Averaging Essential and Fundamental Matrices in Collinear Camera Settings -Supplementary Material-

Amnon Geifman* Yoni Kasten* Meirav Galun Ronen Basri
Weizmann Institute of Science
{amnon.geifman,yoni.kasten,meirav.galun,ronen.basri}@weizmann.ac.il

1. Supporting lemmas

Below are a few supporting lemmas which Thms. 1 and 2 in the paper rely on.

Lemma 1. *Let $E \in \mathbb{S}^{3n}$ of rank 4, and $\Sigma \in \mathbb{R}^{2\times 2}$, a diagonal matrix, with positive elements on the diagonal. Let $X, Y, U, V \in \mathbb{R}^{3n\times 2}$, and we define the mapping $(X,Y) \leftrightarrow (U,V) : X = \sqrt{0.5}(\hat{U} + \hat{V})$, $Y = \sqrt{0.5}(\hat{V} - \hat{U})$, $\hat{U} = \sqrt{0.5}(X - Y), \hat{V} = \sqrt{0.5}(X + Y)$.*

Then, the (thin) SVD of E is of the form

$$E = \left[\hat{U}, \hat{V}\right] \begin{pmatrix} \Sigma & \\ & \Sigma \end{pmatrix} \begin{bmatrix} \hat{V}^T \\ \hat{U}^T \end{bmatrix}$$

if and only if the (thin) spectral decomposition of E is of the form

$$E = [X, Y] \begin{pmatrix} \Sigma & \\ & -\Sigma \end{pmatrix} \begin{bmatrix} X^T \\ Y^T \end{bmatrix}.$$

Proof. The proof is similar to Lemma 6 presented in the supplementary material of [13]. □

Lemma 2. *Let $F \in \mathbb{S}^{3n}$ be a matrix of rank 4. Then, the following three conditions are equivalent.*

(i) *F has exactly 2 positive and 2 negative eigenvalues.*

(ii) *$F = XX^T - YY^T$ with $X, Y \in \mathbb{R}^{3n\times 2}$ and rank$(X) =$ rank$(Y) = 2$.*

(iii) *$F = UV^T + VU^T$ with $U, V \in \mathbb{R}^{3n\times 2}$ and rank$(U) =$ rank$(V) = 2$.*

Proof. The proof is similar to the proof given in [14], Lemma 1. □

Lemma 3. *Let $A, B \in \mathbb{R}^{3\times 3}$ with rank$(A) = 2$, rank$(B) = 3$ and AB^T is skew symmetric, then $T = B^{-1}A$ is skew symmetric.*

Proof. See Lemma 4 in the supplementary material of [14]. □

Lemma 4. *Let $A, B \in \mathbb{R}^{2\times 2}$ s.t $AB^T \neq 0$ and skew-symmetric. Then it follows that $A = BT_y$ where T_y is 2×2 skew symmetric.*

Proof. First we note that $AB^T = T_x$, where

$$T_x = \begin{bmatrix} 0 & x \\ -x & 0 \end{bmatrix}$$

*Equal contributors

for some $0 \neq x \in \mathbb{R}$ and $\text{rank}(A) = \text{rank}(B) = 2$. Therefore,

$$A = T_x B^{-T} = BB^{-1}T_x B^{-T}$$

Denote by $T_y = B^{-1}T_x B^{-T}$. Clearly T_y is skew symmetric, since

$$T_y^T = (B^{-1}T_x B^{-T})^T = B^{-1}T_x^T B^{-T} = -B^{-1}T_x B^{-T}$$

where the rightmost equality is due to the skew symmetry of T_x.

$\square$

Lemma 5. *Let $A, B \in \mathbb{R}^{3\times 2}$ with $rank(A) = rank(B) = 2$ and AB^T is skew symmetric. Then $B = AT$ for some skew symmetric matrix $T \in \mathbb{R}^{2\times 2}$.*

Proof.

$$AB^T = \begin{bmatrix} A_u \\ \mathbf{a}^T \end{bmatrix} \begin{bmatrix} B_u^T & \mathbf{b} \end{bmatrix} = \begin{bmatrix} A_u B_u^T & A_u \mathbf{b} \\ \mathbf{a}^T B_u^T & \mathbf{a}^T \mathbf{b} \end{bmatrix}$$

Where $A_u, B_u \in \mathbb{R}^{2\times 2}$ and $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$. As a result

$$\mathbf{a}^T \mathbf{b} = 0, \; A_u \mathbf{b} = -B_u \mathbf{a}$$

Assuming first that $A_u B_u^T \neq 0$ we get (using Lemma 4) that

$$A_u = B_u T$$

with $T \in \mathbb{R}^{2\times 2}$ skew symmetric. Consequently,

$$AB^T = \begin{bmatrix} B_u T B_u^T & B_u T \mathbf{b} \\ -\mathbf{b}^T T^T B_u^T & 0 \end{bmatrix}$$

Since $\mathbf{b}^T T \mathbf{b} = 0$ we can write

$$AB^T = \begin{bmatrix} B_u T B_u^T & B_u T \mathbf{b} \\ \mathbf{b}^T T B_u^T & \mathbf{b}^T T \mathbf{b} \end{bmatrix} = \begin{bmatrix} B_u \\ \mathbf{b}^T \end{bmatrix} T \begin{bmatrix} B_u^T & \mathbf{b} \end{bmatrix} = BTB^T$$

Since $\text{rank}(B) = 2$, multiplying by $B(B^T B)^{-1}$ yields,

$$AB^T B(B^T B)^{-1} = BT_0 B^T B(B^T B)^{-1}$$

and consequently,

$$A = BT$$

A similar construction can be made in case $A_u B_u^T = 0$.

$\square$

Lemma 6. *Suppose that $T \in \mathbb{R}^{2\times 2}$ is a non-zero skew symmetric matrix, and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ are orthonormal. Then, (1) it holds that for any vector $\mathbf{x} \in \mathbb{R}^2$, $T\mathbf{x}$ is perpendicular to $\mathbf{x}$ and (2) if $\mathbf{x}$ is expressed in terms of the orthonormal basis as $\mathbf{x} = \alpha\mathbf{u} + \beta\mathbf{v}$ then $T\mathbf{x}$ is parallel to $\beta\mathbf{u} - \alpha\mathbf{v}$.*

Proof. Since $T \in \mathbb{R}^{2\times 2}$ is skew-symmetric, then for all $\mathbf{x} \in \mathbb{R}^2$ it holds that $\mathbf{x}^T T\mathbf{x} = 0$, implying that $T\mathbf{x}$ is perpendicular to $\mathbf{x}$. If $\mathbf{x} = \alpha\mathbf{u} + \beta\mathbf{v}$, where $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ are orthonormal then clearly $(\beta\mathbf{u} - \alpha\mathbf{v})^T\mathbf{x} = 0$. Due to the uniqueness of orthogonality in $\mathbb{R}^2$ (up to scale), we obtain that $T\mathbf{x}$ is parallel to $\beta\mathbf{u} - \alpha\mathbf{v}$. $\square$

Lemma 7. *Let $F \in \mathbb{S}^{3n}$ such that (1) $F = UV^T + VU^T$ with $U, V \in \mathbb{R}^{3n\times 2}$ and (2) $\forall i \in [n]$, $F_{ii} = U_i V_i^T + V_i U_i^T = 0$. Suppose that $\exists i$ such that $rank(U_i) = rank(V_i) = 1$ and $U_i V_i^T = 0$, and $\exists j \neq i$ $rank(U_j) = rank(V_j) = 2$, then $rank(F_{ij}) = U_i V_j^T + V_i U_j^T \leq 1$.*

Proof. W.l.o.g. assume $\text{rank}(U_1) = \text{rank}(V_1) = 1$ and $\text{rank}(U_2) = \text{rank}(V_2) = 2$. Since $V_2 U_2^T$ is skew-symmetric and $\text{rank}(V_2) = \text{rank}(U_2) = 2$, it follows from Lemma 5, that U_2 can be expressed as

$$U_2 = V_2 T \tag{1}$$

where $T \in \mathbb{R}^{2\times 2}$ is a skew-symmetric matrix.

Since $\text{rank}(U_i) = \text{rank}(V_i) = 1$, they can be expressed as

$$U_1 = \begin{bmatrix} \alpha_0 \mathbf{u}^T \\ \alpha_1 \mathbf{u}^T \\ \alpha_2 \mathbf{u}^T \end{bmatrix}, \quad V_1 = \begin{bmatrix} \beta_0 \mathbf{v}^T \\ \beta_1 \mathbf{v}^T \\ \beta_2 \mathbf{v}^T \end{bmatrix} \tag{2}$$

where $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ with $\|\mathbf{u}\| = \|\mathbf{v}\| = 1$, $\{\alpha_i\}$ and $\{\beta_i\}$ are scalars. Moreover, since we assume that $U_1 V_1^T = 0$, it follows that $\mathbf{u}^T \mathbf{v} = 0$. Therefore, $\{\mathbf{u}, \mathbf{v}\}$ form an orthonormal basis in $\mathbb{R}^2$ and thus $\exists \{\gamma_i\}$ and $\{\delta_i\}$ such that

$$V_2^T = \begin{bmatrix} \gamma_1 \mathbf{u} + \delta_1 \mathbf{v}, & \gamma_2 \mathbf{u} + \delta_2 \mathbf{v}, & \gamma_3 \mathbf{u} + \delta_3 \mathbf{v} \end{bmatrix}_{2\times 3} \tag{3}$$

and by (1) and using Lemma 6 also

$$U_2^T = -T V_2^T = a \begin{bmatrix} \gamma_1 \mathbf{v} - \delta_1 \mathbf{u}, & \gamma_2 \mathbf{v} - \delta_2 \mathbf{u}, & \gamma_3 \mathbf{v} - \delta_3 \mathbf{u} \end{bmatrix}_{2\times 3} \tag{4}$$

for some scalar a.

Using (2) and (3), we obtain

$$U_1 V_2^T = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix} \begin{bmatrix} \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix}.$$

In a similar way, using (2) and (4) we obtain

$$V_1 U_2^T = a \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix} \begin{bmatrix} \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix}.$$

Finally,

$$F_{12} = U_1 V_2^T + V_1 U_2^T = \left(\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} + a \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix} \right) \begin{bmatrix} \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix}$$

implying that $\text{rank}(F_{12}) \le 1$

□

Lemma 8. *Let $F \in \mathcal{F}$, a n-view fundamental matrix, with $\text{rank}(F) = 4$ such that (1) $F = UV^T + VU^T$ with $U, V \in \mathbb{R}^{3n\times 2}$, $\text{rank}(U) = \text{rank}(V) = 2$, (2) $\text{rank}(F_i) = 2$. Then, F can be written also as $F = \hat{U}\hat{V}^T + \hat{V}\hat{U}^T$ with $\hat{U}, \hat{V} \in \mathbb{R}^{3n\times 2}$ such that for all $i \in [n]$, $\hat{U}_i \hat{V}_i^T$ is skew-symmetric and $\text{rank}(\hat{U}_i \hat{V}_i^T) = 2$.*

Proof. Since $F_{ii} = 0$ for all $i \in [n]$, $U_i V_i^T + V_i U_i^T = 0$, and hence $U_i V_i^T$ is skew-symmetric. Therefore, either $\text{rank}(U_i V_i^T) = 2$ or $U_i V_i^T = 0$. We thus need to show that there exists a decomposition of F such that $\hat{U}_i \hat{V}_i^T \neq 0$ for all $i \in [n]$.

Suppose that $U_i V_i^T = 0$ for some $i \in [n]$. First, we note that both cases, that $\text{rank}(U_i) = \text{rank}(V_i) = 2$ and $\text{rank}(U_i) = 1$, $\text{rank}(V_i) = 2$, are not feasible. The former case is infeasible because the null space of U_i only includes the zero vector. The latter case is infeasible because the null space of U_i is of dimension 1, and the space spanned by the rows of V_i is of dimension 2. Consequently, it remains to analyze the case that either (i) $\text{rank}(U_i) = \text{rank}(V_i) = 1$, and (ii) $U_i = 0$ (or $V_i = 0$) for some $i \in [n]$. We further prove that it is possible to generate a valid decomposition, i.e., for all $i \in [n]$, $\hat{U}_i \hat{V}_i^T$ is skew-symmetric and $\text{rank}(\hat{U}_i \hat{V}_i^T) = 2$.

Following the assumptions, $\text{rank}(F) = 4$ and $F = UV^T + VU^T$, such that $\text{rank}(U) = \text{rank}(V) = 2$, and using Lemma 2, it holds that $F = XX^T - YY^T$ where $X, Y \in \mathbb{R}^{3n\times 2}$, $\text{rank}(X) = \text{rank}(Y) = 2$ and we denote $X = [\mathbf{x}^1, \mathbf{x}^2]$ and $Y = [\mathbf{y}^1, \mathbf{y}^2]$. The entries of the vectors are enumerated by subscript i.

Based on the X, Y decomposition, we will construct $\hat{U}, \hat{V} \in \mathbb{R}^{3n\times 2}$, such that $F = \hat{U}\hat{V}^T + \hat{V}\hat{U}^T$, where for any $i \in [n]$, it holds that $\hat{U}_i\hat{V}_i^T$ is skew-symmetric with $\text{rank}(\hat{U}_i\hat{V}_i^T) = 2$. More precisely, we use either

$$\hat{U} = \frac{1}{\sqrt{2}}[\mathbf{x}^1 + \mathbf{y}^1, \mathbf{x}^2 - \mathbf{y}^2], \quad \hat{V} = \frac{1}{\sqrt{2}}[\mathbf{x}^1 - \mathbf{y}^1, \mathbf{x}^2 + \mathbf{y}^2] \tag{5}$$

or

$$\hat{U} = \frac{1}{\sqrt{2}}[\mathbf{x}^1 + \mathbf{y}^1, \mathbf{x}^2 + \mathbf{y}^2], \quad \hat{V} = \frac{1}{\sqrt{2}}[\mathbf{x}^1 - \mathbf{y}^1, \mathbf{x}^2 - \mathbf{y}^2] \tag{6}$$

to construct a decomposition for F that satisfies the conditions of the lemma. Both decompositions satisfy that $F = XX^T - YY^T = \mathbf{x}^1 \otimes \mathbf{x}^1 + \mathbf{x}^2 \otimes \mathbf{x}^2 - \mathbf{y}^1 \otimes \mathbf{y}^1 - \mathbf{y}^2 \otimes \mathbf{y}^2 = \hat{U}\hat{V}^T + \hat{V}\hat{U}^T$, where we use the symbol $\otimes$ to the denote the outer product.

Special case: $\exists i$ such that $U_i = 0$ or $V_i = 0$. Assuming that for one of the decompositions, (5) and (6), there exists i where w.l.o.g. $U_i = 0$. Then we note that $\text{rank}(V_i) = 2$ since otherwise $\text{rank}(F_i) < 2$. Moreover, $\forall j \neq i$ $\text{rank}(U_j) = 2$ since otherwise $\text{rank}(F_{ij}) < 2$, and therefore for all $j \neq i$ $\text{rank}(V_j) \in \{0, 2\}$. Consequently, we can use Lemma 10 to construct the desired decomposition. Hence from now on we discard the case that $\exists i$ such that $U_i = 0$ or $V_i = 0$.

First step: for any $i \in [n]$ in one of the decompositions, (5) or (6), it holds that rank$(\hat{U}_i) =$ rank$(\hat{V}_i) = 2$. W.l.o.g. we prove this for U_1, V_1. Assume by contradiction that in both decompositions $\text{rank}(\hat{U}_1) = \text{rank}(\hat{V}_1) = 1$. First, we note that in both decompositions, the situation that one of the columns of $\hat{U}_1$ or $\hat{V}_1$ is identically zero, is not possible. W.l.o.g., assume that the first column of $\hat{U}_1$ in the first representation, $\mathbf{x}^1_{(1:3)} + \mathbf{y}^1_{(1:3)}$, is zero. Then due to the rank 1 assumption and the relation $\hat{U}_1\hat{V}_1^T = 0$, this implies that $\mathbf{x}^2_{(1:3)} + \mathbf{y}^2_{(1:3)}$ is zero, yielding U_1 which is identically zero in the second representation, contradicting the rank1 assumption.

Now, due to the rank1 assumption, we obtain that the following vectors (none of them is zero) are parallel

$$\mathbf{x}^1_{(1:3)} + \mathbf{y}^1_{(1:3)}, \quad \mathbf{x}^1_{(1:3)} - \mathbf{y}^1_{(1:3)}, \quad \mathbf{x}^2_{(1:3)} + \mathbf{y}^2_{(1:3)}, \quad \mathbf{x}^2_{(1:3)} - \mathbf{y}^2_{(1:3)} \Rightarrow$$

$\exists \mathbf{u} \neq 0 \in \mathbb{R}^{3\times 1}$ and scalars $\{\gamma_i \neq 0\}_{i=1}^4$ such that

$$U_1 = [\gamma_1\mathbf{u}, \gamma_2\mathbf{u}] \;\; V_1 = [\gamma_3\mathbf{u}, \gamma_4\mathbf{u}] \tag{7}$$

Therefore,

$$\begin{aligned} F_1 &= U_1V^T + V_1U^T \\ &= \begin{bmatrix}\gamma_1\mathbf{u} & \gamma_2\mathbf{u}\end{bmatrix} V^T + \begin{bmatrix}\gamma_3\mathbf{u} & \gamma_4\mathbf{u}\end{bmatrix} U^T \\ &= \begin{bmatrix}\gamma_1\mathbf{u} & \gamma_2\mathbf{u} & \gamma_3\mathbf{u} & \gamma_4\mathbf{u}\end{bmatrix} \begin{bmatrix} V^T \\ U^T \end{bmatrix} \end{aligned}$$

yielding $\text{rank}(F_1) \leq 1$, contradicting the assumption that $\text{rank}(F_i) = 2$.

Second step: Each decomposition satisfies $\forall i$ rank$(\hat{U}_i) =$ rank$(\hat{V}_i) = 1$ or $\forall i$ rank$(\hat{U}_i) =$ rank$(\hat{V}_i) = 2$. Following the first step, we get that for any $i \in [n]$, it holds that $\text{rank}(\hat{U}_i) = \text{rank}(\hat{V}_i) = 2$ in one of the decompositions. Select $j \in [n]$ and assume w.l.o.g. that $\text{rank}(\hat{U}_j) = \text{rank}(\hat{V}_j) = 2$ in the first decomposition. Now, for each i, we know that $\hat{U}_i\hat{V}_i^T$ is a skew symmetric matrix. Therefore, for any $i \neq j$ $\text{rank}(\hat{U}_i\hat{V}_i^T)$ is either 2 or 0. If the rank is 2, then immediately $\text{rank}(\hat{U}_i) = \text{rank}(\hat{V}_i) = 2$. If the rank is 0, i.e., $\hat{U}_i\hat{V}_i^T = 0$, then $\text{rank}(\hat{U}_i) = \text{rank}(\hat{V}_i) = 1$ and this is contradicted using Lemma 7, and therefore $\text{rank}(\hat{U}_i) = \text{rank}(\hat{V}_i) = 2$.

These proof shows that there is a decomposition such that for any i, it holds that $\text{rank}(\hat{U}_i) = \text{rank}(\hat{V}_i) = 2$.

□

Lemma 9. *Let $F \in \mathcal{F}$, rank$(F) = 4$ and $F = UV^T + VU^T$ where $U, V \in \mathbb{R}^{3n\times 2}$. Let $Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \in \mathbb{R}^{4\times 4}$ such that it is subject to the following constraints:*

1. *$Q_{12}Q_{11}^T + Q_{11}Q_{12}^T = 0$, i.e., $Q_{12}Q_{11}^T$ is skew symmetric (3 constraints)*
2. *$Q_{22}Q_{21}^T + Q_{21}Q_{22}^T = 0$, i.e., $Q_{22}Q_{21}^T$ is skew symmetric (3 constraints)*
3. *$Q_{12}Q_{21}^T + Q_{11}Q_{22}^T = I$ (4 constraints)*

It then follows that $F = \tilde{U}\tilde{V}^T + \tilde{V}\tilde{U}^T$ where $[\tilde{U}\ \tilde{V}] = [U\ V]Q$.

This Lemma is adapted from [25] that proved a similar result for the case that rank$(F) = 6$.

Proof. Denote by $J = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \in \mathbb{R}^{4\times4}$, then,

$$QJQ^T = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}^T = \begin{bmatrix} Q_{12} & Q_{11} \\ Q_{22} & Q_{21} \end{bmatrix} \begin{bmatrix} Q_{11}^T & Q_{21}^T \\ Q_{12}^T & Q_{22}^T \end{bmatrix} =$$

$$\begin{bmatrix} Q_{12}Q_{11}^T + Q_{11}Q_{12}^T & Q_{12}Q_{21}^T + Q_{11}Q_{22}^T \\ Q_{22}Q_{11}^T + Q_{21}Q_{12}^T & Q_{22}Q_{21}^T + Q_{21}Q_{22}^T \end{bmatrix} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$$

where the rightmost equality is obtained by using the constraints on Q. Let $\tilde{U}, \tilde{V} \in \mathbb{R}^{3n\times2}$ defined as $[\tilde{U}\ \tilde{V}] = [U\ V]Q$. It follows that:

$$F = UV^T + VU^T = [U\ V]J[U\ V]^T = [U\ V]QJQ^T[U\ V]^T = [\tilde{U}\ \tilde{V}]J[\tilde{U}\ \tilde{V}]^T = \tilde{U}\tilde{V}^T + \tilde{V}\tilde{U}^T$$

□

Lemma 10. *Let $F \in \mathcal{F}$, for which the following conditions hold:*

1. *rank$(F) = 4$ and $F = UV^T + VU^T$, where $U, V \in \mathbb{R}^{3n\times2}$*
2. *rank$(F_i) = 2$, where F_i denotes the i^{th} block-row of F, $i \in [n]$.*
3. *For all $i \in [n]$, rank(U_i), rank$(V_i) \in \{0, 2\}$, where U_i and V_i respectively denote the 3×2 blocks of U, V, and there exists at least one block of rank 0.*

Then there exists a new decomposition $F = \tilde{U}\tilde{V}^T + \tilde{V}\tilde{U}^T$ such that all the 3×2 blocks of $\tilde{U}, \tilde{V}$ are of rank 2.

Proof. Following Lemma 9, let $\alpha \neq 0$ we set Q as follows:

$$Q_{11} = \frac{1}{\alpha}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, Q_{12} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, Q_{21} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}, Q_{22} = \alpha\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

It can be readily verified that Q satisfies the conditions above, i.e,

- $Q_{12}Q_{11}^T = \frac{1}{\alpha}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
- $Q_{22}Q_{21}^T = \alpha\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$
- $Q_{22}Q_{11}^T + Q_{21}Q_{12}^T = I$

and moreover, the rank of each sub matrix of Q is 2. Denote by

$$\begin{aligned} \tilde{U} &= UQ_{11} + VQ_{21} \\ \tilde{V} &= UQ_{12} + VQ_{22} \\ \tilde{U}_i &= U_iQ_{11} + V_iQ_{21} \\ \tilde{V}_i &= U_iQ_{12} + V_iQ_{22} \end{aligned}$$

Let $i \in [n]$, then $F_i = U_iV^T + V_iU^T = \begin{bmatrix} U_i & V_i \end{bmatrix} \begin{bmatrix} V^T \\ U^T \end{bmatrix}$. As a result rank$\left(\begin{bmatrix} U_i & V_i \end{bmatrix}\right) \geq$ rank$(F_i) = 2$. Consequently

$$U_i = 0 \Rightarrow \text{rank}(V_i) = 2, V_i = 0 \Rightarrow \text{rank}(U_i) = 2$$

We consider the following 3 possibilities:

1. $U_i = 0 \Rightarrow \text{rank}(V_i) = \text{rank}(Q_{22}) = \text{rank}(Q_{21}) = 2$. It follows that:

$$\text{rank}(\tilde{U_i}) = \text{rank}(U_i Q_{11} + V_i Q_{21}) = \text{rank}(V_i Q_{21}) = 2$$

$$\text{rank}(\tilde{V_i}) = \text{rank}(U_i Q_{12} + V_i Q_{22}) = \text{rank}(V_i Q_{22}) = 2$$

2. $V_i = 0 \Rightarrow \text{rank}(U_i) = \text{rank}(Q_{11}) = \text{rank}(Q_{12}) = 2$ we get that:

$$\text{rank}(\tilde{U_i}) = \text{rank}(U_i Q_{11} + V_i Q_{21}) = \text{rank}(U_i Q_{11}) = 2$$

$$\text{rank}(\tilde{V_i}) = \text{rank}(U_i Q_{12} + V_i Q_{22}) = \text{rank}(U_i Q_{12}) = 2$$

3. $\text{rank}(U_i) = \text{rank}(V_i) = 2$. Since there exists $j \neq i$ that satisfies one of the two possibilities above it holds that $\text{rank}(\tilde{U_j}) = \text{rank}(\tilde{V_j}) = 2$. This implies, due to Lemma 7, that $\text{rank}(\tilde{U_i}) \neq 1$ and $\text{rank}(\tilde{V_i}) \neq 1$. Next we exclude the case that either $\tilde{U_i} = 0$ or $\tilde{V_i} = 0$. Assume w.l.o.g. that $\alpha = 1$, $\text{rank}(\tilde{U_i}) = 2$ and $\text{rank}(\tilde{V_i}) = 0$, we have that $\tilde{V_i} = 0 = U_i Q_{12} + V_i Q_{22}$. Now, if we instead select any $\hat{\alpha} \neq \alpha = 1$ then we obtain $\hat{V_i} = U_i Q_{12} + V_i \hat{Q}_{22}$ where $\hat{Q}_{22} = \hat{\alpha} Q_{22}$, from which we obtain $\hat{V_i} = (\hat{\alpha} - 1) V_i Q_{22} \neq 0$ where the latter inequality is due to the full rank of V_i and Q_{22}.

   Next, if $\tilde{U_i} = 0$ we can choose yet a different value for α, obtaining $\text{rank}(\tilde{U_i}) = \text{rank}(\tilde{V_i}) = 2$. Overall, there are at most $2(n-1)$ possible choices of values for α that make either $\tilde{U_i}$ or $\tilde{V_i}$ for some $i \in [n]$ zero. Therefore we can always choose a value for α that will keep all of these blocks rank 2.

$\square$

2. Supplementary Results

In the following pages we present additional results for the experiments presented in the main paper. Specifically, we provide tables with complete running time and with median evaluation for the KITTI dataset, in both calibrated and uncalibrated setups (Tables 1-4 in the paper). We further include here the complete results for the experiments in Table 7 in the paper, which involved uncalibrated, unordered internet photos. Below we refer to the paper "Algebraic Characterization of Essential Matrices and Their Averaging in Multiview Settings" [13] as GESFM, which stands for "Global Essentials SFM".

Table 1. KITTI, calibrated: Mean position error in meters before BA. For each dataset and number of cameras we show the median of mean errors.

Dataset	5 Cameras					10 Cameras					20 Cameras				
	VP	R4	GESFM[13]	LUD[20]	1DSFM[34]	VP	R4	GESFM[13]	LUD[20]	1DSFM[34]	VP	R4	GESFM[13]	LUD[20]	1DSFM[34]
00	**0.02**	**0.02**	0.30	0.04	0.27	**0.04**	**0.04**	1.42	0.06	0.70	**0.09**	**0.09**	2.92	0.10	1.47
01	**0.37**	1.22	2.47	1.06	2.08	**0.63**	3.12	5.70	1.59	4.69	**1.24**	5.82	12.1	2.32	4.66
02	**0.02**	**0.02**	0.59	0.08	0.86	0.08	**0.05**	1.90	0.11	1.64	0.22	**0.11**	4.46	0.19	1.90
03	**0.03**	**0.03**	0.21	0.06	0.31	0.12	0.10	0.52	**0.05**	0.79	0.42	1.32	2.33	**0.11**	2.11
04	**0.03**	0.07	0.83	0.10	0.79	**0.07**	0.09	2.70	0.14	1.65	**0.14**	0.76	6.14	0.28	4.94
05	**0.01**	**0.01**	0.47	0.04	0.36	**0.03**	**0.03**	1.51	0.10	0.51	**0.08**	0.09	3.33	0.12	1.92
06	**0.02**	**0.02**	1.03	0.09	0.80	**0.05**	0.06	2.78	0.09	1.48	0.18	0.76	6.14	**0.12**	3.91
07	**0.01**	0.02	0.18	0.04	0.21	**0.03**	**0.03**	0.72	0.04	0.35	**0.08**	0.12	1.53	**0.08**	1.36
08	**0.02**	**0.02**	0.27	0.04	0.61	**0.04**	**0.04**	0.79	0.05	0.92	**0.09**	0.10	1.87	0.15	2.05
09	**0.02**	**0.02**	0.51	0.09	0.46	0.05	**0.04**	2.06	0.07	0.60	0.13	0.11	3.85	**0.10**	2.24
10	**0.02**	**0.02**	0.42	0.05	0.46	**0.03**	0.04	1.64	0.05	0.91	0.11	0.12	2.45	**0.09**	2.29

Table 2. KITTI, calibrated: average execution time in seconds, KITTI. [34]'s results are unavailable.

Dataset	**5 Cameras**				**10 Cameras**				**20 Cameras**			
	VP	R4	GESFM[13]	LUD[20]	VP	R4	GESFM[13]	LUD[20]	VP	R4	GESFM[13]	LUD[20]
00	0.42	0.48	**0.22**	0.78	1.18	1.18	**0.57**	2.30	2.96	2.24	**1.47**	5.99
01	0.42	0.41	**0.22**	0.19	1.15	0.95	**0.57**	0.77	2.97	1.94	**1.42**	2.29
02	0.43	0.45	**0.22**	0.58	1.16	1.03	**0.57**	1.65	2.97	2.12	**1.45**	4.15
03	0.43	0.55	**0.21**	1.77	1.20	1.24	**0.57**	5.88	3.20	2.45	**1.50**	17.65
04	0.42	0.46	**0.22**	0.63	1.22	1.05	**0.57**	1.94	3.04	2.12	**1.43**	5.29
05	0.43	0.46	**0.22**	0.66	1.21	1.04	**0.62**	2.09	3.15	2.12	**1.49**	5.65
06	0.43	0.42	**0.22**	0.33	1.27	0.95	**0.59**	0.95	3.18	1.92	**1.52**	2.54
07	0.42	0.49	**0.22**	0.99	1.24	1.10	**0.60**	3.25	3.07	2.26	**1.48**	9.63
08	0.42	0.47	**0.22**	0.79	1.24	1.07	**0.57**	2.41	3.02	2.16	**1.47**	6.51
09	0.41	0.47	**0.22**	0.71	1.24	1.07	**0.60**	2.00	3.11	2.11	**1.46**	5.26
10	0.43	0.47	**0.22**	0.84	1.22	1.05	**0.57**	2.53	3.06	2.21	**1.50**	6.68

Table 3. KITTI, uncalibrated: Mean reprojection error in pixels after BA. for each dataset and number of cameras we show the median of mean errors.

	5 Cameras				**10 Cameras**				**20 Cameras**			
	VC	R4	GPSFM[14]	PPSFM[17]	VP	R4	GPSFM[14]	PPSFM[17]	VP	R4	GPSFM[14]	PPSFM[17]
00	**0.12**	**0.12**	**0.12**	0.13	**0.15**	**0.15**	0.42	0.16	**0.16**	**0.16**	3.70	0.18
01	**0.18**	0.19	0.51	0.21	**0.25**	1.00	1.17	0.29	5.84	6.77	6.38	**0.41**
02	**0.12**	**0.12**	0.14	0.13	**0.15**	**0.15**	2.41	0.16	**0.16**	**0.16**	10.89	0.19
03	**0.14**	**0.14**	0.15	0.16	**0.17**	**0.17**	0.18	0.2	**0.2**	**0.2**	1.11	0.26
04	**0.12**	**0.12**	0.13	0.13	**0.15**	**0.15**	4.39	**0.15**	**0.18**	**0.18**	27.03	0.2
05	**0.13**	**0.13**	0.14	0.15	**0.16**	**0.16**	2.74	0.18	**0.18**	**0.18**	18.73	0.22
06	**0.12**	**0.12**	0.14	0.13	**0.15**	**0.15**	0.26	0.16	**0.17**	**0.17**	32.43	0.21
07	**0.14**	**0.14**	0.15	**0.16**	**0.17**	**0.17**	3.80	0.2	0.21	**0.20**	3.60	0.26
08	**0.13**	**0.13**	0.14	**0.13**	**0.16**	**0.16**	0.29	0.18	**0.19**	**0.19**	1.85	0.21
09	**0.11**	**0.11**	0.13	0.12	**0.14**	**0.14**	1.10	0.16	**0.16**	**0.16**	7.15	0.18
10	**0.12**	**0.12**	0.13	0.13	**0.15**	**0.15**	1.26	0.17	**0.16**	**0.16**	5.65	0.19

Table 4. KITTI, uncalibrated: average execution time in seconds.

	5 Cameras				**10 Cameras**				**20 Cameras**			
	VP	R4	GPSFM[14]	PPSFM[17]	VP	R4	GPSFM[14]	PPSFM[17]	VP	R4	GPSFM[14]	PPSFM[17]
00	0.69	1.39	**0.67**	1.99	1.81	2.81	**1.71**	5.50	4.29	5.80	**4.28**	11.05
01	0.56	0.85	**0.50**	2.37	1.48	1.88	**1.31**	5.47	3.89	4.09	**2.82**	10.10
02	**0.62**	1.17	0.63	1.71	1.62	2.38	**1.57**	4.55	4.14	5.01	**3.43**	10.00
03	**0.83**	1.71	0.95	2.79	**2.08**	3.53	2.33	7.51	**5.17**	8.21	5.84	14.00
04	**0.64**	1.18	0.67	1.62	**1.60**	2.35	1.64	4.44	4.21	4.85	**3.27**	10.49
05	**0.65**	1.21	0.66	1.84	1.68	2.47	**1.55**	4.67	4.26	5.27	**3.54**	10.64
06	0.55	0.93	**0.53**	1.16	1.44	1.92	**1.29**	3.11	3.67	4.15	**2.69**	7.25
07	0.75	1.40	**0.70**	3.17	**1.87**	2.97	1.92	11.02	5.04	6.65	**4.52**	22.41
08	**0.68**	1.33	**0.68**	1.93	**1.71**	2.75	1.78	5.31	4.57	6.16	**4.41**	10.52
09	**0.66**	1.25	**0.66**	2.06	**1.68**	2.54	1.74	5.03	4.33	5.32	**3.91**	11.11
10	**0.66**	1.25	0.68	1.80	1.70	2.61	**1.64**	4.94	4.55	5.99	**4.10**	10.42

Table 5. Unordered internet photos, uncalibrated: Mean reprojection error and execution times.

Dataset	#points	#Images	Error(pixels)			Time(s)		
			OURS	**GPSFM**[14]	**PPSFM**[17]	**Ours**	**GPSFM**[14]	**PPSFM**[17]
Dino 319	319	36	**0.43**	**0.43**	0.47	12.32	3.64	**3.46**
Dino 4983	4983	36	0.43	**0.42**	0.47	15.75	**4.65**	13.00
Corridor	737	11	**0.26**	**0.26**	0.27	2.48	**1.03**	1.55
House	672	10	**0.34**	**0.34**	0.40	1.82	**0.94**	1.03
Gustav Vasa	4249	18	**0.16**	**0.16**	0.17	4.90	**2.47**	6.64
Folke Filbyter	21150	40	**0.26**	0.82	0.31	14.30	**6.70**	102.77
Park Gate	9099	34	**0.31**	**0.31**	0.45	19.68	**9.25**	31.58
Nijo	7348	19	**0.39**	**0.39**	0.44	7.02	**3.80**	12.68
Drinking Fountain	5302	14	**0.28**	**0.28**	0.31	4.64	**2.12**	9.37
Golden Statue	39989	18	**0.22**	**0.22**	0.23	10.08	**5.05**	36.21
Jonas Ahls	2021	40	**0.18**	**0.18**	0.20	13.84	**5.49**	13.40
De Guerre	13477	35	**0.26**	**0.26**	0.28	34.32	**11.19**	32.67
Dome	84792	85	**0.24**	**0.24**	0.25	108.18	**65.12**	226.13
Alcatraz Courtyard	23674	133	**0.52**	**0.52**	0.57	126.40	**63.94**	151.28
Alcatraz Water Tower	14828	172	**0.47**	**0.47**	0.59	169.08	90.24	**71.80**
Cherub	72784	65	0.75	**0.74**	0.81	48.52	**27.30**	101.64
Pumpkin	69335	195	**0.38**	**0.38**	0.44	203.06	**93.32**	222.09
Sphinx	32668	70	**0.34**	**0.34**	0.36	39.63	**31.41**	79.91
Toronto University	7087	77	**0.24**	0.54	0.26	30.47	**26.59**	91.26
Sri Thendayuthapani	88849	98	**0.31**	0.51	0.33	**219.11**	220.25	325.58
Porta san Donato	25490	141	**0.40**	**0.40**	3.56	126.54	**82.18**	157.96
Buddah Tooth	27920	162	**0.60**	**0.60**	0.71	142.02	**59.75**	81.05
Tsar Nikolai I	37857	98	**0.29**	0.32	0.31	89.93	**70.79**	101.01
Smolny Cathedral	51115	131	**0.46**	0.48	0.50	303.62	**210.75**	263.60
Skansen Kronan	28371	131	**0.41**	0.44	0.44	118.60	**83.43**	161.81