Optical Solutions for Spectral Imaging Inverse Problems with a Shift-Variant System.

Proof of Lemma 1

Lemma 1 *Let* $\hat{u}_0(\bar{x},\bar{y},\lambda)=u_0(\bar{x}-a,\bar{y}-b,\lambda)$ *be the source incident wavefront and the respective output of the double phase coding system* $\hat{u}_3(x,y,\lambda)$*, and* $u_3(x,y,\lambda)$ *the output for* $u_0(\bar{x},\bar{y},\lambda)$*. Then, given that the* $\hat{u}_3(x,y,\lambda)\neq u_3(x-a,y-b,\lambda)$*, it is possible to conclude that the double-phase codification system is shift-variant.*

Suppose a monochromatic wavefront $u_0(\bar{x},\bar{y},\lambda)$ for a wavelength λ, this wavefron is propagated through the air a distance z_1 to the first DOE with a heightmap $h_1(x'',y'')$ and a refractive index change per λ defined by Δ_{n_λ}, with an effect $\phi_1(x'',y'',\lambda)=e^{\frac{j2\pi\Delta_{n_\lambda}h_1(x'',y'')}{\lambda}}$ the wavefront after the effect of the DOE is

$$u_1(x'',y'',\lambda)=\frac{e^{jkz_1}}{jz_1\lambda}\phi_1(x'',y'',\lambda)\iint u_0(\bar{x},\bar{y},\lambda)e^{\frac{jk}{2z_1}((x''-\bar{x})^2+(y''-\bar{y})^2)}\,d\bar{x}\,d\bar{y}, \tag{1}$$

This equation can be rewritten using the convolution operator as

$$u_1(x'',y'',\lambda)=\frac{e^{jkz_1}}{jz_1\lambda}\phi_1(x'',y'',\lambda)\left(u_0(\bar{x},\bar{y},\lambda)*e^{\frac{jk}{2z_1}(\bar{x}^2+\bar{y}^2)}\right), \tag{2}$$

Then, the wavefront is propagated a distance z_2 to the second DOE with a heightmap $h_2(x',y')$ and an effect $\phi_2(x',y',\lambda)$ similar to the effect of the DOE 1, the resulting wavefront after the second DOE is dedscribed as:

$$u_2(x',y',\lambda)=\frac{e^{jkz_2}}{jz_2\lambda}\phi_2(x',y',\lambda)\iint u_1(x'',y'',\lambda)e^{\frac{jk}{2z_2}((x'-x'')^2+(y'-y'')^2)}\,dx''\,dy'', \tag{3}$$

Similar to Eq,1, Eq,3 can be expressed as

$$\begin{aligned}u_2(x',y',\lambda)=&\frac{e^{jkz_2}}{jz_2\lambda}\phi_2(x',y',\lambda)\left(u_1(x'',y'',\lambda)*e^{\frac{jk}{2z_1}(x''^2+y''^2)}\right)\\=&\frac{e^{jkz_2}}{jz_2\lambda}\phi_2(x',y',\lambda)\left(\left(\frac{e^{jkz_1}}{jz_1\lambda}\phi_1(x'',y'',\lambda)\left(u_0(\bar{x},\bar{y},\lambda)*e^{\frac{jk}{2z_1}(\bar{x}^2+\bar{y}^2)}\right)\right)*e^{\frac{jk}{2z_2}(x''^2+y''^2)}\right),\\=&\frac{e^{jk(z_2+z_1)}}{j^2z_2z_1\lambda^2}\phi_2(x',y',\lambda)\left(\left(\phi_1(x'',y'',\lambda)\left(u_0(\bar{x},\bar{y},\lambda)*e^{\frac{jk}{2z_1}(\bar{x}^2+\bar{y}^2)}\right)\right)*e^{\frac{jk}{2z_2}(x''^2+y''^2)}\right)\end{aligned} \tag{4}$$

Finally propagating the double coded wave a distance z_3 to the sensor, and following the same analysis of Eq.4,the wavefront in the sensor is described as:

$$\begin{aligned}&u_3(x,y,\lambda)=\\&\frac{e^{jk(z3+z_2+z_1)}}{j^3z_3z_2z_1\lambda^3}\left(\left(\phi_2(x',y',\lambda)\left(\left(\phi_1(x'',y'',\lambda)\left(u_0(\bar{x},\bar{y},\lambda)*e^{\frac{jk}{2z_1}(\bar{x}^2+\bar{y}^2)}\right)\right)*e^{\frac{jk}{2z_2}(x''^2+y''^2)}\right)\right)*e^{\frac{jk}{2z_3}(x'^2+y'^2)}\right)\end{aligned} \tag{5}$$

Then, for the case of $u_0(\bar{x},\bar{y}) = \delta(0,0)$

$$
\begin{aligned}
&u_3(x,y,\lambda) = \\
&\frac{e^{jk(z3+z_2+z_1)}}{j^3 z_3 z_2 z_1 \lambda^3}\left(\left(\phi_2(x',y',\lambda)\left(\left(\phi_1(x'',y'',\lambda)\left(\delta(0,0) * e^{\frac{jk}{2z_1}(\bar{x}^2+\bar{y}^2)}\right)\right) * e^{\frac{jk}{2z_2}(x''^2+y''^2)}\right)\right) * e^{\frac{jk}{2z_3}(x'^2+y'^2)}\right) \\
&\frac{e^{jk(z3+z_2+z_1)}}{j^3 z_3 z_2 z_1 \lambda^3}\left(\left(\phi_2(x',y',\lambda)\left(\left(\phi_1(x'',y'',\lambda)e^{\frac{jk}{2z_1}(\bar{x}^2+\bar{y}^2)}\right)\right) * e^{\frac{jk}{2z_2}(x''^2+y''^2)}\right) * e^{\frac{jk}{2z_3}(x'^2+y'^2)}\right) \\
&\frac{e^{jk(z3+z_2+z_1)}}{j^3 z_3 z_2 z_1 \lambda^3}\left(\left(\phi_2(x',y',\lambda)\left(\int\int \phi_1(x'',y'',\lambda)e^{\frac{jk}{2z_1}(x''^2+y''^2)}e^{\frac{jk}{2z_2}((x'-x'')^2+(y'-y'')^2)}dx''dy''\right)\right) * e^{\frac{jk}{2z_3}(x'^2+y'^2)}\right)
\end{aligned}
\quad , \tag{6}
$$

Expanding the quadratic term of Eq. 6 is possible to get:

$$
\begin{aligned}
&u_3(x,y,\lambda) = \\
&\frac{e^{jk(z3+z_2+z_1)}}{j^3 z_3 z_2 z_1 \lambda^3}\left(\left(\phi_2(x',y',\lambda)e^{\frac{jk}{2z_2}(x'^2+y'^2)}\mathcal{F}\left\{\phi_1(x'',y'',\lambda)e^{\frac{jk}{2}(x''^2+y''^2)(\frac{1}{z_1}+\frac{1}{z_2})}\right\}\right) * e^{\frac{jk}{2z_3}(x'^2+y'^2)}\right),
\end{aligned}
\tag{7}
$$

Where $\mathcal{F}$ is the 2D Fourier transform. Then, for the case of $u_0(\bar{x},\bar{y}) = \delta(a,b)$

$$
\begin{aligned}
&u_3(x,y,\lambda) = \\
&\frac{e^{jk(z3+z_2+z_1)}}{j^3 z_3 z_2 z_1 \lambda^3}\left(\left(\phi_2(x',y',\lambda)\left(\left(\phi_1(x'',y'',\lambda)\left(\delta(a,b) * e^{\frac{jk}{2z_1}(\bar{x}^2+\bar{y}^2)}\right)\right) * e^{\frac{jk}{2z_2}(x''^2+y''^2)}\right)\right) * e^{\frac{jk}{2z_3}(x'^2+y'^2)}\right) \\
&\frac{e^{jk(z3+z_2+z_1)}}{j^3 z_3 z_2 z_1 \lambda^3}\left(\left(\phi_2(x',y',\lambda)\left(\left(\phi_1(x'',y'',\lambda)e^{\frac{jk}{2z_1}((\bar{x}-a)^2+(\bar{y}-b)^2)}\right)\right) * e^{\frac{jk}{2z_2}(x''^2+y''^2)}\right) * e^{\frac{jk}{2z_3}(x'^2+y'^2)}\right) \\
&\frac{e^{jk(z3+z_2+z_1)}}{j^3 z_3 z_2 z_1 \lambda^3}\left(\left(\phi_2(x',y',\lambda)\left(\int\int \phi_1(x'',y'',\lambda)e^{\frac{jk}{2z_1}((x''-a)^2+(y''-b)^2)}e^{\frac{jk}{2z_2}((x'-x'')^2+(y'-y'')^2)}dx''dy''\right)\right) * e^{\frac{jk}{2z_3}(x'^2+y'^2)}\right)
\end{aligned}
\quad , \tag{8}
$$

Expanding the quadratic term of Eq. 8 is possible to get:

$$
\begin{aligned}
&u_3(x,y,\lambda) = \\
&\frac{e^{jk(z3+z_2+z_1)}}{j^3 z_3 z_2 z_1 \lambda^3}\left(\left(\phi_2(x',y',\lambda)e^{\frac{jk}{2z_2}(x'^2+y'^2)}\mathcal{F}\left\{\phi_1(x'',y'',\lambda)e^{\frac{jk}{2z_1}((x''-a)^2+(y''-b)^2)}e^{\frac{jk}{2z_2}(x''^2+y''^2)}\right\}\right) * e^{\frac{jk}{2z_3}(x'^2+y'^2)}\right) \\
&\frac{e^{jk(z3+z_2+z_1)}}{j^3 z_3 z_2 z_1 \lambda^3}\left(\left(\phi_2(x',y',\lambda)e^{\frac{jk}{2z_2}(x'^2+y'^2)}e^{\frac{jk}{2z_1}(a^2+b^2)}\mathcal{F}\left\{\phi_1(x'',y'',\lambda)e^{\frac{jk}{2}(x''^2+y''^2)(\frac{1}{z_1}+\frac{1}{z_2})}e^{\frac{jk}{2z_1}(ax'+by')}\right\}\right) * e^{\frac{jk}{2z_3}(x'^2+y'^2)}\right), \\
&\frac{e^{jk(z3+z_2+z_1)}}{j^3 z_3 z_2 z_1 \lambda^3}e^{\frac{jk}{2z_1}(a^2+b^2)}\left(\left(\phi_2(x',y',\lambda)e^{\frac{jk}{2z_2}(x'^2+y'^2)}\mathcal{F}\left\{\phi_1(x'',y'',\lambda)e^{\frac{jk}{2}(x''^2+y''^2)(\frac{1}{z_1}+\frac{1}{z_2})}\right\} * \delta(a,b)\right) * e^{\frac{jk}{2z_3}(x'^2+y'^2)}\right)
\end{aligned}
\tag{9}
$$

By visual comparison of Eq. 7 and Eq. 9 to analyze the spatial variant characteristics of a double DOE system, it is possible to see that the red term $\frac{jk}{2z_1}(a^2+b^2)$ in the exponential, converted the system to shift-variant. If only the amplitude is analyzed, is possible to see that the terms $\phi_2(x',y',\lambda)e^{\frac{jk}{2z_2}(x'^2+y'^2)}$ multiplying the Fourier transform before the last two convolutions does not allow the shift to reach the final output. From this analysis, it is possible to conclude that the system cannot be modeled by a general PSF as traditional single DOE systems.