1. Proof of Theorem 1

**Theorem** For any $x$ in $[0, 1]$ and any two positive integers $a > b$,

$$ [2^a x] >> (a - b) = [2^b x] \quad (S1) $$

**Proof** We need to prove that

$$\left\lfloor \frac{[2^a x]}{2^{a-b}} \right\rfloor = [2^b x] \quad (S2) $$

We first notice that

$$\frac{[2^a x]}{2^{a-b}} \leq \frac{2^a x}{2^{a-b}} = 2^b x \quad (S3) $$

so we have

$$\left\lfloor \frac{2^a x}{2^{a-b}} \right\rfloor < [2^b x] \quad (S4) $$

due to the non-decreasing monotonicity of the floor function.

At the same time, we have

$$\frac{[2^a x]}{2^{a-b}} = \frac{2^{a-b} \cdot [2^b x]}{2^{a-b}} \quad (S5a) $$

$$\geq \frac{2^{a-b} \cdot [2^b x]}{2^{a-b}} \quad (S5b) $$

$$= \frac{2^{a-b} \cdot [2^b x]}{2^{a-b}} \quad (S5c) $$

$$= [2^b x] \quad (S5d) $$

where in the penultimate equality we have used the fact that $a > b$. Thus we have

$$\left\lfloor \frac{2^a x}{2^{a-b}} \right\rfloor \geq [2^b x] \quad (S6) $$

From (S4) and (S6), we can get the desired result (S1).