Appendices

A. Appendix A

We prove the invariance under local diffeomorphism that was used in Section 4 of the main paper and we then provide qualitative results on the NRSfM challenge dataset.

A.1. Invariance under Local Diffeomorphism

In figure 1, η is the image registration function, which means that we have $\mathbf{x} = \eta(\overline{\mathbf{x}})$. We can then write

$$\begin{pmatrix} \mathrm{d}u \\ \mathrm{d}v \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial \overline{u}} & \frac{\partial u}{\partial \overline{v}} \\ \frac{\partial v}{\partial \overline{u}} & \frac{\partial v}{\partial \overline{v}} \end{pmatrix} \begin{pmatrix} \mathrm{d}\overline{u} \\ \mathrm{d}\overline{v} \end{pmatrix} = \mathbf{J}_\eta \begin{pmatrix} \mathrm{d}\overline{u} \\ \mathrm{d}\overline{v} \end{pmatrix}, \tag{23}$$

where $\mathbf{J}_\eta$ is the Jacobian of the function η. Thus, under a change of variable from $\mathbf{x}$ to $\overline{\mathbf{x}}$, we write $E(\phi\circ\eta)$ in terms of $E(\phi)$ (written as (2)) as

$$E(\phi\circ\eta) = \left(\begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{pmatrix} \mathbf{J}_\eta \quad |\mathbf{J}_\eta|\mathbf{e}_3 \right) = E(\phi)\mathrm{diag}(\mathbf{J}_\eta, |\mathbf{J}_\eta|). \tag{24}$$

We now introduce a theorem that describes the relation between moving frames related by a local diffeomorphism.

Theorem 1 (Moving frames under local diffeomorphism)**.** *Given two surfaces $\mathcal{S}$ and $\overline{\mathcal{S}}$ related by a local diffeomorphic mapping ψ, their respective moving frames E and $\overline{E}$ are related by a linear transformation.*

Proof. From figure 1, we can write $\overline{\phi} = \psi\circ\phi\circ\eta$ and therefore,

$$\mathbf{J}_{\overline{\phi}} = \mathbf{J}_{\psi\circ\phi\circ\eta}\mathbf{J}_{\phi\circ\eta}\mathbf{J}_\eta. \tag{25}$$

According to inverse function theorem, $\mathbf{J}_\psi$ is a linear function if ψ is locally diffeomorphic. Thus, we write $\mathbf{J}_\psi = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)\mathbf{R}$ where λ_i are scalars and $\mathbf{R}$ is a rotation matrix. Using this result and the expression of $E(\phi)$ in terms of $\mathbf{J}_\phi$ in (2), we write (25) as

$$\overline{E}(\overline{\phi}) = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)\mathbf{R}E(\phi\circ\eta), \tag{26}$$

where $E(\phi\circ\eta)$ is given by (24). □

Given the relation (25) obtained between moving frames under local diffeomorphism, we now derive the relation between connection components $\overline{\Gamma}^i_{jk}(\overline{\phi})$ and $\Gamma^i_{jk}(\phi)$ in the next theorem.

We write $w^i_j = \Gamma^i_{j1}\mathrm{d}u + \Gamma^i_{j2}\mathrm{d}v$, and thus, the linear system in (3) can be written as

$$\mathrm{d}\mathbf{e}_j = \frac{\partial \mathbf{e}_j}{\partial u}\mathrm{d}u + \frac{\partial \mathbf{e}_j}{\partial v}\mathrm{d}v = w^1_j\mathbf{e}_1 + w^2_j\mathbf{e}_2 + w^3_j\mathbf{e}_3, \quad j = [1,2,3] \tag{27}$$

Theorem 2 (Connection preservation under local diffeomorphism)**.** *Given two surfaces $\mathcal{S}$ and $\overline{\mathcal{S}}$ related by a local diffeomorphic mapping ψ, their respective connection components $\Gamma^i_{jk}(\phi)$ and $\overline{\Gamma}^i_{jk}(\overline{\phi})$ are preserved, i.e., $\overline{\Gamma}^i_{jk}(\overline{\phi}) = \Gamma^i_{jk}(\phi\circ\eta)$.*

Proof. Using the relation between $\overline{E}$ and E obtained in (26) in (27), we get

$$\begin{aligned} &\begin{pmatrix} \lambda_1\overline{\mathbf{e}}_1^\top \\ \lambda_2\overline{\mathbf{e}}_2^\top \\ \lambda_3\overline{\mathbf{e}}_3^\top \end{pmatrix}^\top \mathbf{R}^\top \begin{pmatrix} \overline{w}^1_1 & \overline{w}^1_2 & \overline{w}^1_3 \\ \overline{w}^2_1 & \overline{w}^2_2 & \overline{w}^2_3 \\ \overline{w}^3_1 & \overline{w}^3_2 & \overline{w}^3_3 \end{pmatrix} = \\ &\begin{pmatrix} \lambda_1\mathbf{e}_1^\top \\ \lambda_2\mathbf{e}_2^\top \\ \lambda_3\mathbf{e}_3^\top \end{pmatrix}^\top \mathbf{R}^\top \mathrm{diag}(\mathrm{d}\mathbf{J}_\eta, \mathrm{d}|\mathbf{J}_\eta|) + \\ &\begin{pmatrix} \lambda_1\mathbf{e}_1^\top \\ \lambda_2\mathbf{e}_2^\top \\ \lambda_3\mathbf{e}_3^\top \end{pmatrix}^\top \mathbf{R}^\top \begin{pmatrix} w^1_1 & w^1_2 & w^1_3 \\ w^2_1 & w^2_2 & w^2_3 \\ w^3_1 & w^3_2 & w^3_3 \end{pmatrix} \mathrm{diag}(\mathbf{J}_\eta, |\mathbf{J}_\eta|). \end{aligned} \tag{28}$$

The above relation is thus independent of λ_i and $\mathbf{R}$. We write $\overline{w}^i_j(\overline{\phi}) = w^i_j(\phi\circ\eta)$ and thus, $\overline{\Gamma}^i_j(\overline{\phi}) = \Gamma^i_j(\phi\circ\eta)$. □

Upon expanding the relation between the connections (28), we get

$$\begin{aligned} &\begin{pmatrix} \overline{\Gamma}^1_{11} & \overline{\Gamma}^2_{11} & \overline{\Gamma}^3_{11} \\ \overline{\Gamma}^1_{21} & \overline{\Gamma}^2_{21} & \overline{\Gamma}^3_{21} \\ \overline{\Gamma}^1_{31} & \overline{\Gamma}^2_{31} & \overline{\Gamma}^3_{31} \end{pmatrix} = \begin{pmatrix} \mathbf{J}_\eta & 0 \\ 0 & |\mathbf{J}_\eta| \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial \mathbf{J}_\eta}{\partial \overline{u}} & 0 \\ 0 & \frac{\partial |\mathbf{J}_\eta|}{\partial \overline{u}} \end{pmatrix} + \\ &\frac{\partial u}{\partial \overline{u}} \begin{pmatrix} \mathbf{J}_\eta & 0 \\ 0 & |\mathbf{J}_\eta| \end{pmatrix}^{-1} \begin{pmatrix} \Gamma^1_{11} & \Gamma^2_{11} & \Gamma^3_{11} \\ \Gamma^1_{21} & \Gamma^2_{21} & \Gamma^3_{21} \\ \Gamma^1_{31} & \Gamma^2_{31} & \Gamma^3_{31} \end{pmatrix} \begin{pmatrix} \mathbf{J}_\eta & 0 \\ 0 & |\mathbf{J}_\eta| \end{pmatrix} + \\ &\frac{\partial v}{\partial \overline{u}} \begin{pmatrix} \mathbf{J}_\eta & 0 \\ 0 & |\mathbf{J}_\eta| \end{pmatrix}^{-1} \begin{pmatrix} \Gamma^1_{12} & \Gamma^2_{12} & \Gamma^3_{12} \\ \Gamma^1_{22} & \Gamma^2_{22} & \Gamma^3_{22} \\ \Gamma^1_{32} & \Gamma^2_{32} & \Gamma^3_{32} \end{pmatrix} \begin{pmatrix} \mathbf{J}_\eta & 0 \\ 0 & |\mathbf{J}_\eta| \end{pmatrix}, \\ &\begin{pmatrix} \overline{\Gamma}^1_{12} & \overline{\Gamma}^2_{12} & \overline{\Gamma}^3_{12} \\ \overline{\Gamma}^1_{22} & \overline{\Gamma}^2_{22} & \overline{\Gamma}^3_{22} \\ \overline{\Gamma}^1_{32} & \overline{\Gamma}^2_{32} & \overline{\Gamma}^3_{32} \end{pmatrix} = \begin{pmatrix} \mathbf{J}_\eta & 0 \\ 0 & |\mathbf{J}_\eta| \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial \mathbf{J}_\eta}{\partial \overline{v}} & 0 \\ 0 & \frac{\partial |\mathbf{J}_\eta|}{\partial \overline{v}} \end{pmatrix} + \\ &\frac{\partial u}{\partial \overline{v}} \begin{pmatrix} \mathbf{J}_\eta & 0 \\ 0 & |\mathbf{J}_\eta| \end{pmatrix}^{-1} \begin{pmatrix} \Gamma^1_{11} & \Gamma^2_{11} & \Gamma^3_{11} \\ \Gamma^1_{21} & \Gamma^2_{21} & \Gamma^3_{21} \\ \Gamma^1_{31} & \Gamma^2_{31} & \Gamma^3_{31} \end{pmatrix} \begin{pmatrix} \mathbf{J}_\eta & 0 \\ 0 & |\mathbf{J}_\eta| \end{pmatrix} + \\ &\frac{\partial v}{\partial \overline{v}} \begin{pmatrix} \mathbf{J}_\eta & 0 \\ 0 & |\mathbf{J}_\eta| \end{pmatrix}^{-1} \begin{pmatrix} \Gamma^1_{12} & \Gamma^2_{12} & \Gamma^3_{12} \\ \Gamma^1_{22} & \Gamma^2_{22} & \Gamma^3_{22} \\ \Gamma^1_{32} & \Gamma^2_{32} & \Gamma^3_{32} \end{pmatrix} \begin{pmatrix} \mathbf{J}_\eta & 0 \\ 0 & |\mathbf{J}_\eta| \end{pmatrix}. \end{aligned} \tag{29}$$

We write $\mathbf{J}_\eta = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ and the above expression can be written as

$$\begin{pmatrix}\overline{\Gamma}^1_{11} & \overline{\Gamma}^2_{11} & \overline{\Gamma}^3_{11}\\ \overline{\Gamma}^1_{21} & \overline{\Gamma}^2_{21} & \overline{\Gamma}^3_{21}\\ \overline{\Gamma}^1_{31} & \overline{\Gamma}^2_{31} & \overline{\Gamma}^3_{31}\end{pmatrix} = \begin{pmatrix}\mathbf{J}_\eta & 0\\ 0 & |\mathbf{J}_\eta|\end{pmatrix}^{-1}\left(\begin{pmatrix}\frac{\partial \mathbf{J}_\eta}{\partial \overline{u}} & 0\\ 0 & \frac{\partial|\mathbf{J}_\eta|}{\partial\overline{u}}\end{pmatrix} + \begin{pmatrix} a\Gamma^1_{11}+b\Gamma^1_{12} & a\Gamma^2_{11}+b\Gamma^2_{12} & a\Gamma^3_{11}+b\Gamma^3_{12}\\ a\Gamma^1_{21}+b\Gamma^1_{22} & a\Gamma^2_{21}+b\Gamma^2_{22} & a\Gamma^3_{21}+b\Gamma^3_{22}\\ a\Gamma^1_{31}+b\Gamma^1_{32} & a\Gamma^2_{31}+b\Gamma^2_{32} & a\Gamma^3_{31}+b\Gamma^3_{32}\end{pmatrix}\begin{pmatrix}\mathbf{J}_\eta & 0\\ 0 & |\mathbf{J}_\eta|\end{pmatrix}\right),$$

$$\begin{pmatrix}\overline{\Gamma}^1_{12} & \overline{\Gamma}^2_{12} & \overline{\Gamma}^3_{12}\\ \overline{\Gamma}^1_{22} & \overline{\Gamma}^2_{22} & \overline{\Gamma}^3_{22}\\ \overline{\Gamma}^1_{32} & \overline{\Gamma}^2_{32} & \overline{\Gamma}^3_{32}\end{pmatrix} = \begin{pmatrix}\mathbf{J}_\eta & 0\\ 0 & |\mathbf{J}_\eta|\end{pmatrix}^{-1}\left(\begin{pmatrix}\frac{\partial \mathbf{J}_\eta}{\partial \overline{v}} & 0\\ 0 & \frac{\partial|\mathbf{J}_\eta|}{\partial\overline{v}}\end{pmatrix} + \begin{pmatrix} c\Gamma^1_{11}+d\Gamma^1_{12} & c\Gamma^2_{11}+d\Gamma^2_{12} & c\Gamma^3_{11}+d\Gamma^3_{12}\\ c\Gamma^1_{21}+d\Gamma^1_{22} & c\Gamma^2_{21}+d\Gamma^2_{22} & c\Gamma^3_{21}+d\Gamma^3_{22}\\ c\Gamma^1_{31}+d\Gamma^1_{32} & c\Gamma^2_{31}+d\Gamma^2_{32} & c\Gamma^3_{31}+d\Gamma^3_{32}\end{pmatrix}\begin{pmatrix}\mathbf{J}_\eta & 0\\ 0 & |\mathbf{J}_\eta|\end{pmatrix}\right).$$

Substituting Γ^i_{jk} and $\overline{\Gamma}^i_{jk}$ from (4) gives

$$\overline{\Gamma}^1_{11} = \frac{2\overline{x}_1\overline{\beta}^4}{\overline{D}}(1+(1+\overline{v}^2)\overline{x}_2^2+\overline{u}\overline{x}_1+2\overline{v}\overline{x}_2+\overline{u}\overline{v}\overline{x}_1\overline{x}_2) = \frac{2\beta^4(ax_1+bx_2)d}{D\mathbf{J}_\eta}\left(a(1+vx_2)^2+ax_2^2-b(1+u^2)x_1x_2+u(1+vx_2)(ax_1-bx_2)\right) -$$
$$\frac{2\beta^4(ax_1+bx_2)c}{D\mathbf{J}_\eta}\left(b(1+ux_1)^2+bx_1^2-a(1+v^2)x_1x_2-v(1+ux_1)(ax_1-bx_2)\right)+\frac{1}{\mathbf{J}_\eta}\left(d\frac{\partial a}{\partial\overline{u}}-c\frac{\partial b}{\partial\overline{u}}\right),$$

$$\overline{\Gamma}^2_{11} = \frac{2\overline{x}_1^2\overline{\beta}^4}{\overline{D}}(\overline{v}+(1+\overline{v}^2)\overline{x}_2^2+\overline{u}\overline{v}\overline{x}_1) = \frac{2\beta^4(ax_1+bx_2)b}{D\mathbf{J}_\eta}\left(a(1+vx_2)^2+ax_2^2-b(1+u^2)x_1x_2+u(1+vx_2)(ax_1-bx_2)\right) -$$
$$\frac{2\beta^4(ax_1+bx_2)d}{D\mathbf{J}_\eta}\left(b(1+ux_1)^2+bx_1^2-a(1+v^2)x_1x_2-v(1+ux_1)(ax_1-bx_2)\right)+\frac{1}{\mathbf{J}_\eta}\left(b\frac{\partial a}{\partial\overline{u}}-a\frac{\partial b}{\partial\overline{u}}\right),$$

$$\overline{\Gamma}^1_{12} = \overline{\Gamma}^1_{21} = \frac{\overline{x}_2\overline{\beta}^4}{\overline{D}}(1+(1+\overline{v}^2)\overline{x}_2^2-(1+\overline{u}^2)\overline{x}_1^2+2\overline{v}\overline{x}_2) = \frac{\beta^4(ax_1+bx_2)d}{D\mathbf{J}_\eta}\left(c(1+vx_2)^2+cx_2^2-d(1+u^2)x_1x_2+u(1+vx_2)(cx_1-dx_2)\right) -$$
$$\frac{\beta^4(ax_1+bx_2)c}{D\mathbf{J}_\eta}\left(d(1+ux_1)^2+dx_1^2-c(1+v^2)x_1x_2-u(1+vx_2)(ax_1-bx_2)\right)+$$
$$\frac{\beta^4(cx_1+dx_2)d}{D\mathbf{J}_\eta}\left(a(1+vx_2)^2+ax_2^2-b(1+u^2)x_1x_2+u(1+vx_2)(ax_1-bx_2)\right)-$$
$$\frac{\beta^4(cx_1+dx_2)c}{D\mathbf{J}_\eta}\left(b(1+ux_1)^2+bx_1^2-a(1+v^2)x_1x_2-v(1+ux_1)(cx_1-dx_2)\right)+\frac{1}{\mathbf{J}_\eta}\left(d\frac{\partial c}{\partial\overline{u}}-c\frac{\partial d}{\partial\overline{u}}\right),$$

$$\overline{\Gamma}^2_{12} = \overline{\Gamma}^2_{21} = \frac{\overline{x}_1\overline{\beta}^4}{\overline{D}}(1+(1+\overline{u}^2)\overline{x}_1^2-(1+\overline{v}^2)\overline{x}_2^2+2\overline{u}\overline{x}_1) = \frac{\beta^4(ax_1+bx_2)a}{D\mathbf{J}_\eta}\left(d(1+ux_1)^2+dx_1^2-c(1+v^2)x_1x_2-v(1+ux_1)(ax_1-bx_2)\right) -$$
$$\frac{\beta^4(ax_1+bx_2)a}{D\mathbf{J}_\eta}\left(c(1+vx_2)^2+cx_2^2-d(1+u^2)x_1x_2+u(1+vx_2)(cx_1-dx_2)\right)+$$
$$\frac{\beta^4(cx_1+dx_2)a}{D\mathbf{J}_\eta}\left(b(1+ux_1)^2+bx_1^2-a(1+v^2)x_1x_2-v(1+ux_1)(cx_1-dx_2)\right)-$$
$$\frac{\beta^4(cx_1+dx_2)b}{D\mathbf{J}_\eta}\left(a(1+vx_2)^2+cx_2^2-d(1+u^2)x_1x_2+u(1+vx_2)(ax_1-bx_2)\right)-\frac{1}{\mathbf{J}_\eta}\left(b\frac{\partial c}{\partial\overline{u}}-a\frac{\partial d}{\partial\overline{u}}\right),$$

$$\overline{\Gamma}^1_{22} = -\frac{2\overline{x}_2^2\overline{\beta}^4}{\overline{D}}(\overline{u}+(1+\overline{u}^2)\overline{x}_1^2+\overline{u}\overline{v}\overline{x}_2) = \frac{2\beta^4(cx_1+dx_2)d}{D\mathbf{J}_\eta}\left(c(1+vx_2)^2+cx_2^2-d(1+u^2)x_1x_2+u(1+vx_2)(cx_1-dx_2)\right) -$$
$$\frac{2\beta^4(cx_1+dx_2)c}{D\mathbf{J}_\eta}\left(d(1+ux_1)^2+dx_1^2-c(1+v^2)x_1x_2-v(1+ux_1)(cx_1-dx_2)\right)+\frac{1}{\mathbf{J}_\eta}\left(d\frac{\partial c}{\partial\overline{v}}-c\frac{\partial d}{\partial\overline{v}}\right),$$

$$\overline{\Gamma}^2_{22} = \frac{2\overline{x}_2\overline{\beta}^4}{\overline{D}}(1+(1+\overline{u}^2)\overline{x}_1^2+\overline{v}\overline{x}_2+2\overline{u}\overline{x}_1+\overline{u}\overline{v}\overline{x}_1\overline{x}_2) = \frac{2\beta^4(cx_1+dx_2)a}{D\mathbf{J}_\eta}\left(d(1+ux_1)^2+dx_1^2-c(1+v^2)x_1x_2-v(1+ux_1)(cx_1-dx_2)\right) -$$
$$\frac{2\beta^4(cx_1+dx_2)b}{D\mathbf{J}_\eta}\left(c(1+vx_2)^2+cx_2^2-d(1+u^2)x_1x_2+u(1+vx_2)(cx_1-dx_2)\right)+\frac{1}{\mathbf{J}_\eta}\left(a\frac{\partial d}{\partial\overline{v}}-b\frac{\partial c}{\partial\overline{v}}\right),$$

$$\overline{\Gamma}^3_{31} = \overline{x}_1+\frac{2\overline{\beta}^4\overline{x}_1}{\overline{D}}(1+\overline{u}\overline{x}_1+\overline{v}\overline{x}_2) = (ax_1+bx_2)\left(1+\frac{2\beta^4}{D}(1+ux_1+vx_2)\right)+|\mathbf{J}_\eta|^{-1}\frac{\partial|\mathbf{J}_\eta|}{\partial\overline{u}},$$

$$\overline{\Gamma}^3_{32} = \overline{x}_2+\frac{2\overline{\beta}^4\overline{x}_2}{\overline{D}}(1+\overline{u}\overline{x}_1+\overline{v}\overline{x}_2) = (cx_1+dx_2)\left(1+\frac{2\beta^4}{D}(1+ux_1+vx_2)\right)+|\mathbf{J}_\eta|^{-1}\frac{\partial|\mathbf{J}_\eta|}{\partial\overline{v}},$$

$$\overline{\Gamma}^3_{11} = \frac{\overline{x}_1^2\overline{\beta}^3}{\overline{D}} = \frac{\beta^3(ax_1+bx_2)^2}{|\mathbf{J}_\eta|D},$$

$$\overline{\Gamma}_{12}^3 = \overline{\Gamma}_{21}^3 = \frac{\overline{x}_1\overline{x}_2\overline{\beta}^3}{\overline{D}} = \frac{\beta^3(ax_1+bx_2)(cx_1+dx_2)}{|\mathbf{J}_\eta|D},$$

$$\overline{\Gamma}_{22}^3 = \frac{\overline{x}_2^2\overline{\beta}^3}{\overline{D}} = \frac{\beta^3(cx_1+dx_2)^2}{|\mathbf{J}_\eta|D},$$

$$\overline{\Gamma}_{31}^1 = \frac{\overline{\beta}^5(\overline{x}_1^2+\overline{v}\overline{x}_1^2\overline{x}_2-\overline{u}\overline{x}_1\overline{x}_2^2)}{\overline{D}} = \frac{\beta^5(ax_1+bx_2)}{D}\left(d(x_1+vx_1x_2-ux_2^2)-c(x_2+ux_1x_2-vx_1^2)\right),$$

$$\overline{\Gamma}_{31}^2 = \frac{\overline{\beta}^5(\overline{x}_1\overline{x}_2+\overline{u}\overline{x}_1^2\overline{x}_2-\overline{v}\overline{x}_1^3)}{\overline{D}} = \frac{\beta^5(ax_1+cx_2)}{D}\left(-b(x_1+vx_1x_2-ux_2^2)+a(x_2+ux_1x_2-vx_1^2)\right),$$

$$\overline{\Gamma}_{32}^1 = \frac{\overline{\beta}^5(\overline{x}_1\overline{x}_2+\overline{v}\overline{x}_1\overline{x}_2^2-\overline{u}\overline{x}_2^3)}{\overline{D}} = \frac{\beta^5(cx_1+dx_2)}{D}\left(d(x_1+vx_1x_2-ux_2^2)-c(x_2+ux_1x_2-vx_1^2)\right),$$

$$\overline{\Gamma}_{32}^2 = \frac{\overline{\beta}^5(\overline{x}_2^2+\overline{u}\overline{x}_1\overline{x}_2^2-\overline{v}\overline{x}_1^2\overline{x}_2)}{\overline{D}} = \frac{\beta^5(cx_1+dx_2)}{D}\left(-b(x_1+vx_1x_2-ux_2^2)+a(x_2+ux_1x_2-vx_1^2)\right).$$

From the last expressions, we can that the last 8 connections ($\overline{\Gamma}_{11}^3, \overline{\Gamma}_{12}^3, \overline{\Gamma}_{12}^3, \overline{\Gamma}_{21}^3, \overline{\Gamma}_{22}^3, \overline{\Gamma}_{31}^1, \overline{\Gamma}_{31}^2, \overline{\Gamma}_{32}^1, \overline{\Gamma}_{32}^2$), do not contain second order derivatives of η.