1. Numerical solution for ODEs

In this section, we briefly revisit the numerical methods of ODEs for those who are not familiar with it, including Leapfrog method and Runge-Kutta method that we used to develop the CNN blocks.

1.1. Leapfrog method

First, we reformulate the initial value problem of the following first-order differential equation

\[ \frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0. \]  

Consider the second-degree Taylor polynomial approximation at \( x_n \)

\[ y(x_{n+1}) = y(x_n) + hy'(x_n) + \frac{h^2}{2!} y''(x_n) + \frac{1}{3!} y'''(\xi_1), \]

\[ y(x_{n-1}) = y(x_n) - hy'(x_n) + \frac{h^2}{2!} y''(x_n) - \frac{1}{3!} y'''(\xi_2), \]

where \( h \) is the step size, \( x_n < \xi_1 < x_{n+1}, x_{n-1} < \xi_2 < x_n \). Subtract (3) from (2), drop the higher order term and make use of (1), we will obtain the Leapfrog scheme

\[ y_{n+1} = y_{n-1} + 2hf(x_n, y_n). \]

The local truncation error is \( O(h^3) \), while the total error will be accumulated to \( O(h^2) \). Therefore, Leapfrog is a second-order method.

1.2. Runge-Kutta method

2-stage Runge-Kutta: We use the trapezoid formula to approximate \( y_{n+1} \)

\[ y_{n+1} = y_n + \frac{h}{2} [f(x_n, y_n) + f(x_{n+1}, y_{n+1})]. \]  

However, we have no knowledge of the exact value of \( y_{n+1} \), thus we turn to the first-order approximation

\[ y_{n+1} \approx y_n + hf(x_n, y_n). \]  

These formulas can be rewritten into a 2-stage Runge-Kutta scheme

\[ y_{n+1} = y_n + \frac{1}{2}(G_1 + G_2), \]

\[ G_1 = hf(x_n, y_n), \]

\[ G_2 = hf(x_n + h, y_n + G_1). \]

The local truncation error is \( O(h^3) \), and it is a second-order scheme which is also known as Heun’s method.

Runge-Kutta family: Arbitrary \( s \) (a positive integer) stage Runge-Kutta method takes the form

\[ y_{n+1} = y_n + \sum_{i=1}^{s} \gamma_i G_i, \]

\[ G_1 = hf(x_n, y_n), \]

\[ G_i = hf(x_n + \alpha_i h, y_n + \sum_{j=1}^{i-1} \beta_{ij} G_j) \]

(we recommend [1] for further readings). As shown above, Runge-Kutta schemes make use of multiple steps to improve approximation accuracy. The coefficient \( \alpha, \beta, \gamma \) can be determined by Taylor series. Here, we take \( s = 3 \) as an example, which is the same case we used to develop RK3-block. First we reformulate the 3-stage Runge-Kutta scheme

\[ y_{n+1} = y_n + \gamma_1 G_1 + \gamma_2 G_2 + \gamma_3 G_3, \]

\[ G_1 = hf(x_n, y_n), \]

\[ G_2 = hf(x_n + \alpha_2 h, y_n + \beta_{21} G_1), \]

\[ G_3 = hf(x_n + \alpha_3 h, y_n + \beta_{31} G_1 + \beta_{32} G_2). \]
For the sake of clarity, we omit the variable in the derivatives. Consider the Taylor expansion of \( y(x) \) for \( G_2 \) at \((x_n, y_n)\), drop the higher-order terms and substitute \( G_1 \) with formula (12), we have

\[
G_2 = h[f + (\alpha_2 hf_x + \beta_{21} hf_y)] + \frac{1}{2}(\alpha_2^2 f_{xx} + \beta_{21}^2 f_{yy} + 2\alpha_2 \beta_{21} f_{xy} f_x) h^2
\]

\[= fh + (\alpha_2 f_x + \beta_{21} f_y)h^2 + \frac{1}{2}(\alpha_2^2 f_{xx} + \beta_{21}^2 f_{yy} + 2\alpha_2 \beta_{21} f_{xy} f_x) h^3.\]

Similarly, we obtain

\[
G_3 = fh + (\alpha_3 f_x + \beta_{21} f_y + \beta_{32} f_{xy} f_x) h^2 + \{ \beta_{32} f_{fy} (\alpha_2 f_x + \beta_{21} f_y) f_x + \frac{1}{2}(\alpha_3^2 f_{xx} + ((\beta_{31})^2 f_x + 2\beta_{31} \beta_{32} f_x^2) + 2\beta_3 (\beta_{31} f_{xy} + \beta_{32} f_{xy} f_x)\}) h^3.\]

Note that we have dropped the \( O(h^4) \) term, then consider the derivatives of \( y \) at \( x_n \)

\[
\frac{dy(x_n)}{dx_n} = f(x_n, y_n) = f,
\]

\[
\frac{d^2 y(x_n)}{dx_n^2} = \frac{df(x_n, y_n)}{dx_n} = f_x + f_y f,
\]

\[
\frac{d^3 y(x_n)}{dx_n^3} = f_{xx} + 2f_{xy} f + f_y f_x + f_y^2 f. \tag{19}
\]

Use the Taylor expansion

\[
y_{n+1} = y_n + y'(x_n)h + \frac{h^2}{2} y''(x_n) + \frac{h^3}{6} y'''(x_n) + O(h^4), \tag{20}
\]

and compare the coefficients of \( (11) \) and \( (20) \), we have the following equations

\[
\gamma_1 + \gamma_2 + \gamma_3 = 1, \tag{21}
\]

\[
\gamma_2 \beta_{21} + \gamma_3 (\beta_{31} + \beta_{32}) = \frac{1}{2}, \]

\[
\gamma_2 \alpha_2 + \gamma_3 \alpha_3 = \frac{1}{2}, \]

\[
\gamma_2 \alpha_2^2 + \gamma_3 \alpha_3^2 = \frac{1}{2}, \]

\[
\gamma_2 \beta_{21}^2 + \gamma_3 (\beta_{31}^2 + \beta_{32}^2 + 2\beta_{31} \beta_{32}) = \frac{1}{2}, \]

\[
\alpha_2 \beta_{32} \gamma_3 = \frac{1}{6}, \]

\[
\beta_{21} \beta_{32} \gamma_3 = \frac{1}{6}, \]

\[
\alpha_2 \beta_{21} \gamma_2 + \alpha_3 (\beta_{31} + \beta_{32}) \gamma_3 = \frac{1}{3}, \]

The solution is not unique. In our designs, we take

\[
\gamma_1 = \frac{1}{6}, \gamma_2 = \frac{2}{3}, \gamma_3 = \frac{1}{6}, \tag{22}
\]

\[
\beta_{21} = \frac{1}{2}, \beta_{31} = -1, \beta_{32} = 2, \]

\[
\alpha_2 = \frac{1}{2}, \alpha_3 = 1,
\]

which is also known as Kutta’s method. Since we have dropped the term with order higher than \( O(h^4) \), the local truncation error is \( O(h^4) \), i.e., a third-order method.

### 2. Dynamical system

In this section, we briefly revisit the concept of dynamical system used in this paper. Here we adopt the definition in [2]. Generally speaking, a dynamical system is a monoid \( G \) acting on a set \( M \). More precisely, there is a map

\[
\Phi : G \times M \to M \tag{23}
\]

\[
(g, x) \mapsto \Phi_g(x) \tag{24}
\]

for \( \forall g, t \in G \) and the identity element \( e \in G \), satisfies

\[
\Phi_g \circ \Phi_t = \Phi_{gt}, \quad \Phi_e = I. \tag{25}
\]

In the semantics of SISR, \( \Phi_t(x) \) can be granted as a map from the input sample \( x \) to the output high-resolution image through time \( t \), since we regard \( t \) as the element of \( G \). When \( t = 0, \Phi_0(x) = x \) is an identity mapping, then we would like to approach \( \Phi_t(x) \) through an ODE (e.g. formula (1)) and interpret it as a CNN. In order to achieve this process, we design finer CNN blocks and choose appropriate \( t \), which is corresponded to block numbers (detailed in Table 1).

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References
