

APPENDIX

The Alignment of the Spheres: Globally-Optimal Spherical Mixture Alignment for Camera Pose Estimation

A. Proof of the L_2 Distance Objective Function

Lemma 1, the L_2 objective function, is reproduced below and the proof is given in full.

Lemma 1. (*L_2 objective function*) The L_2 distance between qPNMM and vMFMM models with rotation $\mathbf{R} \in SO(3)$ and translation $\mathbf{t} \in \mathbb{R}^3$ can be minimized using the function

$$f(\mathbf{R}, \mathbf{t}) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_1} \frac{\phi_{1i} \phi_{1j} Z(K_{1i1j}(\mathbf{t}))}{Z(\kappa_{1i}(\mathbf{t})) Z(\kappa_{1j}(\mathbf{t}))} - 2 \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \frac{\phi_{1i} \phi_{2j} Z(K_{1i2j}(\mathbf{R}, \mathbf{t}))}{Z(\kappa_{1i}(\mathbf{t})) Z(\kappa_{2j}(\mathbf{t}))}, \quad (1)$$

where

$$K_{1i1j}(\mathbf{t}) = \left\| \kappa_{1i}(\mathbf{t}) \frac{\boldsymbol{\mu}_{1i} - \mathbf{t}}{\|\boldsymbol{\mu}_{1i} - \mathbf{t}\|} + \kappa_{1j}(\mathbf{t}) \frac{\boldsymbol{\mu}_{1j} - \mathbf{t}}{\|\boldsymbol{\mu}_{1j} - \mathbf{t}\|} \right\|, \quad (2)$$

$$K_{1i2j}(\mathbf{R}, \mathbf{t}) = \left\| \kappa_{1i}(\mathbf{t}) \mathbf{R} \frac{\boldsymbol{\mu}_{1i} - \mathbf{t}}{\|\boldsymbol{\mu}_{1i} - \mathbf{t}\|} + \kappa_{2j} \hat{\boldsymbol{\mu}}_{2j} \right\|, \quad (3)$$

$$\kappa_{1i}(\mathbf{t}) = \left(\frac{\|\boldsymbol{\mu}_{1i} - \mathbf{t}\|}{\sigma_{1i}} \right)^2 + 1, \text{ and} \quad (4)$$

$$Z(x) = \frac{e^x - e^{-x}}{x}. \quad (5)$$

Proof. Given qPNMM and vMFMM models of the input data with parameter sets $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ respectively, and a rigid transformation function $T(\boldsymbol{\theta}_1, \mathbf{R}, \mathbf{t}) = \{\mathbf{R}(\boldsymbol{\mu}_{1i} - \mathbf{t}), \sigma_{1i}^2, \phi_{1i}\}_{i=1}^{n_1}$, the L_2 distance between densities for a rotation \mathbf{R} and translation \mathbf{t} is given by

$$d_{L_2} = \int_{\mathbb{S}^2} [p(\mathbf{f} \mid T(\boldsymbol{\theta}_1, \mathbf{R}, \mathbf{t})) - p(\mathbf{f} \mid \boldsymbol{\theta}_2)]^2 d\mathbf{f} \quad (6)$$

$$= \int_{\mathbb{S}^2} [p(\mathbf{f} \mid T(\boldsymbol{\theta}_1, \mathbf{R}, \mathbf{t}))]^2 + [p(\mathbf{f} \mid \boldsymbol{\theta}_2)]^2 - 2p(\mathbf{f} \mid T(\boldsymbol{\theta}_1, \mathbf{R}, \mathbf{t}))p(\mathbf{f} \mid \boldsymbol{\theta}_2) d\mathbf{f} \quad (7)$$

$$= \int_{\mathbb{S}^2} [p(\mathbf{f} \mid T(\boldsymbol{\theta}_1, \mathbf{R}, \mathbf{t}))]^2 d\mathbf{f} - 2 \int_{\mathbb{S}^2} p(\mathbf{f} \mid T(\boldsymbol{\theta}_1, \mathbf{R}, \mathbf{t}))p(\mathbf{f} \mid \boldsymbol{\theta}_2) d\mathbf{f} + C \quad (8)$$

$$= \int_{\mathbb{S}^2} \sum_{i=1}^{n_1} \phi_{1i} \text{qPN}(\mathbf{f} \mid \boldsymbol{\mu}_{1i} - \mathbf{t}, \sigma_{1i}^2) \sum_{j=1}^{n_1} \phi_{1j} \text{qPN}(\mathbf{f} \mid \boldsymbol{\mu}_{1j} - \mathbf{t}, \sigma_{1j}^2) d\mathbf{f} \\ - 2 \int_{\mathbb{S}^2} \sum_{i=1}^{n_1} \phi_{1i} \text{qPN}(\mathbf{f} \mid \mathbf{R}(\boldsymbol{\mu}_{1i} - \mathbf{t}), \sigma_{1i}^2) \sum_{j=1}^{n_2} \phi_{2j} \text{vMF}(\mathbf{f} \mid \hat{\boldsymbol{\mu}}_{2j}, \kappa_{2j}) d\mathbf{f} + C \quad (9)$$

$$= \sum_{i=1}^{n_1} \sum_{j=1}^{n_1} \phi_{1i} \phi_{1j} \int_{\mathbb{S}^2} \text{qPN}(\mathbf{f} \mid \boldsymbol{\mu}_{1i} - \mathbf{t}, \sigma_{1i}^2) \text{qPN}(\mathbf{f} \mid \boldsymbol{\mu}_{1j} - \mathbf{t}, \sigma_{1j}^2) d\mathbf{f} \\ - 2 \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \phi_{1i} \phi_{2j} \int_{\mathbb{S}^2} \text{qPN}(\mathbf{f} \mid \mathbf{R}(\boldsymbol{\mu}_{1i} - \mathbf{t}), \sigma_{1i}^2) \text{vMF}(\mathbf{f} \mid \hat{\boldsymbol{\mu}}_{2j}, \kappa_{2j}) d\mathbf{f} + C \quad (10)$$

$$= \sum_{i=1}^{n_1} \sum_{j=1}^{n_1} \phi_{1i} \phi_{1j} \int_{\mathbb{S}^2} \text{vMF}\left(\mathbf{f} \mid \frac{\boldsymbol{\mu}_{1i} - \mathbf{t}}{\|\boldsymbol{\mu}_{1i} - \mathbf{t}\|}, \frac{\|\boldsymbol{\mu}_{1i} - \mathbf{t}\|^2}{\sigma_{1i}^2} + 1\right) \text{vMF}\left(\mathbf{f} \mid \frac{\boldsymbol{\mu}_{1j} - \mathbf{t}}{\|\boldsymbol{\mu}_{1j} - \mathbf{t}\|}, \frac{\|\boldsymbol{\mu}_{1j} - \mathbf{t}\|^2}{\sigma_{1j}^2} + 1\right) d\mathbf{f} \\ - 2 \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \phi_{1i} \phi_{2j} \int_{\mathbb{S}^2} \text{vMF}\left(\mathbf{f} \mid \mathbf{R} \frac{\boldsymbol{\mu}_{1i} - \mathbf{t}}{\|\boldsymbol{\mu}_{1i} - \mathbf{t}\|}, \frac{\|\boldsymbol{\mu}_{1i} - \mathbf{t}\|^2}{\sigma_{1i}^2} + 1\right) \text{vMF}(\mathbf{f} \mid \hat{\boldsymbol{\mu}}_{2j}, \kappa_{2j}) d\mathbf{f} + C \quad (11)$$

$$= \sum_{i=1}^{n_1} \sum_{j=1}^{n_1} \phi_{1i} \phi_{1j} \int_{\mathbb{S}^2} \frac{\exp\left(\left(\frac{\|\boldsymbol{\mu}_{1i} - \mathbf{t}\|^2}{\sigma_{1i}^2} + 1\right) \left(\frac{\boldsymbol{\mu}_{1i} - \mathbf{t}}{\|\boldsymbol{\mu}_{1i} - \mathbf{t}\|}\right)^\top \mathbf{f}\right)}{2\pi Z\left(\frac{\|\boldsymbol{\mu}_{1i} - \mathbf{t}\|^2}{\sigma_{1i}^2} + 1\right)} \frac{\exp\left(\left(\frac{\|\boldsymbol{\mu}_{1j} - \mathbf{t}\|^2}{\sigma_{1j}^2} + 1\right) \left(\frac{\boldsymbol{\mu}_{1j} - \mathbf{t}}{\|\boldsymbol{\mu}_{1j} - \mathbf{t}\|}\right)^\top \mathbf{f}\right)}{2\pi Z\left(\frac{\|\boldsymbol{\mu}_{1j} - \mathbf{t}\|^2}{\sigma_{1j}^2} + 1\right)} d\mathbf{f} \\ - 2 \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \phi_{1i} \phi_{2j} \int_{\mathbb{S}^2} \frac{\exp\left(\left(\frac{\|\boldsymbol{\mu}_{1i} - \mathbf{t}\|^2}{\sigma_{1i}^2} + 1\right) \left(\mathbf{R} \frac{\boldsymbol{\mu}_{1i} - \mathbf{t}}{\|\boldsymbol{\mu}_{1i} - \mathbf{t}\|}\right)^\top \mathbf{f}\right)}{2\pi Z\left(\frac{\|\boldsymbol{\mu}_{1i} - \mathbf{t}\|^2}{\sigma_{1i}^2} + 1\right)} \frac{\exp(\kappa_{2j} \hat{\boldsymbol{\mu}}_{2j}^\top \mathbf{f})}{2\pi Z(\kappa_{2j})} d\mathbf{f} + C \quad (12)$$

$$= \sum_{i=1}^{n_1} \sum_{j=1}^{n_1} \phi_{1i} \phi_{1j} \int_{\mathbb{S}^2} \frac{\exp\left(\kappa_{1i}(\mathbf{t}) \left(\frac{\boldsymbol{\mu}_{1i} - \mathbf{t}}{\|\boldsymbol{\mu}_{1i} - \mathbf{t}\|}\right)^\top \mathbf{f}\right)}{2\pi Z(\kappa_{1i}(\mathbf{t}))} \frac{\exp\left(\kappa_{1j}(\mathbf{t}) \left(\frac{\boldsymbol{\mu}_{1j} - \mathbf{t}}{\|\boldsymbol{\mu}_{1j} - \mathbf{t}\|}\right)^\top \mathbf{f}\right)}{2\pi Z(\kappa_{1j}(\mathbf{t}))} d\mathbf{f} \\ - 2 \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \phi_{1i} \phi_{2j} \int_{\mathbb{S}^2} \frac{\exp\left(\kappa_{1i}(\mathbf{t}) \left(\mathbf{R} \frac{\boldsymbol{\mu}_{1i} - \mathbf{t}}{\|\boldsymbol{\mu}_{1i} - \mathbf{t}\|}\right)^\top \mathbf{f}\right)}{2\pi Z(\kappa_{1i}(\mathbf{t}))} \frac{\exp(\kappa_{2j} \hat{\boldsymbol{\mu}}_{2j}^\top \mathbf{f})}{2\pi Z(\kappa_{2j})} d\mathbf{f} + C \quad (13)$$

$$= \sum_{i=1}^{n_1} \sum_{j=1}^{n_1} \phi_{1i} \phi_{1j} \int_{\mathbb{S}^2} \frac{\exp\left(\left(\kappa_{1i}(\mathbf{t}) \frac{\boldsymbol{\mu}_{1i} - \mathbf{t}}{\|\boldsymbol{\mu}_{1i} - \mathbf{t}\|} + \kappa_{1j}(\mathbf{t}) \frac{\boldsymbol{\mu}_{1j} - \mathbf{t}}{\|\boldsymbol{\mu}_{1j} - \mathbf{t}\|}\right)^\top \mathbf{f}\right)}{2\pi Z(\kappa_{1i}(\mathbf{t})) 2\pi Z(\kappa_{1j}(\mathbf{t}))} d\mathbf{f} \\ - 2 \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \phi_{1i} \phi_{2j} \int_{\mathbb{S}^2} \frac{\exp\left(\left(\kappa_{1i}(\mathbf{t}) \mathbf{R} \frac{\boldsymbol{\mu}_{1i} - \mathbf{t}}{\|\boldsymbol{\mu}_{1i} - \mathbf{t}\|} + \kappa_{2j} \hat{\boldsymbol{\mu}}_{2j}\right)^\top \mathbf{f}\right)}{2\pi Z(\kappa_{1i}(\mathbf{t})) 2\pi Z(\kappa_{2j})} d\mathbf{f} + C \quad (14)$$

$$= \sum_{i=1}^{n_1} \sum_{j=1}^{n_1} \phi_{1i} \phi_{1j} \frac{2\pi Z\left(\left\|\kappa_{1i}(\mathbf{t}) \frac{\boldsymbol{\mu}_{1i} - \mathbf{t}}{\|\boldsymbol{\mu}_{1i} - \mathbf{t}\|} + \kappa_{1j}(\mathbf{t}) \frac{\boldsymbol{\mu}_{1j} - \mathbf{t}}{\|\boldsymbol{\mu}_{1j} - \mathbf{t}\|}\right\|\right)}{2\pi Z(\kappa_{1i}(\mathbf{t})) 2\pi Z(\kappa_{1j}(\mathbf{t}))} \\ - 2 \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \phi_{1i} \phi_{2j} \frac{2\pi Z\left(\left\|\kappa_{1i}(\mathbf{t}) \mathbf{R} \frac{\boldsymbol{\mu}_{1i} - \mathbf{t}}{\|\boldsymbol{\mu}_{1i} - \mathbf{t}\|} + \kappa_{2j} \hat{\boldsymbol{\mu}}_{2j}\right\|\right)}{2\pi Z(\kappa_{1i}(\mathbf{t})) 2\pi Z(\kappa_{2j})} + C \quad (15)$$

$$= C' \left(\sum_{i=1}^{n_1} \sum_{j=1}^{n_1} \phi_{1i} \phi_{1j} \frac{Z(K_{1i1j}(\mathbf{t}))}{Z(\kappa_{1i}(\mathbf{t})) Z(\kappa_{1j}(\mathbf{t}))} - 2 \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \phi_{1i} \phi_{2j} \frac{Z(K_{1i2j}(\mathbf{R}, \mathbf{t}))}{Z(\kappa_{1i}(\mathbf{t})) Z(\kappa_{2j})} \right) + C. \quad (16)$$

The first term of (7) is invariant under rotations and the second term is independent of the rotation and translation. Equation (8) replaces the integral of the second term with a constant C and applies the integral termwise; equation (9) substitutes the probability density functions for the qPNMM and vMFMM (equations (8) and (9) from the main paper) into (8); equation (10) rearranges the integrals and summations; equation (11) substitutes the definition for the qPN distribution (equation (4) from the main paper) into (10); equation (12) substitutes the definition for the vMF distribution (equation (1) from the main paper) into (11); equation (13) uses the definition of $\kappa_{1i}(\mathbf{t})$ from (4); and equation (14) simplifies the expression using the product-to-sum property of exponential functions. Equation (15) uses the observation that integrals of the form $\int_{\mathbb{S}^2} \exp(\mathbf{x}^\top \mathbf{f}) d\mathbf{f}$ are equal to the normalization constant of a vMF density with $\kappa = \|\mathbf{x}\|$ and $\hat{\boldsymbol{\mu}} = \mathbf{x}/\kappa$. Finally, equation (16) uses the definition of K_{1i1j} and K_{1i2j} from (2) and (3), and lets $C' = \frac{1}{2\pi}$. The objective function (1) is obtained by removing constant summands and factors. \square

B. Proof of the Objective Function Bounds

Theorem 1, the L_2 objective function bounds, is reproduced below and the proof is given in full.

Theorem 1. (Objective function bounds) *For the transformation domain $\mathcal{C}_r \times \mathcal{C}_t$ centered at $(\mathbf{r}_0, \mathbf{t}_0)$, the minimum of the objective function (10) has an upper bound*

$$\bar{d} \triangleq f(\mathbf{R}_{\mathbf{r}_0}, \mathbf{t}_0) \quad (17)$$

and a lower bound

$$\underline{d} \triangleq \sum_{i=1}^{n_1} \sum_{j=1}^{n_1} \phi_{1i} \phi_{1j} \min_{\mathbf{t} \in \mathcal{C}_t} \frac{Z(\underline{K}_{1i1j}(\mathbf{t}))}{Z(\kappa_{1i}(\mathbf{t}))Z(\kappa_{1j}(\mathbf{t}))} - 2 \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \phi_{1i} \phi_{2j} \max_{\mathbf{t} \in \mathcal{C}_t} \frac{Z(\bar{K}_{1i2j}(\mathbf{t}))}{Z(\kappa_{1i}(\mathbf{t}))Z(\kappa_{2j}(\mathbf{t}))} \quad (18)$$

where

$$\underline{K}_{1i1j}(\mathbf{t}) = \sqrt{\kappa_{1i}^2(\mathbf{t}) + \kappa_{1j}^2(\mathbf{t}) + 2\kappa_{1i}(\mathbf{t})\kappa_{1j}(\mathbf{t}) \cos A} \quad (19)$$

$$\bar{K}_{1i2j}(\mathbf{t}) = \sqrt{\kappa_{1i}^2(\mathbf{t}) + \kappa_{2j}^2(\mathbf{t}) + 2\kappa_{1i}(\mathbf{t})\kappa_{2j}(\mathbf{t}) \cos B} \quad (20)$$

$$A = \min \{ \pi, \angle(\boldsymbol{\mu}_{1i} - \mathbf{t}_0, \boldsymbol{\mu}_{1j} - \mathbf{t}_0) + \psi_t(\boldsymbol{\mu}_{1i}, \mathcal{C}_t) + \psi_t(\boldsymbol{\mu}_{1j}, \mathcal{C}_t) \} \quad (21)$$

$$B = \max \{ 0, \angle(\mathbf{R}_{\mathbf{r}_0}(\boldsymbol{\mu}_{1i} - \mathbf{t}_0), \hat{\boldsymbol{\mu}}_{2j}) - \psi_t(\boldsymbol{\mu}_{1i}, \mathcal{C}_t) - \psi_r(\hat{\boldsymbol{\mu}}_{2j}, \mathcal{C}_r) \} \quad (22)$$

Proof. The validity of the upper bound follows from

$$f(\mathbf{R}_{\mathbf{r}_0}, \mathbf{t}_0) \geq \min_{\substack{\mathbf{r} \in \mathcal{C}_r \\ \mathbf{t} \in \mathcal{C}_t}} f(\mathbf{R}_{\mathbf{r}}, \mathbf{t}). \quad (23)$$

That is, the function value at a specific point within the domain is greater than or equal to the minimum within the domain. The validity of the lower bound follows from

$$\min_{\substack{\mathbf{r} \in \mathcal{C}_r \\ \mathbf{t} \in \mathcal{C}_t}} f(\mathbf{R}_{\mathbf{r}}, \mathbf{t}) = \min_{\substack{\mathbf{r} \in \mathcal{C}_r \\ \mathbf{t} \in \mathcal{C}_t}} \left(\sum_{i=1}^{n_1} \sum_{j=1}^{n_1} \phi_{1i} \phi_{1j} \frac{Z(K_{1i1j}(\mathbf{t}))}{Z(\kappa_{1i}(\mathbf{t}))Z(\kappa_{1j}(\mathbf{t}))} - 2 \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \phi_{1i} \phi_{2j} \frac{Z(K_{1i2j}(\mathbf{R}, \mathbf{t}))}{Z(\kappa_{1i}(\mathbf{t}))Z(\kappa_{2j}(\mathbf{t}))} \right) \quad (24)$$

$$\geq \sum_{i=1}^{n_1} \sum_{j=1}^{n_1} \phi_{1i} \phi_{1j} \min_{\mathbf{t} \in \mathcal{C}_t} \frac{Z(K_{1i1j}(\mathbf{t}))}{Z(\kappa_{1i}(\mathbf{t}))Z(\kappa_{1j}(\mathbf{t}))} - 2 \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \phi_{1i} \phi_{2j} \max_{\substack{\mathbf{r} \in \mathcal{C}_r \\ \mathbf{t} \in \mathcal{C}_t}} \frac{Z(K_{1i2j}(\mathbf{R}, \mathbf{t}))}{Z(\kappa_{1i}(\mathbf{t}))Z(\kappa_{2j}(\mathbf{t}))} \quad (25)$$

$$\geq \sum_{i=1}^{n_1} \sum_{j=1}^{n_1} \phi_{1i} \phi_{1j} \min_{\mathbf{t} \in \mathcal{C}_t} \frac{Z(\underline{K}_{1i1j}(\mathbf{t}))}{Z(\kappa_{1i}(\mathbf{t}))Z(\kappa_{1j}(\mathbf{t}))} - 2 \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \phi_{1i} \phi_{2j} \max_{\mathbf{t} \in \mathcal{C}_t} \frac{Z(\bar{K}_{1i2j}(\mathbf{t}))}{Z(\kappa_{1i}(\mathbf{t}))Z(\kappa_{2j}(\mathbf{t}))}, \quad (26)$$

where the second inequality follows termwise with $Z(x) = (e^x - e^{-x})x^{-1}$ monotonically increasing for $x \geq 0$. This can be seen by taking the derivative and observing that $\frac{dZ}{dx} \geq 0$ for $x \geq 0$. Hence, $Z(\underline{K}) \leq Z(K) \leq Z(\bar{K})$ for $0 \leq \underline{K} \leq K \leq \bar{K}$.

We will now derive $\underline{K}_{1i1j}(\mathbf{t})$ and $\overline{K}_{1i2j}(\mathbf{t})$. Observe that $\forall \mathbf{t} \in \mathcal{C}_t$,

$$K_{1i1j}(\mathbf{t}) = \left\| \kappa_{1i}(\mathbf{t}) \frac{\boldsymbol{\mu}_{1i} - \mathbf{t}}{\|\boldsymbol{\mu}_{1i} - \mathbf{t}\|} + \kappa_{1j}(\mathbf{t}) \frac{\boldsymbol{\mu}_{1j} - \mathbf{t}}{\|\boldsymbol{\mu}_{1j} - \mathbf{t}\|} \right\| \quad (27)$$

$$= \sqrt{\left\| \kappa_{1i}(\mathbf{t}) \frac{\boldsymbol{\mu}_{1i} - \mathbf{t}}{\|\boldsymbol{\mu}_{1i} - \mathbf{t}\|} + \kappa_{1j}(\mathbf{t}) \frac{\boldsymbol{\mu}_{1j} - \mathbf{t}}{\|\boldsymbol{\mu}_{1j} - \mathbf{t}\|} \right\|^2} \quad (28)$$

$$= \sqrt{\kappa_{1i}^2(\mathbf{t}) + \kappa_{1j}^2(\mathbf{t}) + 2\kappa_{1i}(\mathbf{t})\kappa_{1j}(\mathbf{t}) \frac{\boldsymbol{\mu}_{1i} - \mathbf{t}}{\|\boldsymbol{\mu}_{1i} - \mathbf{t}\|} \cdot \frac{\boldsymbol{\mu}_{1j} - \mathbf{t}}{\|\boldsymbol{\mu}_{1j} - \mathbf{t}\|}} \quad (29)$$

$$= \sqrt{\kappa_{1i}^2(\mathbf{t}) + \kappa_{1j}^2(\mathbf{t}) + 2\kappa_{1i}(\mathbf{t})\kappa_{1j}(\mathbf{t}) \cos \angle(\boldsymbol{\mu}_{1i} - \mathbf{t}, \boldsymbol{\mu}_{1j} - \mathbf{t})} \quad (30)$$

$$\geq \sqrt{\kappa_{1i}^2(\mathbf{t}) + \kappa_{1j}^2(\mathbf{t}) + 2\kappa_{1i}(\mathbf{t})\kappa_{1j}(\mathbf{t}) \cos A} \quad (31)$$

$$= \underline{K}_{1i1j}(\mathbf{t}) \quad (32)$$

where (31) follows, for $\mathbf{t} \in \mathcal{C}_t$, from

$$\angle(\boldsymbol{\mu}_{1i} - \mathbf{t}, \boldsymbol{\mu}_{1j} - \mathbf{t}) \leq \min \{ \pi, \angle(\boldsymbol{\mu}_{1i} - \mathbf{t}_0, \boldsymbol{\mu}_{1j} - \mathbf{t}_0) + \angle(\boldsymbol{\mu}_{1i} - \mathbf{t}, \boldsymbol{\mu}_{1i} - \mathbf{t}_0) + \angle(\boldsymbol{\mu}_{1j} - \mathbf{t}, \boldsymbol{\mu}_{1j} - \mathbf{t}_0) \} \quad (33)$$

$$\leq \min \{ \pi, \angle(\boldsymbol{\mu}_{1i} - \mathbf{t}_0, \boldsymbol{\mu}_{1j} - \mathbf{t}_0) + \psi_t(\boldsymbol{\mu}_{1i}, \mathcal{C}_t) + \psi_t(\boldsymbol{\mu}_{1j}, \mathcal{C}_t) \} \quad (34)$$

$$= A, \quad (35)$$

applying the triangle inequality in spherical geometry twice to obtain (33) (see Figure B.1(a)) and Lemma 3 to obtain (34). Also observe that $\forall (\mathbf{r}, \mathbf{t}) \in (\mathcal{C}_r \times \mathcal{C}_t)$,

$$K_{1i2j}(\mathbf{R}_r, \mathbf{t}) = \left\| \kappa_{1i}(\mathbf{t}) \frac{\boldsymbol{\mu}_{1i} - \mathbf{t}}{\|\boldsymbol{\mu}_{1i} - \mathbf{t}\|} + \kappa_{2j} \mathbf{R}_r^{-1} \hat{\boldsymbol{\mu}}_{2j} \right\| \quad (36)$$

$$= \sqrt{\left\| \kappa_{1i}(\mathbf{t}) \frac{\boldsymbol{\mu}_{1i} - \mathbf{t}}{\|\boldsymbol{\mu}_{1i} - \mathbf{t}\|} + \kappa_{2j} \mathbf{R}_r^{-1} \hat{\boldsymbol{\mu}}_{2j} \right\|^2} \quad (37)$$

$$= \sqrt{\kappa_{1i}^2(\mathbf{t}) + \kappa_{2j}^2 + 2\kappa_{1i}(\mathbf{t})\kappa_{2j} \frac{\boldsymbol{\mu}_{1i} - \mathbf{t}}{\|\boldsymbol{\mu}_{1i} - \mathbf{t}\|} \cdot \mathbf{R}_r^{-1} \hat{\boldsymbol{\mu}}_{2j}} \quad (38)$$

$$= \sqrt{\kappa_{1i}^2(\mathbf{t}) + \kappa_{2j}^2 + 2\kappa_{1i}(\mathbf{t})\kappa_{2j} \cos \angle(\boldsymbol{\mu}_{1i} - \mathbf{t}, \mathbf{R}_r^{-1} \hat{\boldsymbol{\mu}}_{2j})} \quad (39)$$

$$\leq \sqrt{\kappa_{1i}^2(\mathbf{t}) + \kappa_{2j}^2 + 2\kappa_{1i}(\mathbf{t})\kappa_{2j} \cos B} \quad (40)$$

$$= \overline{K}_{1i2j}(\mathbf{t}) \quad (41)$$

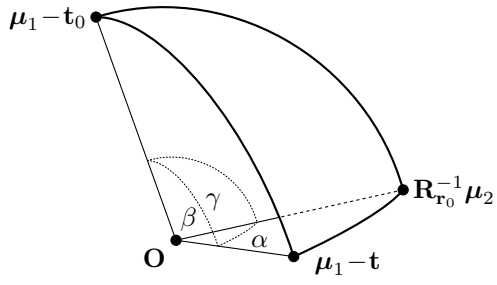
where (40) follows, for $\mathbf{t} \in \mathcal{C}_t$ and $\mathbf{r} \in \mathcal{C}_r$, from

$$\angle(\boldsymbol{\mu}_{1i} - \mathbf{t}, \mathbf{R}_r^{-1} \hat{\boldsymbol{\mu}}_{2j}) \geq \max \{ 0, \angle(\boldsymbol{\mu}_{1i} - \mathbf{t}_0, \mathbf{R}_{r_0}^{-1} \hat{\boldsymbol{\mu}}_{2j}) - \angle(\boldsymbol{\mu}_{1i} - \mathbf{t}, \boldsymbol{\mu}_{1i} - \mathbf{t}_0) - \angle(\mathbf{R}_r^{-1} \hat{\boldsymbol{\mu}}_{2j}, \mathbf{R}_{r_0}^{-1} \hat{\boldsymbol{\mu}}_{2j}) \} \quad (42)$$

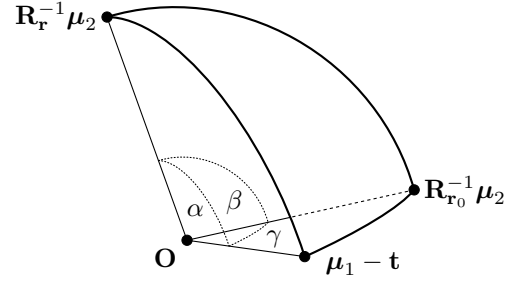
$$\geq \max \{ 0, \angle(\boldsymbol{\mu}_{1i} - \mathbf{t}_0, \mathbf{R}_{r_0}^{-1} \hat{\boldsymbol{\mu}}_{2j}) - \psi_t(\boldsymbol{\mu}_{1i}, \mathcal{C}_t) - \psi_r(\hat{\boldsymbol{\mu}}_{2j}, \mathcal{C}_r) \} \quad (43)$$

$$= B, \quad (44)$$

applying the triangle inequality in spherical geometry twice to obtain (42) (see Figures B.1(a) and B.1(b)) and Lemma 2 and 3 to obtain (43). \square



(a) Triangle inequality for (33) and (42)



(b) Triangle inequality for (42)

Figure B.1. The triangle inequality in spherical geometry, given by $\gamma \leq \min \{\pi, \alpha + \beta\}$. The transformed points have been normalized to lie on the unit sphere.

C. Implementation Details

C.1. Optimizing the Lower Bound

Computing the lower bound (18) involves solving the optimization problems

$$\min_{\mathbf{t} \in \mathcal{C}_t} \frac{Z(\underline{\kappa}_{1i1j}(\mathbf{t}))}{Z(\kappa_{1i}(\mathbf{t}))Z(\kappa_{1j}(\mathbf{t}))} = \min_{\mathbf{t} \in \mathcal{C}_t} \frac{Z\left(\sqrt{\kappa_{1i}^2(\mathbf{t}) + \kappa_{1j}^2(\mathbf{t}) + 2\kappa_{1i}(\mathbf{t})\kappa_{1j}(\mathbf{t})\cos A}\right)}{Z(\kappa_{1i}(\mathbf{t}))Z(\kappa_{1j}(\mathbf{t}))} \quad (45)$$

and

$$\max_{\mathbf{t} \in \mathcal{C}_t} \frac{Z(\overline{\kappa}_{1i2j}(\mathbf{t}))}{Z(\kappa_{1i}(\mathbf{t}))Z(\kappa_{2j}(\mathbf{t}))} = \max_{\mathbf{t} \in \mathcal{C}_t} \frac{Z\left(\sqrt{\kappa_{1i}^2(\mathbf{t}) + \kappa_{2j}^2(\mathbf{t}) + 2\kappa_{1i}(\mathbf{t})\kappa_{2j}(\mathbf{t})\cos B}\right)}{Z(\kappa_{1i}(\mathbf{t}))Z(\kappa_{2j}(\mathbf{t}))}. \quad (46)$$

These can be optimized by first defining the smallest and largest values that $\kappa_{1i}(\mathbf{t})$ can attain over the translation cuboid \mathcal{C}_t , given by

$$\underline{\kappa}_{1i}(\mathcal{C}_t) \triangleq \min_{\mathbf{t} \in \mathcal{C}_t} \kappa_{1i}(\mathbf{t}) = \frac{\min_{\mathbf{t} \in \mathcal{C}_t} \|\boldsymbol{\mu}_{1i} - \mathbf{t}\|^2}{\sigma_{1i}^2} + 1, \text{ and} \quad (47)$$

$$\overline{\kappa}_{1i}(\mathcal{C}_t) \triangleq \max_{\mathbf{t} \in \mathcal{C}_t} \kappa_{1i}(\mathbf{t}) = \frac{\max_{\mathbf{t} \in \mathcal{V}_t} \|\boldsymbol{\mu}_{1i} - \mathbf{t}\|^2}{\sigma_{1i}^2} + 1, \quad (48)$$

where \mathcal{V}_t is the set of vertices of \mathcal{C}_t . These values can be computed easily in constant time. Now (45) is minimized by choosing $\kappa_{1i}(\mathbf{t}) = \underline{\kappa}_{1i}(\mathcal{C}_t)$ and $\kappa_{1j}(\mathbf{t}) = \overline{\kappa}_{1j}(\mathcal{C}_t)$ for $A \geq \frac{\pi}{2}$, since the function in this range is monotonically decreasing. For $0 \leq A \leq \frac{\pi}{2}$, (45) is a concave minimization problem over the convex set $[\underline{\kappa}_{1i}(\mathcal{C}_t), \overline{\kappa}_{1i}(\mathcal{C}_t)] \times [\underline{\kappa}_{1j}(\mathcal{C}_t), \overline{\kappa}_{1j}(\mathcal{C}_t)]$, and so all four combinations of extreme points must be tested to evaluate the minimum. Similarly, (46) is maximized by choosing $\kappa_{1i}(\mathbf{t}) = \underline{\kappa}_{1i}(\mathcal{C}_t)$ for $B \geq \frac{\pi}{2}$, since the function in this range is monotonically decreasing. For $0 \leq B \leq \frac{\pi}{2}$, (46) is a concave maximization problem over the convex set $[\underline{\kappa}_{1i}(\mathcal{C}_t), \overline{\kappa}_{1i}(\mathcal{C}_t)]$. We solve this maximization problem by inspecting the gradient at the extreme points and, if necessary, computing the maximum using bisection search.

C.2. Objective Function Normalizer

Since the L_2 distance depends on the mixture model parameters, it cannot be directly compared across datasets. In order to use a consistent ϵ value across datasets, a normalizer constant is used for the L_2 objective function. It maps a putative worst-case alignment to 1 as $\kappa \rightarrow \infty$ and $\sigma^2 \rightarrow 0$, and is given by

$$\frac{1}{2} \left(\sum_{i=1}^{n_1} \phi_{1i}^2 \kappa'_{1i} + \sum_{i=1}^{n_2} \phi_{2i}^2 \kappa_{2i} \right) \quad (49)$$

where $\kappa'_{1i} = \delta^2 / \sigma_{1i}^2 + 1$, for a fixed maximum camera-to-model distance δ .

D. Objective Function Gradient

Spherical Mixture Alignment (SMA) local optimization employs gradient-based optimization to find the nearest local minimum. The gradient of the objective function (1) is given below as Lemma A.

Lemma A. (Derivative of the L_2 objective function) Given a rotation \mathbf{R}_r and a translation \mathbf{t} , then the derivative of the objective function

$$f(\mathbf{R}_r, \mathbf{t}) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_1} f_{1i1j}(\mathbf{t}) - 2 \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} f_{1i2j}(\mathbf{R}_r, \mathbf{t}), \quad (50)$$

where

$$f_{1i1j}(\mathbf{t}) = \frac{\phi_{1i}\phi_{1j}Z(K_{1i1j}(\mathbf{t}))}{Z(\kappa_{1i}(\mathbf{t}))Z(\kappa_{1j}(\mathbf{t}))}, \text{ and} \quad (51)$$

$$f_{1i2j}(\mathbf{t}) = \frac{\phi_{1i}\phi_{2j}Z(K_{1i2j}(\mathbf{R}_r, \mathbf{t}))}{Z(\kappa_{1i}(\mathbf{t}))Z(\kappa_{2j}(\mathbf{t}))}, \quad (52)$$

with respect to \mathbf{t} and \mathbf{r} , is given by

$$\frac{df}{d\mathbf{t}} = \sum_{i=1}^{n_1} \sum_{j=1}^{n_1} \frac{df_{1i1j}}{d\mathbf{t}} - 2 \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \frac{df_{1i2j}}{d\mathbf{t}}, \text{ and} \quad (53)$$

$$\frac{df}{d\mathbf{r}} = -2 \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \frac{df_{1i2j}}{d\mathbf{r}}, \quad (54)$$

where

$$\frac{df_{1i1j}}{d\mathbf{t}} = f_{1i1j} \left[\frac{Z'(K_{1i1j})}{Z(K_{1i1j})} \frac{dK_{1i1j}}{d\mathbf{t}} - \frac{Z'(\kappa_{1i})}{Z(\kappa_{1i})} \frac{d\kappa_{1i}}{d\mathbf{t}} - \frac{Z'(\kappa_{1j})}{Z(\kappa_{1j})} \frac{d\kappa_{1j}}{d\mathbf{t}} \right], \quad (55)$$

$$\frac{df_{1i2j}}{d\mathbf{t}} = f_{1i2j} \left[\frac{Z'(K_{1i2j})}{Z(K_{1i2j})} \frac{dK_{1i2j}}{d\mathbf{t}} - \frac{Z'(\kappa_{1i})}{Z(\kappa_{1i})} \frac{d\kappa_{1i}}{d\mathbf{t}} \right], \text{ and} \quad (56)$$

$$\frac{df_{1i2j}}{d\mathbf{r}} = f_{1i2j} \left[\frac{Z'(K_{1i2j})}{Z(K_{1i2j})} \frac{dK_{1i2j}}{d\mathbf{r}} \right], \quad (57)$$

where

$$Z'(x) = \frac{dZ}{dx} = \frac{e^x + e^{-x}}{x} - \frac{Z(x)}{x}, \quad (58)$$

$$\begin{aligned} \frac{dK_{1i1j}}{d\mathbf{t}} = \frac{\mathbf{k}_{1i1j}^\top}{K_{1i1j}} & \left[\frac{1}{\|\boldsymbol{\mu}_{1i} - \mathbf{t}\|} \left((2 - \kappa_{1i}) \left(\frac{\boldsymbol{\mu}_{1i} - \mathbf{t}}{\|\boldsymbol{\mu}_{1i} - \mathbf{t}\|} \right) \left(\frac{\boldsymbol{\mu}_{1i} - \mathbf{t}}{\|\boldsymbol{\mu}_{1i} - \mathbf{t}\|} \right)^\top - \kappa_{1i} \mathbf{I} \right) \right. \\ & \left. + \frac{1}{\|\boldsymbol{\mu}_{1j} - \mathbf{t}\|} \left((2 - \kappa_{1j}) \left(\frac{\boldsymbol{\mu}_{1j} - \mathbf{t}}{\|\boldsymbol{\mu}_{1j} - \mathbf{t}\|} \right) \left(\frac{\boldsymbol{\mu}_{1j} - \mathbf{t}}{\|\boldsymbol{\mu}_{1j} - \mathbf{t}\|} \right)^\top - \kappa_{1j} \mathbf{I} \right) \right], \end{aligned} \quad (59)$$

$$\frac{dK_{1i2j}}{d\mathbf{t}} = \frac{\mathbf{k}_{1i2j}^\top \mathbf{R}}{K_{1i2j}} \left[\frac{1}{\|\boldsymbol{\mu}_{1i} - \mathbf{t}\|} \left((2 - \kappa_{1i}) \left(\frac{\boldsymbol{\mu}_{1i} - \mathbf{t}}{\|\boldsymbol{\mu}_{1i} - \mathbf{t}\|} \right) \left(\frac{\boldsymbol{\mu}_{1i} - \mathbf{t}}{\|\boldsymbol{\mu}_{1i} - \mathbf{t}\|} \right)^\top - \kappa_{1i} \mathbf{I} \right) \right], \quad (60)$$

$$\frac{dK_{1i2j}}{d\mathbf{r}} = \frac{-\kappa_{1i} \mathbf{k}_{1i2j}^\top \mathbf{R}}{K_{1i2j}} \left[\frac{\boldsymbol{\mu}_{1i} - \mathbf{t}}{\|\boldsymbol{\mu}_{1i} - \mathbf{t}\|} \right]_\times \frac{\mathbf{r} \mathbf{r}^\top + (\mathbf{R}_\mathbf{r}^\top - \mathbf{I})[\mathbf{r}]_\times}{\|\mathbf{r}\|^2}, \text{ and} \quad (61)$$

$$\frac{d\kappa_1}{d\mathbf{t}} = -2 \frac{\kappa_1 - 1}{\|\boldsymbol{\mu}_1 - \mathbf{t}\|^2} (\boldsymbol{\mu}_1 - \mathbf{t})^\top, \quad (62)$$

using the notation

$$K_{1i1j} = \|\mathbf{k}_{1i1j}\|. \quad (63)$$

Proof. The gradient can be derived using the chain rule and Result 1 from [1], applied to obtain (61). \square

E. Additional Results

All qualitative results for the experiment in Section 7.2 are presented in the “qualitative_results” folder, in the format “results_real_2d_<pose_id>_<method>.png”. For clarity, only movable object points (as defined in the dataset) are projected onto the image (in red), using the camera pose found by the different algorithms. This allows the viewer to immediately distinguish whether the 3D projection aligns with the 2D image.

There are many semantic segmentation errors in the dataset, in both 2D and 3D, ranging from incorrect segmentations to mislabelings. This adds significantly to the complexity of the pose estimation task and makes outlier robustness very desirable in any algorithm used to solve the task. Some of the incorrect 3D labels are made apparent in these qualitative results. For example, two chairs are mislabeled in 3D in “results_real_2d_24_gt.png” and are therefore not projected onto the image.

References

- [1] G. Gallego and A. Yezzi. A compact formula for the derivative of a 3-D rotation in exponential coordinates. *Journal of Mathematical Imaging and Vision*, 51(3):378–384, 2015. 7