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A Lighting-Invariant Point Processor for Shading: Supplementary Material

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1. $\tilde{I}_{yy} \leq 0$ from §4.3.

Lemma 1. *The rotation-induced transformation $R_{\mathbf{I}}$ that maps $I_y \mapsto 0$ always maps I_{yy} to a nonpositive value.*

Proof. Under this transformation,

$$R_{\mathbf{I}} : \mathbf{I} = \begin{pmatrix} I \\ I_x \\ I_y \\ I_{xx} \\ I_{xy} \\ I_{yy} \end{pmatrix} \mapsto \tilde{\mathbf{I}} = \begin{pmatrix} I \\ +\sqrt{I_x^2 + I_y^2} \\ 0 \\ \frac{I_x^2 I_{xx} + 2 I_x I_{xy} I_y + I_y^2 I_{yy}}{I_x^2 + I_y^2} \\ \frac{I_x^2 I_{xy} + I_x I_y (I_{yy} - I_{xx}) - I_{xy} I_y^2}{I_x^2 + I_y^2} \\ \frac{I_x^2 I_{yy} - 2 I_x I_{xy} I_y + I_{xx} I_y^2}{I_x^2 + I_y^2} \end{pmatrix} =: \begin{pmatrix} \tilde{I} \\ \tilde{I}_x \\ \tilde{I}_y \\ \tilde{I}_{xx} \\ \tilde{I}_{xy} \\ \tilde{I}_{yy} \end{pmatrix}$$

Over $\mathbb{R}$, $\tilde{I}_{yy}$ is the same sign as its numerator $W := I_x^2 I_{yy} - 2 I_x I_{xy} I_y + I_{xx} I_y^2$, so we'll study $\text{sgn}(W)$ as a proxy. Recall that the point processor considers the point $(x, y) = (0, 0)$. By the quadratic patch assumption the true surface $f^*(x, y) = ax + by + \frac{1}{2}\left(c^2 x + 2dxy + ey^2\right)$, so at this point we have $\mathbf{f}^* = (a, b, c, d, e)^T$. Recall also that $I(0, 0) = \rho \mathbf{L} \cdot \mathbf{N}^*(x, y)/||\mathbf{N}^*(x, y)||$, where $\mathbf{N}^* = (-(\partial f/\partial x), -(\partial f/\partial y), 1)^T = (-a, -b, 1)^T$ and WLOG $\rho = 1$. Writing $\mathbf{I}(0, 0)$ as a function of $(\mathbf{f}^*, \mathbf{L}^*)$, we have

$$W = I_x^2 I_{yy} - 2 I_x I_{xy} I_y + I_{xx} I_y^2 = \underbrace{\frac{(d^2 - ce)^2}{|(1 + a^2 + b^2)^{9/2} (L_1^2 + L_2^2 + L_3^2)^{3/2}|}}_{\text{strictly positive}} \underbrace{(aL_1 + bL_2 - L_3)}_{\text{strictly negative by assumption}} V,$$

$$\begin{aligned} V :&= L_1^2 + b^2 L_1^2 - 2ab L_1 L_2 + L_2^2 + a^2 L_2^2 + 2a L_1 L_3 + 2b L_2 L_3 + a^2 L_3^2 + b^2 L_3^2 \\ &= (bL_1 - aL_2)^2 + (L_1 + aL_3)^2 + (L_2 + bL_3)^2. \end{aligned}$$

Since V can be written as the sum of squares, the entirety of W is nonpositive; therefore, $\tilde{I}_{yy}$ is always nonpositive. Furthermore, V can only be zero-valued when $(a, b) = (-L_1/L_3, -L_2/L_3)$, which generically will not occur.

□

2. Proof of Theorem 1 from §3.2.

Proof. For brevity let $a, b, c, d, e = f_x, f_y, f_{xx}, f_{xy}, f_{yy}$. We begin with Lambert's law for some fixed $\mathbf{I}$ vector,

$$I(x, y) = \rho \mathbf{L} \cdot \frac{\mathbf{N}(x, y)}{||\mathbf{N}(x, y)||}, \qquad (x, y) \in U, \tag{1}$$

with $\mathbf{N}(x,y) := (-(\partial f/\partial x)(x,y), -(\partial f/\partial y)(x,y), 1)^T = (-a-cx-dy, -b-dx-ey, 1)^T$. Since we do not require $\mathbf{L}$ to be unit, we can effectively absorb ρ into it. This turns (1) into

$$I(x,y) = -\frac{(a+cx+dy)L_1 + (b+dx+ey)L_2 - L_3}{\sqrt{a+cx+dy^2 + b+dx+ey^2+1}} = -w((a+cx+dy)L_1 + (b+dx+ey)L_2 - L_3), \qquad (2)$$

with $w := 1/\sqrt{a+cx+dy^2+b+dx+ey^2+1}$. Rearranging this,

$$0 = w((a+cx+dy)L_1 + (b+dx+ey)L_2 - L_3) - I(x,y), \qquad (3)$$
$$0 = w^2(a+cx+dy^2+b+dx+ey^2+1) - 1. \qquad (4)$$

Taking first and second order partial derivatives of (3) with respect to x and y, then evaluating each resulting derivative at the point $(x,y)=(0,0)$ gives us the local system

$$\begin{aligned}
s &:= w^2\left(a^2+b^2+1\right)-1\\
r_0 &:= w(aL_1+bL_2-L_3)+I\\
r_x &:= w^3(-(ac+bd))(aL_1+bL_2-L_3)+w(cL_1+dL_2)+I_x\\
r_y &:= w^3(-(ad+be))(aL_1+bL_2-L_3)+w(dL_1+eL_2)+I_y\\
r_{xx} &:= w^3\left(3w^2(ac+bd)^2-c^2-d^2\right)(aL_1+bL_2-L_3)-2w^3(ac+bd)(cL_1+dL_2)+I_{xx}\\
r_{xy} &:= 3w^5(ac+bd)(ad+be)(aL_1+bL_2-L_3)-dw^3(c+e)(aL_1+bL_2-L_3)\\
&\quad -w^3(ad+be)(cL_1+dL_2)-w^3(ac+bd)(dL_1+eL_2)+I_{xy}\\
r_{yy} &:= w^3\left(3w^2(ad+be)^2-d^2-e^2\right)(aL_1+bL_2-L_3)-2w^3(ad+be)(dL_1+eL_2)+I_{yy}
\end{aligned}$$

and we refer to the vectors (r_0, r_x, r_y, r_{xx}), (r_0, r_x, r_y, r_{xy}), and (r_0, r_x, r_y, r_{yy}) as $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$, respectively. Each of these seven polynomials is linear in the L_i; thus, if $\mathbf{L} := (L_1, L_2, L_3, 1)$, we can write this system as $\mathbf{r}_i = A_i\mathbf{L}$, where $A_i = A_i(w, \mathbf{f}, \mathbf{I})$ is a square functional-entried matrix. Then

$$A_i = \begin{pmatrix} aw & bw & -w & I\\ cw\left(1-a^2w^2\right)-abdw^3 & dw\left(1-b^2w^2\right)-abcw^3 & w^3(ac+bd) & I_x\\ dw\left(1-a^2w^2\right)-abew^3 & ew\left(1-b^2w^2\right)-abdw^3 & w^3(ad+be) & I_y\\ \rho_{i1} & \rho_{i2} & \rho_{i3} & \rho_{i4}\end{pmatrix},$$

$$\begin{aligned}
\rho_{11} &= w^3\left(3a^3c^2w^2+6a^2bcdw^2-a(d^2(1-3b^2w^2)+3c^2)-2bcd\right)\\
\rho_{12} &= w^3\left(bc^2(3a^2w^2-1)+6ab^2cdw^2-2acd+3b^3d^2w^2-3bd^2\right)\\
\rho_{13} &= w^3\left(c^2(1-3a^2w^2)-6abcdw^2+d^2(1-3b^2w^2)\right)\\
\rho_{21} &= w^3\left(3a^3d^2w^2+6a^2bdew^2-a(e^2(1-3b^2w^2)+3d^2)-2bde\right)\\
\rho_{22} &= w^3\left(bd^2(3a^2w^2-1)+6ab^2dew^2-2ade+3b^3e^2w^2-3be^2\right)\\
\rho_{23} &= w^3\left(d^2(1-3a^2w^2)-6abdew^2+e^2(1-3b^2w^2)\right)\\
\rho_{31} &= w^3\Big(3a^3cdw^2+3a^2bw^2(ce+d^2)\\
&\qquad -ad(-3b^2ew^2+3c+e)-b(ce+d^2)\Big)\\
\rho_{32} &= w^3\Big(-bd(-3a^2cw^2+c+3e)+3ab^2w^2(ce+d^2)\\
&\qquad -a(ce+d^2)+3b^3dew^2\Big)\\
\rho_{33} &= w^3\left(c(-3a^2dw^2-3abew^2+d)+d(-3abdw^2-3b^2ew^2+e)\right)\\
\rho_{14} &= I_{xx} \qquad \rho_{24}=I_{yy} \qquad \rho_{34}=I_{xy}.
\end{aligned}$$

Notice that for each i, $\det A_i = w^3(d^2 - ce)C_i$, and that $s = 0 \implies w \neq 0$. Suppose $\neg(C_i = 0\ \forall i)$. That is, $\exists i : C_i \neq 0$. Due to the non-degeneracy assumption and the constraint imposed by s that $w \neq 0$, this is equivalent to $\exists i : w^3(d^2 - ce)C_i = \det A_i \neq 0 \iff \exists i : \ker(A_i) = \{\mathbf{0}\}$. This is equivalent to $\exists i : \forall \mathbf{L} \neq \mathbf{0},\ A_i\mathbf{L} = \mathbf{r}_i \neq \mathbf{0}$, which is to say that there exists an i which, regardless of $\mathbf{L}$, will always violate one of the Lambert partial derivative conditions. This implies $\neg((\mathbf{f}, \mathbf{I})$ are consistent). The contrapositive of this argument is that under the stated assumptions, $(\mathbf{f}, \mathbf{I})$ consistent implies that $C_i = 0\ \forall i$. $\square$

3. Derivation of $T_{\mathbf{I}}$, $S_{\mathbf{I}}$, and $R_{\mathbf{I}}$ from §4.3.

$$T(t) := \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}, \tag{5}$$

$$S(t) := \begin{bmatrix} \cos^2(t) & \sin(2t) & \sin^2(t) \\ -\sin(t)\cos(t) & \cos^2(t) - \sin^2(t) & \sin(t)\cos(t) \\ \sin^2(t) & -2\sin(t)\cos(t) & \cos^2(t) \end{bmatrix}, \tag{6}$$

$$R(t) := \begin{bmatrix} 1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & T_{\mathbf{I}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & S_{\mathbf{I}}^{-1} \end{bmatrix}, \tag{7}$$

for which one can verify that $\mathbf{f} \in F(\mathbf{I})$ if and only if $\hat{R}(t)\mathbf{f} \in F\left(R(t)\mathbf{I}\right)$, with $\hat{R}(t)$ the principal submatrix of $R(t)$ obtained by removing its first row and column. We obtain $T_{\mathbf{I}}$, $S_{\mathbf{I}}$, $R_{\mathbf{I}}$ from setting $t = -\frac{i\log((I_x - iI_y)}{\sqrt{I_x^2 + I_y^2)}}$.