1}^{m} \Big((1+\eta_i)(1-\eta_i)^d - 1 \Big), \tag{46}$$

where $d = \dim Y$ and $\eta_i = j_*(\rho^*[X_i])$.

C.4. Linear systems

In the proof of Theorem 1.7, we use the language of linear systems. We don't go through many details here, instead, we recall basic definitions. An introduction to line bundles and other relevant concepts are given in the lecture notes of Vakil [18]. For a rational function s on a projective variety X, we define $(s) = \sum \mathrm{ord}_Z(s) Z \in \mathrm{CH}(X)$, where $\mathrm{ord}_Z(s)$ is the order of s at the point Z. Two divisors D, D' are *linearly equivalent* if $D' = D + (s)$ for some rational function s.

Definition C.13. *Let X be a smooth variety. A complete linear system $|D_0|$ of an effective divisor D_0 ($\sum n_i D_i$ so that $n_i \geq 0$) is the set of all effective divisors linearly equivalent to it. A linear system is a linear subspace of a complete linear system.*

Definition C.14. *Let D_0 be an effective divisor. $\Gamma(X, \mathcal{O}(D_0))$ is the of global sections s on X with $(s) + D_0 \geq 0$.*

$\Gamma(X, \mathcal{O}(D_0))$ is interpreted as a complete linear system via the map $f \mapsto (f) + D_0 \in \mathrm{CH}(X)$. Since $(f) = (g)$ if and only if they differ by a non-zero scalar (zeros and poles determine a rational function), $\Gamma(X, \mathcal{O}(D_0))$ can be viewed as a projective space. Subspaces of $\Gamma(X, \mathcal{O}(D_0))$ are also called linear systems.

The *base locus* of a linear system is the intersection of the zero sets of all global sections on X of the linear system. A linear system is *basepoint free* if the base locus is empty. In other words, for every point $x \in X$, there is a global section s such that $s(x) \neq 0$.

Lemma C.15. *The restriction of a basepoint free linear system is basepoint free.*

Proof. Let V be a subvariety of X. Take $v \in V$. Since X is basepoint free, there is a global section s of the linear system for X such that $s(x) \neq 0$. Restricting this section to V we get a global section $s|_V$ for the restricted linear system that is non-zero on v. □

A linear system on a smooth variety X is called very ample if it allows the variety to be embedded into a projective space in a way that preserves its geometry. For a basepoint free linear system, let $a_0, \ldots, a_n$ be global sections that do not simultaneously vanish. The linear system is *very ample* if

$$\begin{aligned} g : X &\to \mathbb{P}^n, \\ x &\mapsto (a_0(x) : \cdots : a_n(x)) \end{aligned} \tag{47}$$

is a *closed immersion*. This means that g is isomorphic onto its image, or that $g^*(O(1))$, the pullback of the hyperplane bundle $O(1)$ on $\mathbb{P}^n$, is isomorphic to L. The restriction of a very ample linear system is very ample.

We define a divisor D_0 to be basepoint free if its linear system $\Gamma(X, \mathcal{O}(D_0))$ is basepoint free. We define a divisor to be very ample analogously.

One importance of basepoint free linear systems comes in the form of this celebrated result:

Theorem C.16 (Bertini's Theorem [7, Section 1.1])**.** *Let X be a smooth complex variety and let Γ be a positive dimensional linear system on X. Then the general element of Γ is smooth away from the base locus.*

C.5. Whitney stratification

A natural way to partition a variety X is via the inclusion

$$X \supseteq \mathrm{sing}(X) \supseteq \mathrm{sing}(\mathrm{sing}(X)) \supseteq \cdots, \tag{48}$$

where sing denote the singular locus. However, not all points of $\mathrm{sing}(X) \setminus \mathrm{sing}(\mathrm{sing}(X))$ necessarily look locally the same. A more fine grained version of this partition is called a *Whitney stratification* [17]. We don't recall the definition here, because all we need to know is that a Whitney stratification of a smooth variety X is $\mathcal{S} = \{X\}$ and the Whitney stratification of a variety X whose singular locus is a finite set of point is $\mathcal{S} = \{X_{\mathrm{reg}}, \{s_1\}, \ldots, \{s_r\}\}$, where X_{reg} is the set of smooth points of X and $s_1, \ldots, s_r$ are the singular points of X. By a theorem of Whitney, any algebraic variety has a Whintey stratification [19, 20].

Before we state the main theorem on Whitney stratifications that we use in this article, we define *Milnor fibers* [11, Chapter 10]. Let X be a smooth variety and V a divisor on X. Choose any Whitney stratum $S \in \mathcal{S}$ and any point $x \in S$. In a sufficiently small ball $B_{\epsilon,x}$ centered at x, the hypersurface V is equal to the zero locus of a holomorphic function f. The Milnor fiber of V at $x \in S$ is given by

$$F_x := B_{\epsilon,x} \cap \{f = t\}, \tag{49}$$

for small $|t|$ greater than 0. The Euler characteristic of F_x is independent of the choice of the local equation f at x, and it is constant along the given stratum containing x.

Theorem C.17 ([15], [11, Theorem 10.4.4])**.** *Let X be a smooth complex projective variety, and let V be a very ample divisor in X. Let $V = \sqcup_{s \in \mathcal{S}} S$ be a Whitney stratification of X. Let W be another divisor on X that is linearly equivalent to V. Suppose W is smooth and W intersects V transversally in the stratified sense (with respect to the above Whitney stratification). Then we have*

$$\chi(W) - \chi(V) = \sum_{S \in \mathcal{S}} \mu_S \chi(S \setminus W), \tag{50}$$

where μ_S is the Euler characteristic of the reduced cohomology of the Milnor fiber at any point $x \in S$.

The Euler characteristic of the *reduced cohomology* is the normal Euler characteristic minus one.

D. Computation of Euclidean Distance Degrees

Let $X \in \mathbb{P}^3$ be a point and $L \in \mathrm{Gr}(1, \mathbb{P}^3)$ a line. Let $\mathcal{C}$ be a generic arrangement of m cameras. For the sake of notation, we write $\mathcal{M}_m^L$ for $\mathcal{M}_\mathcal{C}^L$ and $\mathcal{L}_m^X$ for $\mathcal{L}_\mathcal{C}^X$. We assume from now on that $m \geq 3$ for the anchored line multiview variety $\mathcal{L}_m^X$. We recall notation from [12]. Write each $\mathbb{P}^2$ as $\mathbb{C}^2 \cup \mathbb{P}^1_\infty$, where $\mathbb{C}^2$ is the chosen affine chart and $\mathbb{P}^1_\infty$ is the line at infinity. Denote the hypersurface $\mathbb{P}^2 \times \cdots \times \mathbb{P}^1_\infty \times \cdots \times \mathbb{P}^2$ in $(\mathbb{P}^2)^m$ by $H_{\infty,i}$, where $\mathbb{P}^1_\infty$ is the i-th factor. Let $H_\infty = \cup_{i=1}^m H_{\infty,i}$. Denote by H_Q the closure of the hypersurface $\sum_{i=1}^{2m}(z_i - \beta_i)^2 + \beta_0 = 0$ in $(\mathbb{P}^2)^m$. In the remainder of this proof, we will use the following notation:

$$\begin{aligned} &D_Q^L := \mathcal{M}_m^L \cap H_Q, D_{\infty,i}^L := \mathcal{M}_m^L \cap H_{\infty,i}, \\ &D_\infty^L := \mathcal{M}_m^L \cap H_\infty, \end{aligned} \tag{51}$$

for the anchored point multiview variety, and

$$\begin{aligned} &D_Q^X := \mathcal{L}_m^X \cap H_Q, D_{\infty,i}^X := \mathcal{L}_m^X \cap H_{\infty,i}, \\ &D_\infty^X := \mathcal{L}_m^X \cap H_\infty, \end{aligned} \tag{52}$$

for the anchored line multiview variety. Write M_m^L and L_m^X for the corresponding affine varieties. Notice that H_∞ is the complement of the affine chart $\mathbb{C}^{2m}$ in $(\mathbb{P}^2)^m$, thus D_∞^L is the complement of M_m^L in $\mathcal{M}_m^L$ and D_∞^X is the complement of L_m^X in $\mathcal{L}_m^X$ and. As derived in [12], we have,

$$\chi(M_m^L \cap U_\beta) = \tag{53}$$
$$\chi(\mathcal{M}_m^L) - \chi(D_\infty^L) + \chi(D_Q^L \cap D_\infty^L) - \chi(D_Q^L), \tag{54}$$
$$\chi(L_m^X \cap U_\beta) = \tag{55}$$
$$\chi(\mathcal{L}_m^X) - \chi(D_\infty^X) + \chi(D_Q^X \cap D_\infty^X) - \chi(D_Q^X). \tag{56}$$

The structure of the proof of Theorem 1.7 is to calculate the four terms of Equation (54) and **??**.

Lemma D.1. *For a fixed X and L, let $\mathcal{C}$ be a generic arrangement of m cameras.*

1. $\chi(\mathcal{M}_\mathcal{C}^L) = 2.$
2. $\chi(\mathcal{L}_\mathcal{C}^X) = 3 + m.$

Proof.
1. $\mathcal{M}_\mathcal{C}^L$ is isomorphic to $\mathbb{P}^1$ and we are done by Lemma C.4.

2. Recall that we assume $m \geq 3$. By genericity, c_i are not collinear. Therefore the back-projected planes of an element $\ell \in \mathcal{L}_\mathcal{C}^X$ meet in exactly a line. Consider the partition

$$\mathcal{L}_\mathcal{C}^X = U \cup \bigcup_{i=1}^m U_i, \tag{57}$$

where U is the set of $\ell \in \mathcal{L}_\mathcal{C}^X$ whose back-projected planes meet in a line away from any center, and U_i is the set of $\ell \in \mathcal{L}_\mathcal{C}^X$ whose back-projected planes meet in $c_i \vee X$. By Lemma C.2, $\chi(\mathcal{L}_\mathcal{C}^X) = \chi(U) + \sum \chi(U_i)$. Since $\Upsilon_\mathcal{C}$ is injective on the subset of $\Lambda(X) \setminus \cup(c_i \vee X)$, we see via the isomorphism $U \cong \mathbb{P}^2$ that U is isomorphic to $\mathbb{P}^2$ minus m points. By Lemma C.3, $\chi(U) = 3 - m$. If instead $\ell \in \mathcal{L}_\mathcal{C}^X$ meet exactly in $c_i \vee X$, then for $j \neq i$ we have $\ell_j = C_j \cdot (c_i \vee X)$, and ℓ_i is any line in $\Lambda(C_i X)$. However, $\Lambda(C_i X) \cong \mathbb{P}^1$, implying $\chi(U_i) = 2$. In total, $\chi(\mathcal{L}_\mathcal{C}^X) = 3 - m + 2m = 3 + m$. □

In the next step, we compute the second terms of the right-hand sides of Equation (54) and **??**.

Lemma D.2. *For a fixed X and L, let $\mathcal{C}$ be a generic arrangement of cameras of m cameras.*

1. $\chi(D_\infty^L) = m.$
2. $\chi(D_\infty^X) = 2m - \binom{m}{2}.$

Proof.
1. Each $D_{\infty,i}^L$ is a generic point of $\mathcal{M}_m^L$. By additivity of the Euler characteristic, we have that

$$\chi(D_\infty^L) = \chi(\cup_{i=1}^m D_{\infty,i}^L) = \sum_{i=1}^m \chi(D_{\infty,i}^L) = m. \tag{58}$$

2. Each $D_{\infty,i}^X$ is a curve inside $\mathcal{L}_m^X$. This curve corresponds precisely to fixing the i-th back-projected plane H_i to be generic through c_i and X. Such a plane contains no other center, and a line in this plane uniquely determines all other back-projected planes. Therefore $D_{\infty,i}^L$ is isomorphic to the set of lines in H_i through X, which in turn is isomorphic to $\mathbb{P}^1$. We get $\chi(D_{\infty,i}^X) = 2$. Moreover, $\chi(D_{\infty,i}^X)$ only have pairwise intersections, and two generic back-projected planes H_i, H_j through c_i, X and c_j, X, respectively, meet in exactly a generic line through X. Therefore $D_{\infty,i}^X \cap D_{\infty,j}^X$ consists of a single element. We then get

$$\chi(D_\infty^X) = \chi(\cup_{i=1}^m D_{\infty,i}^X) \tag{59}$$
$$= \sum_{i=1}^m \chi(D_{\infty,i}^X) - \sum_{i<j} \chi\left(D_{\infty,i}^X \cap D_{\infty,j}^X\right) \tag{60}$$
$$= \sum_{i=1}^m 2 - \sum_{i<j} 1 = 2m - \binom{m}{2}, \tag{61}$$

by additivity. □

We recall that H_Q is the closure of the affine hypersurface

$$\sum_{1 \leq i \leq 2m} (z_i - \beta_i)^2 + \beta_0 = 0, \tag{62}$$

in $(\mathbb{P}^2)^m$. We introduce homogeneous coordinates x_i, y_{2i-1}, y_{2i} with $z_{2i-1} = y_{2i-1}/x_i$ and $z_{2i} = y_{2i}/x_i$ for $1 \leq i \leq m$. Write $\mathbf{x} = x_1 \cdots x_m$. Then the homogenization of Equation (62), and hence the equation of H_Q, is

$$\sum_{i=1}^m \left((y_{2i-1} - \beta_{2i-1} x_i)^2 + (y_{2i} - \beta_{2i} x_i)^2 \right) \frac{\mathbf{x}^2}{x_i^2} + \\ + \beta_0 \mathbf{x}^2 = 0. \tag{63}$$

Lemma D.3.

1. $\chi(D_Q^L \cap D_\infty^L) = 0;$
2. $\chi(D_Q^X \cap D_\infty^X) = \binom{m}{2}.$

Proof. We homogenize the equation defining H_Q as in Equation (63), and assume without loss of generality that

$H_{\infty,i}$ is defined by the equation $x_i = 0$. We have by inspection,

$$H_Q \cap H_{\infty,i} = \{y_{2i-1} + \sqrt{-1}y_{2i} = x_i = 0\} \cup \tag{64}$$
$$\cup \{y_{2i-1} - \sqrt{-1}y_{2i} = x_i = 0\} \cup \tag{65}$$
$$\cup \bigcup_{j \neq i} \{x_i = x_j = 0\}. \tag{66}$$

Let,

$$K_i^{L,+} := \mathcal{M}_m^L \cap \{y_{2i-1} + \sqrt{-1}y_{2i} = x_i = 0\},$$
$$K_i^{L,-} := \mathcal{M}_m^L \cap \{y_{2i-1} - \sqrt{-1}y_{2i} = x_i = 0\},$$
$$A_{i,j}^L := \mathcal{M}_m^L \cap \{x_i = x_j = 0\}, j \neq i.$$

Write $K_i^{X,+}, K_i^{X,-}$ and $A_{i,j}^X$ analogously for intersecting with $\mathcal{L}_m^X$ instead of $\mathcal{M}_m^L$. With this notation,

$$\mathcal{M}_m^L \cap H_Q \cap H_{\infty,i} = K_i^{L,+} \cup K_i^{L,-} \cup \bigcup_{i \neq j} A_{i,j}^L, \tag{67}$$

$$\mathcal{L}_m^X \cap H_Q \cap H_{\infty,i} = K_i^{X,+} \cup K_i^{X,-} \cup \bigcup_{i \neq j} A_{i,j}^X. \tag{68}$$

As we shall see below, $K_i^{L,\pm}, K_i^{X,\pm} = \emptyset$. Therefore, by inclusion/exclusion,

$$\chi(D_Q^L \cap D_\infty^L) = \chi(\bigcup_{i \neq j} A_{i,j}^L), \tag{69}$$

$$\chi(D_Q^X \cap D_\infty^X) = \chi(\bigcup_{i \neq j} A_{i,j}^X). \tag{70}$$

1. By construction, the i-th factor of any element of $K_i^{L,\pm} \subseteq (\mathbb{P}^2)^m$ is fixed equal to $[0 : \mp\sqrt{-1} : 1]$. However, by genericity, this point does not lie in $C_i \cdot L$, implying that $K_i^{L,\pm} = \emptyset$. Regarding $A_{i,j}^L$, setting $x_i = 0, x_j = 0$ corresponds to fixing two generic image lines in the corresponding image planes. The back-projected planes of those image lines meet in a generic line, and such a line does not meet L. This implies $A_{i,j}^L = \emptyset$, and we are done by Equation (69).

2. By construction, the i-th factor of any element of $K_i^{X,\pm} \subseteq (\mathbb{P}^2)^m$ is fixed equal to $[0 : \mp\sqrt{-1} : 1]$. However, by genericity, the line this vector defines does not contain C_iX, implying that $K_i^{X,\pm} = \emptyset$. Regarding $A_{i,j}^X$, setting $x_i = 0, x_j = 0$ corresponds to intersecting $\Lambda(C_iX), \Lambda(C_jX)$ with generic hyperplanes. They intersect in the single elements ℓ_i, ℓ_j. The back-projected planes of ℓ_i, ℓ_j meet in a generic line through X. Therefore $A_{i,j}^X$ is a generic point of $\mathcal{L}_m^X$. All $A_{i,j}^X$ are disjoint, and we are done by Equation (70). □

The hypersurface H_Q in the hypersurface H_Q is defined by Equation (63). It follows by Appendix C.2 that we have the following linear equivalence of divisors in $(\mathbb{P}^2)^m$:

$$H_Q \equiv 2H_{\infty,1} + \cdots + 2H_{\infty,m}. \tag{71}$$

Then as divisors of the anchored multiview varieties,

$$D_Q^L \equiv 2\mathcal{M}_m^L \cap H_{\infty,1} + \cdots + 2\mathcal{M}_m^L \cap H_{\infty,m}, \tag{72}$$
$$D_Q^X \equiv 2\mathcal{L}_m^X \cap H_{\infty,1} + \cdots + 2\mathcal{L}_m^X \cap H_{\infty,m}. \tag{73}$$

Consider the well-defined projections $\pi_L : \mathcal{M}_m^L \to L$, and $\pi_X : \mathcal{L}_m^X \to \Lambda(X)$, sending image points and image lines to the intersection of their back-projected lines or planes. In the Chow ring of $\mathbb{P}^3$, every element of L is equivalent. We denote by D_H^L the preimage of a generic hyperplane in L, i.e. a generic point of L. In the Chow ring of $\mathrm{Gr}(1, \mathbb{P}^3)$, every element of $\Lambda(X)$ is equivalent. In particular, we have $\pi_X^*[H] = \pi_X^*[H']$, where H, H' are hyperplanes of lines in $\Lambda(X)$, where a hyperplane of lines is the set of lines in $\Lambda(X)$ contained in some hyperplane of $\mathbb{P}^3$ through X. Let H be a generic plane of lines in $\Lambda(X)$ and $D_H^X = \pi_X^*(H)$. Let H' be a generic among the planes of lines that contain $c_i \vee X$ for some i. In the Chow ring of $\mathcal{L}_m^X$, $\pi_X^*(H')$ is the union of $\mathcal{L}_m^X \cap H_{\infty,i}$ and the variety E_i^X of elements ℓ whose back-projected planes meet exactly in the line $c_i \vee X$. In other words, $E_i^X = \pi_X^*(c_i \vee X)$. We get

$$\begin{aligned} \mathcal{M}_m^L \cap H_{\infty,i} &\equiv D_H^L, \\ \mathcal{L}_m^X \cap H_{\infty,i} &\equiv D_H^X - E_i^X. \end{aligned} \tag{74}$$

It follows that,

$$\begin{aligned} D_Q^L &\equiv 2mD_H^L, \\ D_Q^X &\equiv 2mD_H^X - 2E_i^X - \cdots - 2E_i^X. \end{aligned} \tag{75}$$

Note that $E_i^X \not\equiv E_j^X$ for $i \neq j$.

Lemma D.4.

1. $[D_H^L]^2 = 0$.
2. $[E_i^X]^3 = 0$.
3. $[D_H^X]^3 = 0$.
4. $[D_H^X][E_i^X] = 0$ *for* $i \neq j$.
5. $[E_i^X][E_j^X] = 0$ *for* $i \neq j$.

Proof.

1,2,3. This is a consequence of the fact that D_H^L is a proper subvariety of an irreducible variety of dimension 1. Similarly, both E_i^X and D_H^X are proper subvarieties of an irreducible variety of dimension 2.

4. For a generic plane of lines H through X, $D_H^X \cap E_i^X$ is empty. This suffices by Theorem C.5.

5. We use the fact that $E_j^X \equiv D_H^X - \mathcal{L}_m^X \cap H_{\infty,j}$. By *4.*, intersecting the right-hand side with E_i^X yields the following $E_i^X \cap (\mathcal{L}_m^X \cap H_{\infty,j})$. Now the j-th factor of $\mathcal{L}_m^X \cap H_{\infty,j}$ consists of a fixed generic line through C_iX. Its back-projected plane does not contain $c_i \vee X$. It follows that the intersection must be empty. This suffices by Theorem C.5. □

Proposition D.5. *In the chow ring of* $(\mathbb{P}^2)^m$*, we have the following identities.*

1. $c(\mathcal{M}_m^L) = 1 + 2[D_H^L]$;
2. $c(\mathcal{L}_m^X) = (1 + [D_H^X])^3 - \sum_{i=1}^m \Big([E_i^X] + [E_i^X]^2\Big)$.

Proof.

1. Follows from the fact that $\mathcal{M}_m^L$ is isomorphic to $\mathbb{P}^1$, which in turn has Chern class $1 + 2[x]$ by Lemma C.10, where $[x]$ represents a point of $\mathbb{P}^1$.

2. Recalling that we in the proof of Proposition 1.6 viewed $\mathcal{L}_m^X$ as a blow-up, the Chern class formula from Appendix C.3 gives us

$$c(\mathcal{L}_m^X) = (1 + [D_H^X])^3 + \tag{76}$$

$$+ \sum_{i=1}^m \Big((1 + [E_i^X])(1 - [E_i^X])^2 - 1\Big). \tag{77}$$

After simplification and the fact that $[E_i^X]^3 = 0$, we get the statement. □

As a sanity check, we note that Proposition D.5 gives us the correct Euler characteristics via the Chern-Gauss-Bonnet theorem. The top Chern class of $\mathcal{M}_m^L$ is $2[D_H^L]$, where $[D_H^L]$ is the class of a single point. The top Chern class of $\mathcal{L}_m^X$ is $3[D_H^X]^2 + \sum -[E_i^X]^2$. Now $[D_H^X]^2$ corresponds to the preimage of the intersection of two generic planes through X; it corresponds to a single point. Next, due to the fact that $E_i^X \equiv D_H^X - \mathcal{L}_m^X \cap H_{\infty,i}$, we have that $[E_i^X]^2 \equiv [D_H^X]^2 - 2[D_H^X \cap \mathcal{L}_m^X \cap H_{\infty,i}] + [\mathcal{L}_m^X \cap H_{\infty,i}]^2$. However, $[D_H^X \cap \mathcal{L}_m^X \cap H_{\infty,i}]$ is a single point and the intersection $(\mathcal{L}_m^X \cap H_{\infty,i})^2$ is empty. Therefore $[E_i^X]^2$ is equal to minus a single point. The Chern-Gauss-Bonnet theorem then states that

$$\begin{aligned} \chi(\mathcal{M}_m^L) &= 2, \\ \chi(\mathcal{L}_m^X) &= 3 + m. \end{aligned} \tag{78}$$

We aim to use Theorem C.17 to determine the Euler characteristic of D_Q^X and D_Q^L. We start by considering generic divisors in their linear systems.

Lemma D.6.

1. *A generic divisor D'^L in the linear system $\Gamma(\mathcal{M}_m^L, \mathcal{O}(D_Q^L))$ is smooth;*
2. *A generic divisor D'^X in the linear system $\Gamma(\mathcal{L}_m^X, \mathcal{O}(D_Q^X))$ is smooth.*

Proof. We recall that

$$H_Q \equiv 2H_{1,\infty} + \cdots + 2H_{m,\infty}. \tag{79}$$

Any variety that is the union of hypersurfaces from $i = 1, \ldots, m$ of double lines in factor i is linearly equivalent to H_Q. It is clear that intersecting all such unions gives the empty set, and therefore the base locus of the divisor H_Q is empty. By Lemma C.15, the restriction of H_Q to $\mathcal{M}_m^L$ and $\mathcal{L}_m^X$ gives a basepoint-free linear system. We are done by Bertini's theorem. □

Proposition D.7.

1. *If D'^L is a generic divisor in the linear system $\Gamma(\mathcal{M}_m^L, \mathcal{O}(D_Q^L))$, then $\chi(D'^L) = 2m$;*
2. *If D'^X is a generic divisor in the linear system $\Gamma(\mathcal{L}_m^X, \mathcal{O}(D_Q^X))$, then $\chi(D'^X) = -4m^2 + 8m$.*

Proof.

1. For the purpose of applying the Chern-Gauss-Bonnet theorem, we want to find the top Chern class of D'^L, which is $c_0(D'^L) = 1$. Using $D'^X \equiv D_H^X$, it follows that $\chi(D'^L)$ equals

$$\int_{D'^L} 1|_{D'^L} = \int_{\mathcal{M}_m^L} 1[D'^X] = \int_{\mathcal{M}_m^L} 2m[D_H^X], \tag{80}$$

which equals $2m$ since $[D_H^X]$ is the class of one simple point in $\mathcal{M}_m^L$.

2. By the adjunction formula and considering the fact that $D'^X \equiv D_Q^X$, we have

$$\begin{aligned} c(D'^X) &= \frac{c(\mathcal{L}_m^X|_{D'^X})}{(1 + [D'^X])|_{D'^X}} \\ &= \Big(c(\mathcal{L}_m^X)(1 + [D_Q^X])^{-1}\Big)\Big|_{D'^X}. \end{aligned} \tag{81}$$

Throughout this proof, we use Lemma D.4. The identity $(1+u)^{-1} = 1 - u + u^2 - \cdots$ and Equation (75) imply

$$\begin{aligned} (1 + [D_Q^X])^{-1} =& 1 - 2m[D_H^X] + 2\sum[E_i^X] + \\ &+ 4m^2[D_H^X]^2 + 4\sum[E_i^X]^2. \end{aligned} \tag{82}$$

For the purpose of applying the Chern-Gauss-Bonnet theorem, we want to find the top Chern class of D'^X, which is $c_1(D'^X)$. However, this is the first Chern class of $c(\mathcal{L}_m^X)(1 + [D_Q^X])^{-1}$ restricted to D'^X, by Equation (81). Now using Equation (82) and Proposition D.5, the first Chern class of $c(\mathcal{L}_m^X)(1 + [D_Q^X])^{-1}$ can be written

$$(-2m + 3)[D_H^X] + \sum[E_i^X]. \tag{83}$$

It follows that $\chi(D'^X)$ equals

$$\begin{aligned}
&\int_{D'^X}\left((-2m+3)[D_H^X]+\sum[E_i^X]\right)\Big|_{D'^X}=\\
&=\int_{\mathcal{L}_m^X}\left((-2m+3)[D_H^X]+\sum[E_i^X]\right)[D'^X] \qquad (84)\\
&=\int_{\mathcal{L}_m^X}(-4m^2+6m)[D_H^X]^2-2\sum[E_i^X]^2,
\end{aligned}$$

where we in the last equality used Equation (75). Recall then that $\deg[D_H^X]^2=1$ and $\deg[E_i^X]^2=-1$. So Equation (84) adds to $-4m^2+6m+2m$. □

If we consider y_{2i-1}, y_{2i} and x_i as sections of line bundles $\mathcal{M}_m^L$ and $\mathcal{L}_m^X$, then $D_Q^L=\mathcal{M}_m^L\cap H_Q$, respectively $D_Q^X=\mathcal{L}_m^X\cap H_Q$, is a general divisor in the linear system given by the subspace Γ'^L of $\Gamma(\mathcal{M}_m^L,\mathcal{O}(2mD_H^L))$, respectively Γ'^X of $\Gamma(\mathcal{L}_m^X,\mathcal{O}(2mD_H^X-2E_i^X-\dots-2E_i^X))$, generated by the sections

$$\begin{aligned}
&1.\ (y_1^2+y_2^2)x_2^2\cdots x_m^2+\cdots+\\
&\quad+(y_{2m-1}^2+y_{2m}^2)x_1^2\cdots x_{m-1}^2,\\
&2.\ x_1^2\cdots x_m^2,\\
&3.\ \frac{y_{2i-1}}{x_i}x_1^2\cdots x_m^2 \text{ for } i=1,\dots,m,\\
&4.\ \frac{y_{2i}}{x_i}x_1^2\cdots x_m^2 \text{ for } i=1,\dots,m.
\end{aligned} \qquad (85)$$

To be precise, D_Q^L and D_Q^X are defined through global sections that determined are by H_Q, and H_Q is a linear combination of the generators of Equation (85) with generic coefficients as in Equation (63).

Proposition D.8.

1. *D_Q^L is smooth;*
2. *The singular locus of D_Q^X is the set of $\binom{m}{2}$ points*

$$\bigcup_{i\neq j} D_{\infty,i}^X\cap D_{\infty,j}^X. \qquad (86)$$

Proof.
1. The base locus of Γ'^L is $\cup D_{\infty,i}^L\cap D_{\infty,j}^L$. Indeed, this is precisely the zero locus of the polynomials of Equation (85). However, each $D_{\infty,i}^L\cap D_{\infty,j}^L$ is empty. By Bertini's theorem, D_Q^L is smooth away from this empty set.

2. Similarly, the base locus of Γ'^X is $\cup D_{\infty,i}^X\cap D_{\infty,j}^X$, and each $D_{\infty,i}^X\cap D_{\infty,j}^X$ is a point. On the other hand, D_Q^X has multiplicity at least 2 along $\cup D_{\infty,i}^X\cap D_{\infty,j}^X$. We can see this by looking at the Jacobian condition. The vanishing ideal of $\mathcal{L}_m^X$ together with the additional equation of H_Q defines D_Q^X, a variety of dimension 1. At $S_{i,j}$, the gradient of the generators of $\mathcal{L}_m^X$ give the correct corank 2 since it is smooth, but the additional equation has gradient zero so that the corank is not equal to 1. □

Proposition D.9.

1. *A Whitney stratification of D_Q^L is the single stratum $S_{\text{reg}}=D_Q^L$,*
2. *A Whitney stratification of D_Q^X consists of the stratum of smooth points S_{reg} and $S_{i,j}=D_{\infty,i}^X\cap D_{\infty,j}^X$.*

Proof. This is stated in Appendix C.5. □

Proposition D.10. *The Euler characteristics of the reduced cohomology of the Milnor fibers of the points in Equation (86) are* -1.

Proof. Near

$$S_{i,j}=D_{\infty,i}^X\cap D_{\infty,j}^X, \qquad (87)$$

the functions $x=\frac{x_i}{y_{2i}}, y=\frac{x_j}{y_{2j}}$ form a coordinate frame of $\mathcal{L}_m^X$, meaning the values of x,y determine a unique point of $\mathcal{L}_m^X$. This translates to D_Q^X being determined by the equation

$$u_1x^2+u_2y^2=u_3x^2y^2, \qquad (88)$$

for holomorphic locally non-vanishing functions u_1,u_2,u_3 that we can read off from the homogenization of H_Q in Equation (63).

Next look at $G_t=\{x^2+y^2-x^2y^2=t\}\cap B_\epsilon$ and the map

$$\begin{aligned}
&\psi: G_t\to\{x+y-xy=t\}\cap B_{\epsilon^2},\\
&(x,y)\mapsto(x^2,y^2).
\end{aligned} \qquad (89)$$

Denote by G'_t the set $\{x+y-xy=t\}\cap B_{\epsilon^2}$. Consider the disjoint union

$$\begin{aligned}
G'_t=&(G'_t\cap\{x,y\neq 0\})\cup(G'_t\cap\{x=0\})\\
&\cup(G'_t\cap\{y=0\}).
\end{aligned} \qquad (90)$$

Note that G'_t is smooth at every point for small ϵ; the gradient is $(1-x,1-y)$. Observe that $G'_t\cap\{x=0\}$ and $G'_t\cap\{y=0\}$ are by construction single points. We have that ψ is 4-to-1 on the first set and 2-to-1 on the second and third of the disjoint union. This gives us

$$\begin{aligned}
\chi(G_t)=&4\chi((G'_t\cap\{x,y\neq 0\})+\\
&+2\chi(G'_t\cap\{x=0\})+2\chi(G'_t\cap\{y=0\}) \qquad (91)\\
&=4(1-2)+2+2=0.
\end{aligned}$$

We conclude that the reduced Milnor fiber is -1. □

Proof of Theorem 1.7.

1. We use Theorem C.17 and Proposition D.9 to conclude that

$$\chi(D_Q^L) = \chi(D'^L). \tag{92}$$

We get by Equations (53) and (54), and Lemmas D.1 to D.3 and Proposition D.7 that

$$\chi(M_m^L \cap U_\beta) = 2 - m + 0 - 2m, \tag{93}$$

which sums to $2 - 3m$. Since $\dim \mathcal{M}_m^L = 1$, Theorem 1.7 says that $\mathrm{EDD}(\mathcal{M}_m^L) = 3m - 2$.

2. Similarly, by Theorem C.17 and Proposition D.9, we get

$$\begin{aligned}\chi(D'^X) - \chi(D_Q^X) =& \mu_0 \chi(S_{\text{reg}} \setminus D'^X) + \\ &+ \sum_{i \neq j} \mu_{i,j} \chi(S_{i,j} \setminus D'^X),\end{aligned} \tag{94}$$

where μ_0 and $\mu_{i,j}$ are defined as in Theorem C.17. It is not hard to check that μ_0. Observe that D'^X does not meet any singular points, for instance since the linear system $\Gamma(\mathcal{L}_m^X, \mathcal{O}(D_Q^X))$ is basepoint-free. Therefore $\chi(S_{i,j} \setminus D'^X) = \chi(S_{i,j}) = 1$. We get by Equations (53) and (54), and Lemmas D.1 to D.3 and Propositions D.7 and D.10 that

$$\chi(L_m^X \cap U_\beta) = (3 + m) - (2m - \binom{m}{2}) + \tag{95}$$

$$+ \binom{m}{2} - (-4m^2 + 8m + \binom{m}{2}), \tag{96}$$

which sums to $\frac{9}{2}m^2 - \frac{19}{2}m + 3$. Since $\dim \mathcal{L}_m^X = 2$, Theorem 1.7 says that $\mathrm{EDD}(\mathcal{L}_m^X) = \frac{9}{2}m^2 - \frac{19}{2}m + 3$. □

The following is now a direct consequence:

Corollary 1.8. *Let $\widetilde{\mathcal{C}}$ and $\widehat{\mathcal{C}}$ be generic arrangements of cardinality m.*

1. $\mathrm{EDD}(\mathcal{M}_{\widetilde{\mathcal{C}}}^{1,1}) = 3m - 2$.
2. *If $m \geq 3$, then* $\mathrm{EDD}(\mathcal{M}_{\widehat{\mathcal{C}}}^{2,1}) = \frac{9}{2}m^2 - \frac{19}{2}m + 3$.

Proof. Follows from Theorem 1.7 and Theorem B.4. □

E. Pseudocodes

Finally, in the last section we provide the pseudocode that lay the foundation for our numerical results. For each of the plots presented in the main document, we iterate 1000 (or 100) times on each of the 5 different ways of reconstruction and then we plot the relative error and speed. Note that each time we generate new random camera arrangements, a line in $\mathbb{R}^3$ and p points on this line.

In the pseudocodes below, we present one iteration of each method. The input is a randomly generated camera arrangement $\mathcal{C}$ of 3×4 matrices, a projective line L spanned by two vectors of $\mathbb{R}^4$, and p points $X_i \in \mathbb{R}^3$ such that $[X_i;\ 1]$ lie on L. We use the notation that for a column vector $X \in \mathbb{R}^n$, $[X;\ 1] \in \mathbb{R}^{n+1}$ is the vector we get by adding a 1 as the last coordinate. Let ι be the function that scales a vector such that its last coordinate is 1, and then removes that coordinate. When we write $L' : [L';\ 1] \in \mathrm{Gr}(1, \mathbb{P}^3)$, we mean that L' is a line spanned by two column vectors l_0, l_1 such that the 2×2 lower minor of $\begin{bmatrix} l_0 & l_1 \end{bmatrix}$ is non-zero. This corresponds to choosing an affine patch of the Grassmannian of lines in $\mathbb{P}^3$.

In Algorithms 1 to 5 we use the standard approach for simplicity, but we provide in Algorithm 6 the non-standard approach for (L1).1 to emphasize the distinction.

Algorithm 1: Method (L1).0.

Input : $\mathcal{C} = (C_1, \ldots, C_m), X_1, \ldots, X_p$
Output: The log of the average relative error

- **for** j from 1 to m **do**
  - **for** i from 1 to p **do**
    - $q_{i,j} \leftarrow \iota(C_j[X_i;\ 1]) + \sigma(\epsilon)$;
- $Y_i \leftarrow \underset{X \in \mathbb{R}^3}{\mathrm{argmin}} \sum_{j=1}^m (q_{i,j} - \iota(C_j[X;\ 1]))^2$;
- $e \leftarrow \log_{10}\left(\frac{1}{p\epsilon} \sum_{i=1}^p \|Y_i - X_i\|\right)$;

Return: e

Algorithm 2: Method (L1).1 std.

Input : $\mathcal{C} = (C_1, \ldots, C_m), L, X_1, \ldots, X_p$
Output: The log of the average relative error

- **for** j from 1 to m **do**
  - **for** i from 1 to p **do**
    - $q_{i,j} \leftarrow \iota(C_j[X_i;\ 1]) + \sigma(\epsilon)$;
  - $u_j \leftarrow \iota(C_j \cdot L) + \sigma(\epsilon)$;
- $L_0 \leftarrow \mathrm{nullspace}\begin{bmatrix} C_1^T[u_1;\ 1] & C_2^T[u_2;\ 1] \end{bmatrix}^T$;
- **for** i from 1 to p **do**
  - $Y_i \leftarrow \underset{X \in \mathbb{R}^3 : [X;\ 1] \in L_0}{\mathrm{argmin}} \sum_{j=1}^m (q_{i,j} - \iota(C_j[X;\ 1]))^2$;
- $e \leftarrow \log_{10}\left(\frac{1}{p\epsilon} \sum_{i=1}^p \|Y_i - X_i\|\right)$;

Return: e

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Algorithm 3: Method (L1).2 std.

Input : $\mathcal{C} = (C_1, \ldots, C_m), X_1, \ldots, X_p$
Output: The log of the average relative error

for j from 1 to m **do**
  for i from 1 to p **do**
    $q_{i,j} \leftarrow \iota(C_j[X_i;\ 1]) + \sigma(\epsilon)$;
$Y_1 \leftarrow \underset{X \in \mathbb{R}^3}{\operatorname{argmin}} \sum_{j=1}^{m} (q_{1,j} - \iota(C_j[X;\ 1]))^2$;
$Y_2 \leftarrow \underset{X \in \mathbb{R}^3}{\operatorname{argmin}} \sum_{j=1}^{m} (q_{2,j} - \iota(C_j[X;\ 1]))^2$;
$L_0 \leftarrow \operatorname{span}\{[Y_1;\ 1], [Y_2;\ 1]\}$;
for i from 3 to p **do**
  $Y_i \leftarrow$
  $\underset{X \in \mathbb{R}^3 : [X;\ 1] \in L_0}{\operatorname{argmin}} \sum_{j=1}^{m} (q_{i,j} - \iota(C_j[X;\ 1]))^2$;
$e \leftarrow \log_{10} \left(\frac{1}{p\epsilon} \sum_{i=1}^{p} \|Y_i - X_i\| \right)$;
Return: e

Algorithm 4: Method (L1).3 std.

Input : $\mathcal{C} = (C_1, \ldots, C_m), L, X_1, \ldots, X_p$
Output: The log of the average relative error

for j from 1 to m **do**
  for i from 1 to p **do**
    $q_{i,j} \leftarrow \iota(C_j[X_i;\ 1]) + \sigma(\epsilon)$;
  $u_j \leftarrow \iota(C_j \cdot L) + \sigma(\epsilon)$;
$Y_1 \leftarrow \underset{X \in \mathbb{R}^3}{\operatorname{argmin}} \sum_{j=1}^{m} (q_{1,j} - \iota(C_j[X;\ 1]))^2$;
$L_0 \leftarrow \underset{L' : [L';\ 1] \in \Lambda(Y_1)}{\operatorname{argmin}} \sum_{j=1}^{m} (u_j - \iota(C_j \cdot [L';\ 1]))^2$;
for i from 2 to p **do**
  $Y_i \leftarrow$
  $\underset{X \in \mathbb{R}^3 : [X;\ 1] \in L_0}{\operatorname{argmin}} \sum_{j=1}^{m} (q_{i,j} - \iota(C_j[X;\ 1]))^2$;
$e \leftarrow \log_{10} \left(\frac{1}{p\epsilon} \sum_{i=1}^{p} \|Y_i - X_i\| \right)$;
Return: e

Algorithm 5: Method (L1).4.

Input : $\mathcal{C} = (C_1, \ldots, C_m), L, X_1, \ldots, X_p$
Output: The log of the average relative error

for j from 1 to m **do**
  for i from 1 to p **do**
    $q_{i,j} \leftarrow \iota(C_j[X_i;\ 1]) + \sigma(\epsilon)$;
  $u_j \leftarrow \iota(C_j \cdot L) + \sigma(\epsilon)$;
$L_0 \leftarrow \underset{L' : [L';\ 1] \in \mathrm{Gr}(1, \mathbb{P}^3)}{\operatorname{argmin}} \sum_{j=1}^{m} (u_j - \iota(C_j \cdot [L';\ 1]))^2$;
for i from 1 to m **do**
  $Y_i \leftarrow$
  $\underset{X \in \mathbb{R}^3 : [X;\ 1] \in L_0}{\operatorname{argmin}} \sum_{j=1}^{m} (q_{i,j} - \iota(C_j[X;\ 1]))^2$;
$e \leftarrow \log_{10} \left(\frac{1}{p\epsilon} \sum_{i=1}^{p} \|Y_i - X_i\| \right)$;
Return: e

Algorithm 6: Method (L1).1

Input : $\mathcal{C} = (C_1, \ldots, C_m), L, X_1, \ldots, X_p$
Output: The log of the average relative error

for j from 1 to m **do**
  for i from 1 to p **do**
    $q_{i,j} \leftarrow \iota(C_j[X_i;\ 1]) + \sigma(\epsilon)$;
  $u_j \leftarrow \iota(C_j \cdot L) + \sigma(\epsilon)$;
$l_0, l_1 \leftarrow$
ON-basis of nullspace $\begin{bmatrix} C_1^T[u_1;\ 1] & C_2^T[u_2;\ 1] \end{bmatrix}^T$;
for j from 1 to m **do**
  $a_{j1}, a_{j2} \leftarrow$ ON-basis of $\begin{bmatrix} C_j l_0 & C_j l_1 \end{bmatrix}^T$;
  $A_j \leftarrow \begin{bmatrix} a_{j1} & a_{j2} \end{bmatrix}^T$;
  $C_{\mathrm{aug},j} \leftarrow A_j C_j \begin{bmatrix} l_0 & l_1 \end{bmatrix}$;
  for i from 1 to p **do**
    $q_{i,j}^A \leftarrow \iota(A_j q_{i,j})$;
for i from 1 to p **do**
  $Y_i' \leftarrow \underset{X \in \mathbb{R}^1}{\operatorname{argmin}} \sum_{j=1}^{m} (q_{i,j}^A - \iota(C_{\mathrm{aug},j}[X;\ 1]))^2$;
  $Y_i \leftarrow \iota(\begin{bmatrix} l_0 & l_1 \end{bmatrix} [Y_i';\ 1])$;
$e \leftarrow \log_{10} \left(\frac{1}{p\epsilon} \sum_{i=1}^{p} \|Y_i - X_i\| \right)$;
Return: e

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