Radial Distortion Triangulation: Supplementary Material

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1. Additional experiments with real images

In this section we present additional results on two real datasets; the *Rotunda* dataset from [5] and the *Graffiti* dataset from [6]. The datasets contain images with varying levels of radial distortion. Some example images are shown in Figure 3 and Figure 4. Since there is no ground-truth reconstruction available for these datasets, we build a 3D model using the Structure-from-Motion software from [1]. While this does not provide a true ground truth, we can still expect to get reasonably accurate results since the camera poses and 3D points are refined using observations from multiple images. The statistics for the datasets are shown in Table 1 and the results are shown in Tables 2–5.

Dataset	Image pairs	Points tri.
Rotunda [5]	1891	1240085
Graffiti [6]	171	187614

Table 1. Number of image pairs and total number of triangulations for the two real data experiments.

2. Polynomial distortion models

In the main paper, we use the one parameter division model [4] to model the radial distortion. Another popular choice the is polynomial model from Brown [2] and Conrady [3]. In this section, we show that it is simple to derive versions of our iterative scheme for the polynomial distortion models as well. We focus on the case of two parameters[1], *i.e.* when the undistortion function $u(\mathbf{x})$ is given by

$$u(\mathbf{x}) = (1 + k_1\|\mathbf{x}\|^2 + k_2\|\mathbf{x}\|^4)\mathbf{x}. \quad (1)$$

The iterative scheme will be essentially the same. Since we use a different function u, we will get a different matrix $\mathtt{D}$ and different vectors $\mathbf{n}_1$ and $\mathbf{n}_2$. For this u we have $\mathtt{D}_\mathbf{x} =$

$$\left[(1 + k_1\|\mathbf{x}\|^2 + k_2\|\mathbf{x}\|^4)\mathtt{I}_2 + (2k_1 + 4k_2\|\mathbf{x}\|^2)\mathbf{x}\mathbf{x}^T \quad \mathbf{0}\right] \quad (2)$$

[1]For polynomial models it is typically necessary to use at least two parameters to achieve accurate undistortions.

Additionally, the epipolar constraint,

$$u\left(\mathbf{x}_{d_1} - \mathtt{S}^\top(\lambda^k \mathbf{n}_1^k)\right)^\top \mathtt{F}\, u\left(\mathbf{x}_{d_2} - \mathtt{S}^\top(\lambda^k \mathbf{n}_2^k)\right) = 0, \quad (3)$$

now becomes a 10th degree polynomial in the multiplier λ. Again we are interested in the root with the smallest magnitude and we can start Newton iterations at $\lambda = 0$, instead of solving for all roots. In Sections 2.1 and 2.2 we show some experiments with this iterative method.

Note that here we have only considered the case with two parameters for the distortion. However, it is simple to extend this to other variants of the polynomial model. Similarly, it would be simple to derive iterative schemes for higher-order versions of the division model.

2.1. Synthetic experiments for two-parameter polynomial distortion model

We evaluated the iterative solver for the two-parameter polynomial distortion model on synthetic scenes that were generated in a similar way as the scenes from synthetic experiments in the main paper. In these experiments, we set the radial distortion parameters of the two-parameter polynomial model to $k_1 = 0.2683$ and $k_2 = 0.1217$. These parameters approximately correspond to the parameters of the GoPro Hero4 camera with the wide field-of-view setting.

Figure 1 shows the result of our iterative ITD solver and the state-of-the-art IT solver [7] for different image noise contamination. In this case, we added 2% error to both distortion parameters k_1 and k_2. This simulates a calibration error that can be present in real applications. Figure 1 shows the comparison of the 3D error, the reprojection error and the ratio of 3D errors of the IT [7] and the new ITD solver on 1000 different scenes using box plots. For ratios of 3D errors, we also show the results for the 20% of points which have undergone the most distortion (*i.e.* points closest to the borders), to highlight the benefit of performing the triangulation in distorted space.

A similar comparison for radial distortion noise is in Figure 2. In this case we added 1 px noise to image points. It can be seen that in general the proposed method provides more accurate 3D point triangulations compared to

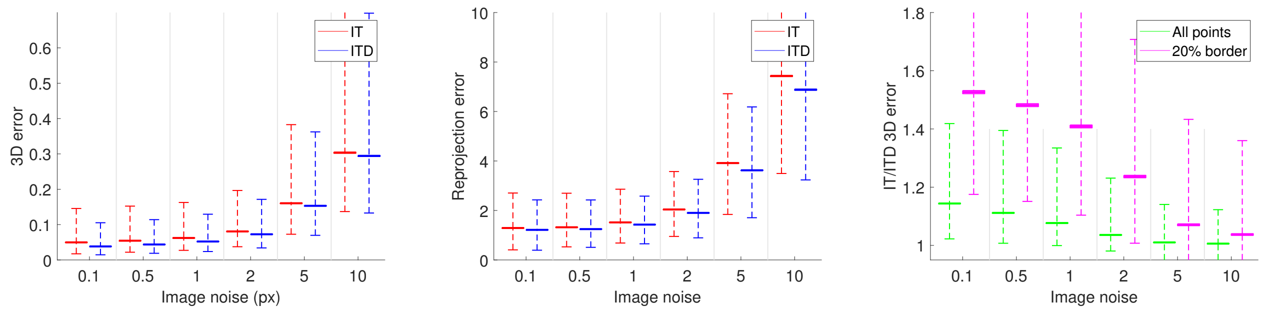

Figure 1. Comparison of the new ITD and the IT [7] solvers for varying image noise and the two-parameter polynomial distortion model with $k_1 = 0.2683$ an $k_2 = 0.1217$, 2% radial distortion error, 3000 px $\times$ 3000 px image size and $f = 1300$ px. These camera parameters approximately correspond to the GoPro Wide setting.

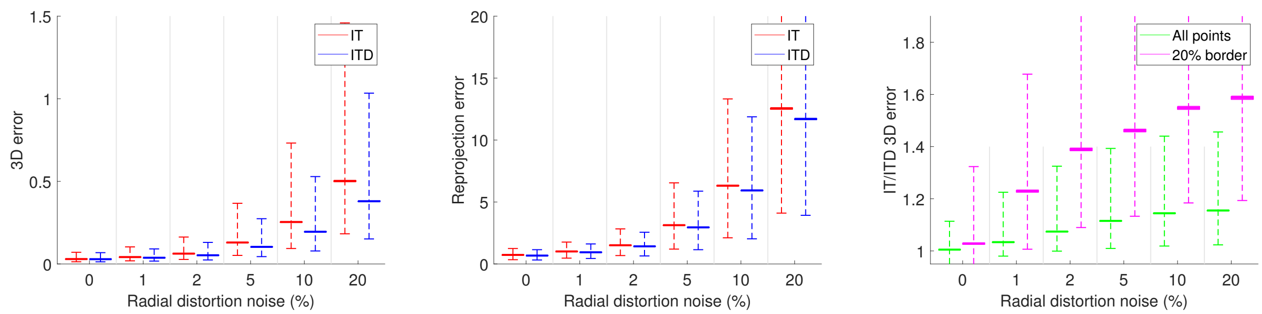

Figure 2. Comparison of the new ITD and the IT [7] solvers for varying radial distortion noise and the two-parameter polynomial distortion model with $k_1 = 0.2683$ an $k_2 = 0.1217$, 1px image noise, 3000 px $\times$ 3000 px image size and $f = 1300$ px. These camera parameters approximately correspond to the GoPro Wide setting.

the IT solver [7] which minimizes ℓ_2 reprojection error in the undistorted image space. The improvement is even larger when we consider points closer to the image border which are more affected by the distortion.

2.2. Real experiment with polynomial model

In this section, we evaluate the iterative solver for the two-parameter polynomial distortion model. We use the checkerboard dataset which was used to evaluate the division model solver in the main paper. Using the ground truth poses we refit a two-parameter polynomial model. In Tables 6–9 we show the results. Again we can see that performing triangulation in the original distorted image space yields improved results. Note that the results for the IT solver from [7] are slightly different than the results of this solver on the same dataset from the main paper since a different model is used for undistorting the image points. For the ITD solver with the polynomial model there were a few failure cases which can be seen from the fact that the reprojection error is not strictly smaller compared to doing triangulation in the undistorted image. For these cases the reprojections were very close ($\approx 10^{-5}$ px) and we believe this to be the result of numerical instabilities.

3. Additional results for experiments

In Tables 10–15 we show more detailed results from the synthetic experiments in the main paper.

References

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[3] Alexander Eugen Conrady. Decentred lens-systems. *Monthly notices of the royal astronomical society*, 79(5):384–390, 1919. 1

[4] Andrew W Fitzgibbon. Simultaneous linear estimation of multiple view geometry and lens distortion. In *Computer Vision and Pattern Recognition (CVPR)*, volume 1, pages I–I. IEEE, 2001. 1

[5] Zuzana Kukelova, Jan Heller, Martin Bujnak, Andrew Fitzgibbon, and Tomas Pajdla. Efficient solution to the epipolar geometry for radially distorted cameras. In *International Conference on Computer Vision (ICCV)*, pages 2309–2317, 2015. 1, 3

[6] Zuzana Kukelova, Jan Heller, Martin Bujnak, and Tomas Pajdla. Radial distortion homography. In *Computer Vision and Pattern Recognition (CVPR)*, pages 639–647, 2015. 1, 4

[7] Peter Lindstrom. Triangulation made easy. In *Computer Vision and Pattern Recognition (CVPR)*, 2010. 1, 2, 6, 7

Figure 3. Example images from the *Rotunda* [5] dataset.

	3D error					Reprojection error [px]				
	mean		median			mean		median		
	IT	ITD	IT	ITD	ITD<IT	IT	ITD	IT	ITD	ITD<IT
0%	0.2000	**0.1852**	0.0104	**0.0102**	54.7%	0.8877	**0.8589**	0.4859	**0.4815**	100%
1%	0.2419	**0.2089**	0.0109	**0.0107**	56.4%	0.9472	**0.9131**	0.5115	**0.5070**	100%
5%	2.2360	**0.4675**	0.0164	**0.0158**	66.6%	1.6680	**1.5829**	0.7052	**0.7001**	100%
10%	**2.2668**	2.2978	0.0241	**0.0230**	72.8%	2.6838	**2.5324**	0.9632	**0.9574**	100%
20%	8.7217	**4.7901**	0.0459	**0.0434**	78.8%	5.7230	**5.3352**	1.6573	**1.6465**	100%

Table 2. *Rotunda* dataset from [5]

	3D error					Reprojection error [px]				
	mean		median			mean		median		
	IT	ITD	IT	ITD	ITD<IT	IT	ITD	IT	ITD	ITD<IT
0%	0.8227	**0.6721**	0.2835	**0.2270**	68.9%	2.4438	**2.1404**	1.4517	**1.3249**	100%
1%	1.5587	**0.9726**	0.3607	**0.2768**	73.3%	2.9890	**2.6025**	1.8931	**1.7325**	100%
5%	3.9813	**2.6891**	0.9641	**0.6758**	85.8%	8.1192	**7.0392**	4.9164	**4.4759**	100%
10%	23.8508	**13.8697**	1.9340	**1.2928**	89.6%	14.1973	**12.3117**	9.5745	**8.6740**	100%
20%	93.9628	**34.9914**	3.9623	**2.7905**	82.9%	34.9048	**29.6772**	21.8035	**19.7227**	100%

Table 3. *Rotunda* dataset from [5]. The 5% of points closest to the border.

Figure 4. Example images from the *Graffiti* [6] dataset.

	3D error					Reprojection error [px]				
	mean		median			mean		median		
	IT	ITD	IT	ITD	ITD<IT	IT	ITD	IT	ITD	ITD<IT
0%	0.3891	**0.3201**	0.0146	**0.0144**	55.9%	0.5833	**0.5754**	0.4352	**0.4313**	100%
1%	0.3669	**0.3668**	0.0156	**0.0154**	57.9%	0.6029	**0.5945**	0.4471	**0.4431**	100%
5%	0.5369	**0.4656**	0.0241	**0.0236**	62.0%	0.8222	**0.8075**	0.5580	**0.5538**	100%
10%	0.4477	**0.4435**	0.0358	**0.0346**	66.6%	1.2873	**1.2613**	0.7497	**0.7450**	100%
20%	1.9290	**1.3707**	0.0606	**0.0583**	71.2%	2.4232	**2.3655**	1.0795	**1.0730**	100%

Table 4. *Graffitti* dataset from [6].

	3D error					Reprojection error [px]				
	mean		median			mean		median		
	IT	ITD	IT	ITD	ITD<IT	IT	ITD	IT	ITD	ITD<IT
0%	5.0896	**3.7441**	0.0659	**0.0644**	59.0%	0.9533	**0.9267**	0.6911	**0.6798**	100%
1%	1.3329	**1.3274**	0.0711	**0.0688**	60.5%	1.0112	**0.9817**	0.7257	**0.7143**	100%
5%	1.6776	**1.6526**	0.1031	**0.0986**	58.7%	1.3353	**1.2875**	0.8518	**0.8379**	100%
10%	1.6355	**1.6173**	0.1573	**0.1502**	61.0%	2.1762	**2.0978**	1.3441	**1.3248**	100%
20%	20.9624	**9.3209**	0.2809	**0.2690**	63.6%	4.7179	**4.4648**	1.6259	**1.6033**	100%

Table 5. *Graffitti* dataset from [6]. The 5% of points closest to the border.

	3D error [mm]					Reprojection error [px]				
	mean		median			mean		median		
	IT	ITD	IT	ITD	ITD<IT	IT	ITD	IT	ITD	ITD<IT
0%	0.0794	**0.0793**	0.0638	**0.0637**	51.3%	0.1240	**0.1234**	0.0919	**0.0915**	99.6%
1%	0.0957	**0.0938**	0.0706	**0.0701**	58.7%	0.1627	**0.1618**	0.1196	**0.1194**	99.7%
5%	0.2491	**0.2355**	0.1390	**0.1344**	77.2%	0.4898	**0.4865**	0.2763	**0.2758**	99.6%
10%	0.4649	**0.4357**	0.2547	**0.2434**	82.9%	0.9367	**0.9299**	0.5116	**0.5102**	99.7%
20%	0.9431	**0.8808**	0.4840	**0.4621**	86.3%	1.8937	**1.8793**	1.0127	**1.0086**	99.6%

Table 6. Checkerboard: GoPro-Medium, (Two-parameter polynomial model).

	3D error [mm]					Reprojection error [px]				
	mean		median			mean		median		
	IT	ITD	IT	ITD	ITD<IT	IT	ITD	IT	ITD	ITD<IT
0%	0.1635	**0.1538**	0.1176	**0.1142**	62.5%	0.3300	**0.3197**	0.2484	**0.2441**	99.9%
1%	0.1950	**0.1794**	0.1311	**0.1263**	65.6%	0.3894	**0.3771**	0.2815	**0.2762**	99.9%
5%	0.5198	**0.4510**	0.3028	**0.2748**	81.4%	1.0836	**1.0471**	0.6630	**0.6526**	99.9%
10%	1.0322	**0.8893**	0.5798	**0.5112**	87.0%	2.1088	**2.0375**	1.3264	**1.3028**	99.9%
20%	2.0327	**1.7400**	1.0899	**0.9685**	88.6%	4.1238	**3.9775**	2.5065	**2.4560**	99.9%

Table 7. Checkerboard: GoPro-Wide, (Two-parameter polynomial model).

	3D error [mm]					Reprojection error [px]				
	mean		median			mean		median		
	IT	ITD	IT	ITD	ITD<IT	IT	ITD	IT	ITD	ITD<IT
0%	0.1413	**0.1369**	0.1035	**0.0993**	62.5%	0.2089	**0.2055**	0.1450	**0.1439**	99.8%
1%	0.2025	**0.1864**	0.1397	**0.1329**	68.3%	0.3545	**0.3495**	0.2866	**0.2801**	99.8%
5%	0.7463	**0.6665**	0.4645	**0.4318**	86.4%	1.3419	**1.3233**	1.0306	**1.0178**	99.8%
10%	1.4370	**1.2668**	0.8714	**0.8046**	89.1%	2.7602	**2.7213**	2.3942	**2.3726**	99.8%
20%	2.9706	**2.5989**	1.7437	**1.6139**	85.5%	5.6592	**5.5746**	4.8192	**4.7538**	100.0%

Table 8. Checkerboard: GoPro-Medium, Top 5% (Two-parameter polynomial model).

	3D error [mm]					Reprojection error [px]				
	mean		median			mean		median		
	IT	ITD	IT	ITD	ITD<IT	IT	ITD	IT	ITD	ITD<IT
0%	0.3913	**0.3303**	0.2147	**0.2004**	69.1%	0.6023	**0.5609**	0.3558	**0.3396**	99.5%
1%	0.5032	**0.4114**	0.2718	**0.2420**	73.7%	0.7792	**0.7286**	0.5622	**0.5354**	99.5%
5%	1.4031	**1.0736**	0.8370	**0.6851**	86.3%	2.5640	**2.4128**	2.1372	**2.0219**	99.3%
10%	2.5535	**1.9666**	1.5732	**1.2527**	86.5%	4.7351	**4.4590**	3.7990	**3.5839**	99.5%
20%	5.3668	**4.1389**	3.0896	**2.4492**	87.5%	9.5623	**8.9792**	7.7458	**7.2960**	99.3%

Table 9. Checkerboard: GoPro-Wide, Top 5% (Two-parameter polynomial model).

	3D error						Reprojection error [px]			
	mean		median		mean		mean		median	
	IT	ITD	IT	ITD	IT/ITD	ITD<IT	IT	ITD	IT	ITD
-0.01	**0.0472**	**0.0472**	0.0213	**0.0212**	**1.0005**	**51.3%**	0.8182	**0.8180**	0.6915	**0.6914**
-0.1	0.1481	**0.1354**	0.0610	**0.0588**	**1.0343**	**68.0%**	1.9626	**1.9151**	1.5109	**1.4961**
-0.2	0.3556	**0.3273**	0.1447	**0.1327**	**1.0960**	**73.1%**	3.9179	**3.5424**	2.7945	**2.6811**
-0.3	0.7299	**0.5834**	0.2631	**0.2336**	**1.1625**	**74.7%**	5.3689	**4.3489**	3.5317	**3.2069**
-0.4	2.0388	**1.6116**	0.3665	**0.3111**	**1.2814**	**78.3%**	7.4106	**5.5524**	3.6027	**3.0603**

Table 10. Comparison of the new ITD and the IT [7] solvers for varying radial distortions, 5% radial distortion error, 1 px image noise w.r.t. 3000 px $\times$ 3000 px image size, and $f = 1300$ px. The radial distortion $k = -0.3$ approximately corresponds to GoPro Wide setting.

	3D error						Reprojection error [px]			
	mean		median		mean		mean		median	
	IT	ITD	IT	ITD	IT/ITD	ITD<IT	IT	ITD	IT	ITD
-0.01	**0.0490**	**0.0490**	**0.0222**	**0.0222**	**1.0012**	**51.8%**	0.8446	**0.8444**	0.7146	**0.7143**
-0.1	0.2370	**0.2307**	0.1148	**0.1102**	**1.0462**	**62.3%**	2.8595	**2.7512**	2.3895	**2.3324**
-0.2	0.7670	**0.6779**	0.3191	**0.2916**	**1.1329**	**63.9%**	5.9807	**5.1350**	4.7738	**4.3346**
-0.3	1.7915	**1.3815**	0.6875	**0.5940**	**1.2913**	**67.1%**	8.7212	**6.2288**	6.5449	**5.1539**
-0.4	7.8512	**6.1978**	1.2046	**0.9310**	**1.6411**	**74.8%**	17.9298	**12.4285**	6.8226	**4.4922**

Table 11. Comparison of the new ITD and the IT [7] solvers for varying radial distortions - top 20% points closest to the border, 5% radial distortion error, 1 px image noise w.r.t. 3000 px $\times$ 3000 px image size, and $f = 1300$ px. The radial distortion $k = -0.3$ approximately corresponds to GoPro Wide setting.

	3D error						Reprojection error [px]			
	mean		median		mean		mean		median	
	IT	ITD	IT	ITD	IT/ITD	ITD<IT	IT	ITD	IT	ITD
0.01px	0.2773	**0.2276**	0.1032	**0.0917**	**1.1582**	**77.6%**	2.1038	**1.7211**	1.3809	**1.2607**
0.5px	0.2599	**0.2276**	0.1071	**0.0953**	**1.1587**	**73.6%**	2.2299	**1.8269**	1.4979	**1.3708**
1px	0.2800	**0.2439**	0.1122	**0.1007**	**1.1567**	**70.6%**	2.3732	**1.9597**	1.6694	**1.5280**
2px	0.3606	**0.2995**	0.1289	**0.1179**	**1.1497**	**65.1%**	2.8308	**2.3980**	2.1507	**1.9587**
5px	**0.5085**	0.6077	0.1934	**0.1823**	**1.1715**	**58.5%**	5.0522	**4.3926**	4.1236	**3.7015**
10px	1.3136	**0.9649**	0.3375	**0.3224**	**1.1798**	**56.0%**	9.3500	**8.1940**	7.7473	**6.9226**

Table 12. Comparison of the new ITD and the IT [7] solvers for varying image noise, $k = -0.3$, 2% radial distortion error, 3000 px $\times$ 3000 px image size and $f = 1300$ px. These camera parameters approximately correspond to the GoPro Wide setting.

	3D error						Reprojection error [px]			
	mean		median		mean		mean		median	
	IT	ITD	IT	ITD	IT/ITD	ITD<IT	IT	ITD	IT	ITD
0.01px	0.6417	**0.5495**	0.2716	**0.2368**	**1.2720**	**67.0%**	3.4421	**2.5156**	2.6145	**2.0887**
0.5px	0.6337	**0.5416**	0.2772	**0.2405**	**1.2787**	**66.7%**	3.5775	**2.6164**	2.7722	**2.2212**
1px	0.6799	**0.5735**	0.2847	**0.2464**	**1.2852**	**66.8%**	3.7121	**2.6932**	2.8425	**2.2664**
2px	0.9525	**0.7093**	0.3065	**0.2667**	**1.2955**	**65.6%**	3.9915	**2.9605**	3.0979	**2.4727**
5px	**1.1111**	1.7031	0.4127	**0.3589**	**1.4370**	**62.1%**	6.2502	**4.7153**	5.0723	**3.9801**
10px	3.2289	**2.1514**	0.6879	**0.6063**	**1.4955**	**59.7%**	11.0007	**8.3626**	9.0402	**7.0616**

Table 13. Comparison of the new ITD and the IT [7] solvers for varying image noise - top 20% points closest to the border, $k = -0.3$, 2% radial distortion error, 3000 px $\times$ 3000 px image size and $f = 1300$ px. These camera parameters approximately correspond to the GoPro Wide setting.

	3D error						Reprojection error [px]			
	mean		median		mean		mean		median	
	IT	ITD	IT	ITD	IT/ITD	ITD<IT	IT	ITD	IT	ITD
0%	**0.0821**	0.0876	0.0317	**0.0304**	**1.1779**	**54.9%**	0.8993	**0.7977**	0.7535	**0.6754**
1%	0.1565	**0.1403**	0.0634	**0.0580**	**1.1498**	**65.1%**	1.4139	**1.2008**	1.0771	**0.9832**
2%	0.2683	**0.2323**	0.1113	**0.0998**	**1.1557**	**70.7%**	2.3785	**1.9717**	1.6799	**1.5382**
5%	0.7866	**0.6053**	0.2679	**0.2373**	**1.1659**	**75.1%**	5.5052	**4.4602**	3.6623	**3.3317**
10%	1.5505	**1.3983**	0.5341	**0.4731**	**1.1821**	**76.3%**	10.9005	**8.6137**	7.0419	**6.3209**
20%	7.8798	**2.9951**	1.0460	**0.9087**	**1.3255**	**77.5%**	22.6661	**17.0477**	14.4391	**12.7194**

Table 14. Comparison of the new ITD and the IT [7] solvers for varying radial distortion noise, $k = -0.3$, 1px image noise, 3000 px $\times$ 3000 px image size and $f = 1300$ px. These camera parameters approximately correspond to the GoPro Wide setting.

	3D error						Reprojection error [px]			
	mean		median		mean		mean		median	
	IT	ITD	IT	ITD	IT/ITD	ITD<IT	IT	ITD	IT	ITD
0%	0.1693	**0.1511**	0.0617	**0.0543**	**1.5018**	**58.6%**	1.0273	**0.7964**	0.8574	**0.6759**
1%	0.3643	**0.3149**	0.1515	**0.1315**	**1.2947**	**65.8%**	2.0063	**1.4902**	1.5610	**1.2455**
2%	0.6516	**0.5402**	0.2821	**0.2436**	**1.2778**	**67.0%**	3.6675	**2.6921**	2.8326	**2.2582**
5%	2.1692	**1.4458**	0.6958	**0.6007**	**1.2962**	**67.9%**	8.9578	**6.4262**	6.8521	**5.4150**
10%	4.2745	**3.6810**	1.3753	**1.1865**	**1.3659**	**67.0%**	18.0080	**12.4447**	13.3929	**10.2920**
20%	30.6281	**7.9721**	2.7296	**2.3087**	**1.9120**	**68.2%**	38.2066	**24.4436**	28.1865	**20.5657**

Table 15. Comparison of the new ITD and the IT [7] solvers for varying radial distortion noise - top 20% closest to the border,, $k = -0.3$, 1px image noise, 3000 px $\times$ 3000 px image size and $f = 1300$ px. These camera parameters approximately correspond to the GoPro Wide setting.