Supplementary Material for
Zero-Assignment Constraint for Graph Matching with Outliers

In the following supplementary document, we present both the detailed proof for proposition 4 in our manuscript and more comparison results. To make it more intuitive and easy to understand, we will present the proof accompanied with an example shown in Fig. 1.

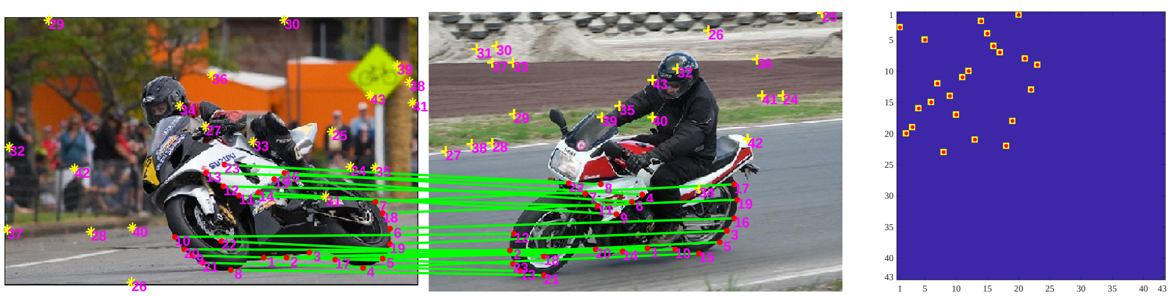


Figure 1: Left: the visualization of the ideal matching $\mathbf{P}^*$ between two graphs. Right: the ideal ground-truth correspondence matrix $\mathbf{P}^*$ (red markers) and a specific correspondence matrix $\mathbf{P} = \mathbf{P}^*$ (yellow markers).

1. Proof for proposition 4

The two graphs $\mathcal{G}$ and $\mathcal{G}'$ in Fig. 1 consist of 23 inliers (red dots) and 20 outliers (yellow signs), respectively. Without ambiguity, we can rerange the inliers as the first $k = 23$ nodes in the two graphs to make the visualization more clear. Therefore, the index sets $\mathscr{A}_I, \mathscr{A}_O, \mathscr{B}_I, \mathscr{B}_O$ defined in our manuscript can be specifically written as

$$\mathscr{A}_I = \{1, 2, ..., 23\}, \quad \mathscr{A}_O = \{24, 25, ..., 43\}, \quad \mathscr{B}_I = \{1, 2, ..., 23\}, \quad \mathscr{B}_O = \{24, 25, ..., 43\}. \tag{1}$$

The numbers shown in pink are the indices of nodes in the two graphs. As we can see, the ideal ground-truth correspondence can be expressed as

$$\begin{array}{ccccccccccccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 20 & 14 & 1 & 15 & 5 & 16 & 17 & 21 & 23 & 12 & 11 & 7 & 22 & 9 & 6 & 4 & 10 & 19 & 3 & 2 & 13 & 18 & 8 \end{array}.$$

To make the proof more clear, we first define some basic notations as follows. Given arbitrary partial permutation $\tau : \mathscr{A} \to \mathscr{B}$ and its compatible partial permutation matrix (*i.e.*, correspondence matrix) $\mathbf{P} \in \mathcal{P}_k$, we define that,

$$\mathcal{A}_I^{\mathbf{P},0} \triangleq \{i \in \mathscr{A}_I; \tau(i) = \varnothing\}, \quad \mathcal{A}_I^{\mathbf{P},1} \triangleq \{i \in \mathscr{A}_I; \tau(i) \neq \tau^*(i)\}, \quad \mathcal{A}_I^{\mathbf{P},*} \triangleq \{i \in \mathscr{A}_I; \tau(i) = \tau^*(i)\},$$
$$\mathcal{A}_O^{\mathbf{P},0} \triangleq \{i \in \mathscr{A}_O; \tau(i) = \varnothing\}, \quad \mathcal{A}_O^{\mathbf{P},1} \triangleq \{i \in \mathscr{A}_O; \tau(i) \neq \varnothing\}.$$

These index sets can be equally defined as

$$\mathcal{A}_I^{\mathbf{P},0} \triangleq \{i \in \mathscr{A}_I; \mathbf{P}_{i,:} \equiv \mathbf{0}\}, \quad \mathcal{A}_I^{\mathbf{P},1} \triangleq \{i \in \mathscr{A}_I; \exists a \neq \tau^*(i), \mathbf{P}_{ia} = 1\}, \quad \mathcal{A}_I^{\mathbf{P},*} \triangleq \{i \in \mathscr{A}_I; \exists a = \tau^*(i), \mathbf{P}_{ia} = 1\},$$
$$\mathcal{A}_O^{\mathbf{P},0} \triangleq \{i \in \mathscr{A}_O; \mathbf{P}_{i,:} \equiv \mathbf{0}\}, \quad \mathcal{A}_O^{\mathbf{P},1} \triangleq \{i \in \mathscr{A}_O; \exists a \in \mathscr{B}, \mathbf{P}_{ia} = 1\}.$$

We can directly obtain some basic properties of the index sets above.

Proposition 1.

- *Complementary*

$$\mathcal{A}_I^{\mathbf{P},0} \cup \mathcal{A}_I^{\mathbf{P},1} \cup \mathcal{A}_I^{\mathbf{P},*} = \mathscr{A}_I, \quad \mathcal{A}_O^{\mathbf{P},0} \cup \mathcal{A}_O^{\mathbf{P},1} = \mathscr{A}_O. \tag{2}$$

- *Disjoint*

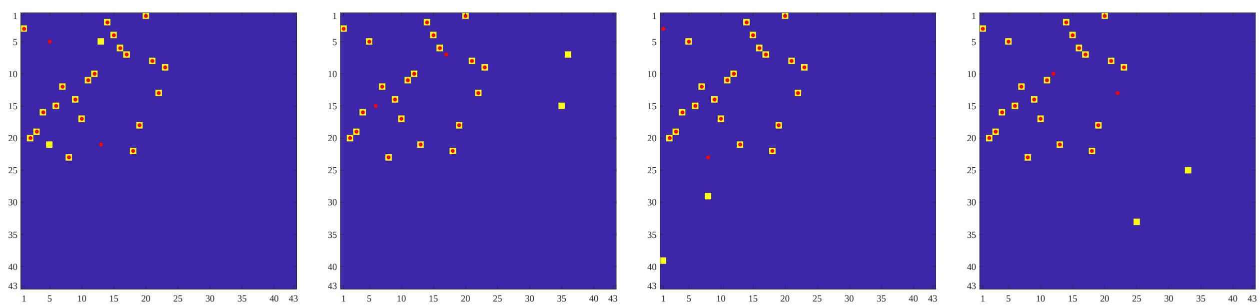
$$\mathcal{A}_I^{\mathbf{P},0} \cap \mathcal{A}_I^{\mathbf{P},1} = \mathcal{A}_I^{\mathbf{P},0} \cap \mathcal{A}_I^{\mathbf{P},*} = \mathcal{A}_I^{\mathbf{P},1} \cap \mathcal{A}_I^{\mathbf{P},*} = \varnothing, \quad \mathcal{A}_O^{\mathbf{P},0} \cap \mathcal{A}_O^{\mathbf{P},1} = \varnothing. \tag{3}$$

- *For the ideal ground-truth correspondence* $\mathbf{P}^*$

$$\mathcal{A}_I^{\mathbf{P}^*,0} = \mathcal{A}_I^{\mathbf{P}^*,1} = \varnothing, \quad \mathcal{A}_I^{\mathbf{P},*} \subseteq \mathcal{A}_I^{\mathbf{P}^*,*} = \mathscr{A}_I, \quad \mathcal{A}_O^{\mathbf{P},0} \subseteq \mathcal{A}_O^{\mathbf{P}^*,0} = \mathscr{A}_O, \quad \mathcal{A}_O^{\mathbf{P}^*,1} = \varnothing. \tag{4}$$

Intuitively, the definitions of sets $\mathcal{A}_I^{\mathbf{P},1}, \mathcal{A}_I^{\mathbf{P},0}, \mathcal{A}_O^{\mathbf{P},1}$ can be expressed as the disturbances to the ground-truth correspondence $\mathbf{P}^*$ caused by the given correspondence $\mathbf{P} \in \mathcal{P}_k$. As shown in Fig. 2, there are 4 typical kinds of disturbances.

- $\mathcal{A}_I^{\mathbf{P},1}$: disturbances resulting in incorrect matchings of inliers in $\mathscr{A}$. There are 2 kinds of disturbances: (I) incorrect matchings between inliers in $\mathscr{A}_I$ and inliers in $\mathscr{B}_I$, (II) incorrect matchings between inliers in $\mathscr{A}_I$ and outliers in $\mathscr{B}_O$.
- $\mathcal{A}_I^{\mathbf{P},0}, \mathcal{A}_O^{\mathbf{P},1}$: disturbances resulting in redundant matchings of outliers in $\mathscr{A}$. There are also 2 kinds of disturbances: (III) redundant matchings between outliers in $\mathscr{A}_O$ and inliers in $\mathscr{B}_I$, (IV) redundant matchings between outliers in $\mathscr{A}_O$ and outliers in $\mathscr{B}_O$. Note that, we have $|\mathcal{A}_I^{\mathbf{P},0}| = |\mathcal{A}_O^{\mathbf{P},1}|$: Eq. (2), (3) mean that $|\mathcal{A}_I^{\mathbf{P},0}| + |\mathcal{A}_I^{\mathbf{P},*}| + |\mathcal{A}_O^{\mathbf{P},1}| = k$; due to the constraint $\mathbf{1}^{\mathrm{T}}\mathbf{P}\mathbf{1} = k$, it has to hold $|\mathcal{A}_I^{\mathbf{P},1}| + |\mathcal{A}_I^{\mathbf{P},*}| + |\mathcal{A}_O^{\mathbf{P},1}| = k$. Therefore, it derives $|\mathcal{A}_I^{\mathbf{P},0}| = |\mathcal{A}_O^{\mathbf{P},1}|$.



(a) Disturbance I (b) Disturbance II (c) Disturbance III (d) Disturbance IV

Figure 2: The 4 typical kinds of disturbances cased by giving arbitrary correspondence matrices $\mathbf{P} \in \mathcal{P}_k$ (yellow signs). The red dots in each subfigure are the ideal ground-truth correspondence matrix $\mathbf{P}^*$.

As mentioned in Sec.3.4 in our manuscript, we first prove that $\forall \mathbf{P} \in \mathcal{P}_k, F_u(\mathbf{P}) \geq F_u(\mathbf{P}^*)$.

Proof.

$$F_u(\mathbf{P}) = \sum_{ia} \mathbf{D}_{ia}\mathbf{P}_{ia} = \sum_{\substack{i \in \mathcal{A}_I^{\mathbf{P},0} \\ a \in \mathscr{B}}} \mathbf{D}_{ia}\mathbf{P}_{ia} + \sum_{\substack{i \in \mathcal{A}_I^{\mathbf{P},1} \\ a \in \mathscr{B}}} \mathbf{D}_{ia}\mathbf{P}_{ia} + \sum_{\substack{i \in \mathcal{A}_I^{\mathbf{P},*} \\ a \in \mathscr{B}}} \mathbf{D}_{ia}\mathbf{P}_{ia} + \sum_{\substack{i \in \mathcal{A}_O^{\mathbf{P},0} \\ a \in \mathscr{B}}} \mathbf{D}_{ia}\mathbf{P}_{ia} + \sum_{\substack{i \in \mathcal{A}_O^{\mathbf{P},1} \\ a \in \mathscr{B}}} \mathbf{D}_{ia}\mathbf{P}_{ia} \tag{5}$$

$$= 0 + \sum_{\substack{i \in \mathcal{A}_I^{\mathbf{P},1} \\ a \in \mathscr{B}}} \mathbf{D}_{ia}\mathbf{P}_{ia} + \sum_{\substack{i \in \mathcal{A}_I^{\mathbf{P},*} \\ a \in \mathscr{B}}} \mathbf{D}_{ia}\mathbf{P}_{ia} + 0 + \sum_{\substack{i \in \mathcal{A}_O^{\mathbf{P},1} \\ a \in \mathscr{B}}} \mathbf{D}_{ia}\mathbf{P}_{ia} \tag{6}$$

$$= \sum_{i \in \mathcal{A}_I^{\mathbf{P},1}} \mathbf{D}_{i,a=\tau(i)\neq\tau^*(i)} + \sum_{i \in \mathcal{A}_I^{\mathbf{P},*}} \mathbf{D}_{i,a=\tau^*(i)} + \sum_{i \in \mathcal{A}_O^{\mathbf{P},1}} \mathbf{D}_{i,a=\tau(i)\neq\tau^*(i)} \tag{7}$$

$$\geq \sum_{i \in \mathcal{A}_I^{\mathbf{P},1}} \mathbf{D}_{i,a=\tau^*(i)} + \sum_{i \in \mathcal{A}_I^{\mathbf{P},*}} \mathbf{D}_{i,a=\tau^*(i)} + \sum_{i \in \mathcal{A}_I^{\mathbf{P},0}} \mathbf{D}_{i,a=\tau^*(i)} = F_u(\mathbf{P}^*). \tag{8}$$

From Eq. (7) to Eq. (8), it holds due to the unary consistency and distinguishability expressed in proposition 2 and 3 in our manuscript. Note that, the equation holds if and only if $\mathcal{A}_I^{\mathbf{P},1} = \varnothing, \mathcal{A}_O^{\mathbf{P},1} = \varnothing$, which means that $\mathbf{P} = \mathbf{P}^*$. □

Next, we demonstrate that it also holds for the pairwise potential $F_p(\mathbf{P})$. We begin with $F_{p_1}(\mathbf{P}) = ||\mathbf{A} - \mathbf{P}\mathbf{B}\mathbf{P}^{\mathrm{T}}||^2_{\mathcal{E}}$ and prove that $\forall \mathbf{P} \in \mathcal{P}_k, F_{p_1}(\mathbf{P}) \geq F_{p_1}(\mathbf{P}^*)$. The proof is mainly organized based on the basic set operations, such as intersection, union, complement, Cartesian product, *etc.*.

Proof. First, we divide $F_{p_1}(\mathbf{P})$ into two disjoint parts according to the index sets defined above,

$$F_{p_1}(\mathbf{P}) = ||\mathbf{A} - \mathbf{P}\mathbf{B}\mathbf{P}^{\mathrm{T}}||^2_{\mathcal{E}} = \sum_{ij} \mathcal{E}_{ij}(\mathbf{A}_{ij} - \sum_{a,b} \mathbf{P}_{ia}\mathbf{B}_{ab}\mathbf{P}_{jb})^2 = \sum_{ij} \mathcal{E}_{ij}(\mathbf{A}_{ij} - \mathbf{P}_{i,:}\mathbf{B}\mathbf{P}^{\mathrm{T}}_{j,:})^2 \tag{9}$$

$$= \sum_{i \in \mathscr{A}_I, j \in \mathscr{A}_I} \mathcal{E}_{ij} ||\mathbf{A}_{ij} - \mathbf{P}_{i,:}\mathbf{B}\mathbf{P}^{\mathrm{T}}_{j,:}||^2 \tag{10}$$

$$+ \sum_{\substack{i \in \mathscr{A}_O, j \in \mathscr{A} \\ \text{or } i \in \mathscr{A}, j \in \mathscr{A}_O}} \mathcal{E}_{ij} ||\mathbf{A}_{ij} - \mathbf{P}_{i,:}\mathbf{B}\mathbf{P}^{\mathrm{T}}_{j,:}||^2. \tag{11}$$

$$(10) = \sum_{\substack{i \in \mathcal{A}_I^{\mathbf{P},0} \\ j \in \mathscr{A}_I}} \mathcal{E}_{ij} ||\mathbf{A}_{ij} - \mathbf{P}_{i,:}\mathbf{B}\mathbf{P}^{\mathrm{T}}_{j,:}||^2 + \sum_{\substack{i \in \mathcal{A}_I^{\mathbf{P},1} \\ j \in \mathscr{A}_I}} \mathcal{E}_{ij} ||\mathbf{A}_{ij} - \mathbf{P}_{i,:}\mathbf{B}\mathbf{P}^{\mathrm{T}}_{j,:}||^2 + \sum_{\substack{i \in \mathcal{A}_I^{\mathbf{P},*} \\ j \in \mathscr{A}_I}} \mathcal{E}_{ij} ||\mathbf{A}_{ij} - \mathbf{P}_{i,:}\mathbf{B}\mathbf{P}^{\mathrm{T}}_{j,:}||^2. \tag{12}$$

$$(11) = \sum_{\substack{i \in \mathcal{A}_O^{\mathbf{P},0}, j \in \mathscr{A} \\ \text{or } i \in \mathscr{A}, j \in \mathcal{A}_O^{\mathbf{P},0}}} \mathcal{E}_{ij} ||\mathbf{A}_{ij} - \mathbf{P}_{i,:}\mathbf{B}\mathbf{P}^{\mathrm{T}}_{j,:}||^2 \quad + \sum_{\substack{i \in \mathcal{A}_O^{\mathbf{P},1}, j \in \mathscr{A} / \mathcal{A}_O^{\mathbf{P},0} \\ \text{or } i \in \mathscr{A} / \mathcal{A}_O^{\mathbf{P},0}, j \in \mathcal{A}_O^{\mathbf{P},1}}} \mathcal{E}_{ij} ||\mathbf{A}_{ij} - \mathbf{P}_{i,:}\mathbf{B}\mathbf{P}^{\mathrm{T}}_{j,:}||^2. \tag{13}$$

Due to the proposition 1 above, we can see that,

$$F_{p_1}(\mathbf{P}^*) = \sum_{\substack{i \in \mathcal{A}_I^{\mathbf{P}^*,*} \\ j \in \mathscr{A}_I}} \mathcal{E}_{ij} ||\mathbf{A}_{ij} - \mathbf{P}^*_{i,:}\mathbf{B}\mathbf{P}^{*\mathrm{T}}_{j,:}||^2 \quad + \sum_{\substack{i \in \mathcal{A}_O^{\mathbf{P}^*,0}, j \in \mathscr{A} \\ \text{or } i \in \mathscr{A}, j \in \mathcal{A}_O^{\mathbf{P}^*,0}}} \mathcal{E}_{ij} ||\mathbf{A}_{ij} - \mathbf{P}^*_{i,:}\mathbf{B}\mathbf{P}^{*\mathrm{T}}_{j,:}||^2 \tag{14}$$

$$= \sum_{\substack{i \in \mathcal{A}_I^{\mathbf{P}^*,*} \\ j \in \mathcal{A}_I^{\mathbf{P}^*,*}}} \mathcal{E}_{ij} ||\mathbf{A}_{ij} - \mathbf{B}_{\tau^*(i)\tau^*(j)}||^2 \quad + \sum_{\substack{i \in \mathcal{A}_O^{\mathbf{P}^*,0}, j \in \mathscr{A} \\ \text{or } i \in \mathscr{A}, j \in \mathcal{A}_O^{\mathbf{P}^*,0}}} \mathcal{E}_{ij} ||\mathbf{A}_{ij} - 0||^2 \tag{15}$$

$$= \sum_{\substack{i \in \mathscr{A}_I \\ j \in \mathscr{A}_I}} \mathcal{E}_{ij} ||\mathbf{A}_{ij} - \mathbf{B}_{\tau^*(i)\tau^*(j)}||^2 \quad + \sum_{\substack{i \in \mathscr{A}_O, j \in \mathscr{A} \\ \text{or } i \in \mathscr{A}, j \in \mathscr{A}_O}} \mathcal{E}_{ij} ||\mathbf{A}_{ij} - 0||^2. \tag{16}$$

Then, for $\forall \mathbf{P} \in \mathcal{P}_k$ and $\mathbf{P} \neq \mathbf{P}^*$, we reorganize Eq. (10) and Eq. (11) based on the index sets.

$$(10) = \sum_{\substack{i \in \mathcal{A}_I^{\mathbf{P},0} \\ j \in \mathscr{A}_I}} \mathcal{E}_{ij} ||\mathbf{A}_{ij} - 0||^2 + \sum_{\substack{i \in \mathcal{A}_I^{\mathbf{P},1} \\ j \in \mathcal{A}_I^{\mathbf{P},0}}} \mathcal{E}_{ij} ||\mathbf{A}_{ij} - 0||^2 + \sum_{\substack{i \in \mathcal{A}_I^{\mathbf{P},1} \\ j \in \mathscr{A}_I / \mathcal{A}_I^{\mathbf{P},0}}} \mathcal{E}_{ij} ||\mathbf{A}_{ij} - \mathbf{B}_{\tau(i)\tau(j)}||^2 \tag{17}$$

$$+ \sum_{\substack{i \in \mathcal{A}_I^{\mathbf{P},*} \\ j \in \mathcal{A}_I^{\mathbf{P},0}}} \mathcal{E}_{ij} ||\mathbf{A}_{ij} - 0||^2 + \sum_{\substack{i \in \mathcal{A}_I^{\mathbf{P},*} \\ j \in \mathcal{A}_I^{\mathbf{P},1}}} \mathcal{E}_{ij} ||\mathbf{A}_{ij} - \mathbf{B}_{\tau(i)\tau(j)}||^2 + \sum_{\substack{i \in \mathcal{A}_I^{\mathbf{P},*} \\ j \in \mathcal{A}_I^{\mathbf{P},*}}} \mathcal{E}_{ij} ||\mathbf{A}_{ij} - \mathbf{B}_{\tau(i)\tau(j)}||^2 \tag{18}$$

$$= \sum_{\substack{i \in \mathcal{A}_I^{\mathbf{P},0} \\ j \in \mathscr{A}_I / \mathcal{A}_I^{\mathbf{P},0}}} \mathcal{E}_{ij} ||\mathbf{A}_{ij}||^2 + \sum_{\substack{i \in \mathcal{A}_I^{\mathbf{P},0} \\ j \in \mathcal{A}_I^{\mathbf{P},0}}} \mathcal{E}_{ij} ||\mathbf{A}_{ij}||^2 + \sum_{\substack{i \in \mathscr{A}_I / \mathcal{A}_I^{\mathbf{P},0} \\ j \in \mathcal{A}_I^{\mathbf{P},0}}} \mathcal{E}_{ij} ||\mathbf{A}_{ij}||^2 \tag{19}$$

$$+ \sum_{\substack{i \in \mathcal{A}_I^{\mathbf{P},1} \\ j \in \mathscr{A}_I / \mathcal{A}_I^{\mathbf{P},0}}} \mathcal{E}_{ij} ||\mathbf{A}_{ij} - \mathbf{B}_{\tau(i)\tau(j)}||^2 + \sum_{\substack{i \in \mathcal{A}_I^{\mathbf{P},*} \\ j \in \mathcal{A}_I^{\mathbf{P},1}}} \mathcal{E}_{ij} ||\mathbf{A}_{ij} - \mathbf{B}_{\tau(i)\tau(j)}||^2 + \sum_{\substack{i \in \mathcal{A}_I^{\mathbf{P},*} \\ j \in \mathcal{A}_I^{\mathbf{P},*}}} \mathcal{E}_{ij} ||\mathbf{A}_{ij} - \mathbf{B}_{\tau(i)\tau(j)}||^2 \tag{20}$$

$$= \sum_{\substack{i \in \mathcal{A}_I^{\mathbf{P},0} \\ j \in \mathscr{A}_I / \mathcal{A}_I^{\mathbf{P},0}}} \mathcal{E}_{ij} ||\mathbf{A}_{ij}||^2 + \sum_{\substack{i \in \mathscr{A}_I / \mathcal{A}_I^{\mathbf{P},0} \\ j \in \mathcal{A}_I^{\mathbf{P},0}}} \mathcal{E}_{ij} ||\mathbf{A}_{ij}||^2 + \sum_{\substack{i \in \mathcal{A}_I^{\mathbf{P},0} \\ j \in \mathcal{A}_I^{\mathbf{P},0}}} \mathcal{E}_{ij} ||\mathbf{A}_{ij}||^2 + \sum_{\substack{i \in \mathcal{A}_I^{\mathbf{P},*} \cup \mathcal{A}_I^{\mathbf{P},1} \\ j \in \mathcal{A}_I^{\mathbf{P},*} \cup \mathcal{A}_I^{\mathbf{P},1}}} \mathcal{E}_{ij} ||\mathbf{A}_{ij} - \mathbf{B}_{\tau(i)\tau(j)}||^2. \tag{21}$$

$$(11) = \sum_{\substack{i\in\mathcal{A}_O^{\mathbf{P},0}, j\in\mathscr{A} \\ \text{or } i\in\mathscr{A}, j\in\mathcal{A}_O^{\mathbf{P},0}}} \mathcal{E}_{ij}||\mathbf{A}_{ij} - 0||^2 + \sum_{\substack{i\in\mathcal{A}_O^{\mathbf{P},1}, j\in\mathcal{A}_I^{\mathbf{P},0} \\ \text{or } i\in\mathcal{A}_I^{\mathbf{P},0}, j\in\mathcal{A}_O^{\mathbf{P},1}}} \mathcal{E}_{ij}||\mathbf{A}_{ij} - 0||^2 \tag{22}$$

$$+ \sum_{\substack{i\in\mathcal{A}_O^{\mathbf{P},1} \\ j\in\mathscr{A}_I/\mathcal{A}_I^{\mathbf{P},0}}} \mathcal{E}_{ij}||\mathbf{A}_{ij} - \mathbf{B}_{\tau(i)\tau(j)}||^2 + \sum_{\substack{i\in\mathscr{A}_I/\mathcal{A}_I^{\mathbf{P},0} \\ j\in\mathcal{A}_O^{\mathbf{P},1}}} \mathcal{E}_{ij}||\mathbf{A}_{ij} - \mathbf{B}_{\tau(i)\tau(j)}||^2 + \sum_{\substack{i\in\mathcal{A}_O^{\mathbf{P},1} \\ j\in\mathcal{A}_O^{\mathbf{P},1}}} \mathcal{E}_{ij}||\mathbf{A}_{ij} - \mathbf{B}_{\tau(i)\tau(j)}||^2. \tag{23}$$

At last, we prove $F_{p_1}(\mathbf{P}) - F_{p_1}(\mathbf{P}^*) \geq 0$. To make it more clear, we still reorganize these terms by their index sets.

$$F_{p_1}(\mathbf{P}) - F_{p_1}(\mathbf{P}^*)$$

$$= [\sum_{\substack{i\in\mathcal{A}_I^{\mathbf{P},0} \\ j\in\mathscr{A}_I/\mathcal{A}_I^{\mathbf{P},0}}} \mathcal{E}_{ij}||\mathbf{A}_{ij}||^2 + \sum_{\substack{i\in\mathscr{A}_I/\mathcal{A}_I^{\mathbf{P},0} \\ j\in\mathcal{A}_I^{\mathbf{P},0}}} \mathcal{E}_{ij}||\mathbf{A}_{ij}||^2 + \sum_{\substack{i\in\mathcal{A}_I^{\mathbf{P},0} \\ j\in\mathcal{A}_I^{\mathbf{P},0}}} \mathcal{E}_{ij}||\mathbf{A}_{ij}||^2] + \sum_{\substack{i\in\mathcal{A}_I^{\mathbf{P},*}\cup\mathcal{A}_I^{\mathbf{P},1} \\ j\in\mathcal{A}_I^{\mathbf{P},*}\cup\mathcal{A}_I^{\mathbf{P},1}}} \mathcal{E}_{ij}||\mathbf{A}_{ij} - \mathbf{B}_{\tau(i)\tau(j)}||^2 \tag{24}$$

$$-[\sum_{\substack{i\in\mathcal{A}_I^{\mathbf{P},0} \\ j\in\mathscr{A}_I/\mathcal{A}_I^{\mathbf{P},0}}} + \sum_{\substack{i\in\mathscr{A}_I/\mathcal{A}_I^{\mathbf{P},0} \\ j\in\mathcal{A}_I^{\mathbf{P},0}}} + \sum_{\substack{i\in\mathcal{A}_I^{\mathbf{P},0} \\ j\in\mathcal{A}_I^{\mathbf{P},0}}}]\mathcal{E}_{ij}||\mathbf{A}_{ij} - \mathbf{B}_{\tau^*(i)\tau^*(j)}||^2 - \sum_{\substack{i\in\mathcal{A}_I^{\mathbf{P},*}\cup\mathcal{A}_I^{\mathbf{P},1} \\ j\in\mathcal{A}_I^{\mathbf{P},*}\cup\mathcal{A}_I^{\mathbf{P},1}}} \mathcal{E}_{ij}||\mathbf{A}_{ij} - \mathbf{B}_{\tau^*(i)\tau^*(j)}||^2 \tag{25}$$

$$+[\sum_{\substack{i\in\mathcal{A}_O^{\mathbf{P},1} \\ j\in\mathscr{A}_I/\mathcal{A}_I^{\mathbf{P},0}}} \mathcal{E}_{ij}||\mathbf{A}_{ij} - \mathbf{B}_{\tau(i)\tau(j)}||^2 + \sum_{\substack{i\in\mathscr{A}_I/\in\mathcal{A}_I^{\mathbf{P},0} \\ j\in\mathcal{A}_O^{\mathbf{P},1}}} \mathcal{E}_{ij}||\mathbf{A}_{ij} - \mathbf{B}_{\tau(i)\tau(j)}||^2 + \sum_{\substack{i\in\mathcal{A}_O^{\mathbf{P},1} \\ j\in\mathcal{A}_O^{\mathbf{P},1}}} \mathcal{E}_{ij}||\mathbf{A}_{ij} - \mathbf{B}_{\tau(i)\tau(j)}||^2] \tag{26}$$

$$-[\sum_{\substack{i\in\mathcal{A}_O^{\mathbf{P},1} \\ j\in\mathscr{A}_I/\mathcal{A}_I^{\mathbf{P},0}}} \mathcal{E}_{ij}||\mathbf{A}_{ij}||^2 + \sum_{\substack{i\in\mathscr{A}_I/\mathcal{A}_I^{\mathbf{P},0} \\ j\in\mathcal{A}_O^{\mathbf{P},1}}} \mathcal{E}_{ij}||\mathbf{A}_{ij}||^2 + \sum_{\substack{i\in\mathcal{A}_O^{\mathbf{P},1} \\ j\in\mathcal{A}_O^{\mathbf{P},1}}} \mathcal{E}_{ij}||\mathbf{A}_{ij}||^2] \tag{27}$$

$$= [\sum_{\substack{i\in\mathcal{A}_I^{\mathbf{P},0} \\ j\in\mathscr{A}_I/\mathcal{A}_I^{\mathbf{P},0}}} + \sum_{\substack{i\in\mathscr{A}_I/\mathcal{A}_I^{\mathbf{P},0} \\ j\in\mathcal{A}_I^{\mathbf{P},0}}} + \sum_{\substack{i\in\mathcal{A}_I^{\mathbf{P},0} \\ j\in\mathcal{A}_I^{\mathbf{P},0}}}]\mathcal{E}_{ij}||\mathbf{A}_{ij}||^2 - [\sum_{\substack{i\in\mathcal{A}_O^{\mathbf{P},1} \\ j\in\mathscr{A}_I/\mathcal{A}_I^{\mathbf{P},0}}} + \sum_{\substack{i\in\mathscr{A}_I/\mathcal{A}_I^{\mathbf{P},0} \\ j\in\mathcal{A}_O^{\mathbf{P},1}}} + \sum_{\substack{i\in\mathcal{A}_O^{\mathbf{P},1} \\ j\in\mathcal{A}_O^{\mathbf{P},1}}}]\mathcal{E}_{ij}||\mathbf{A}_{ij}||^2 \tag{28}$$

$$+ \sum_{\substack{i\in\mathcal{A}_I^{\mathbf{P},*}\cup\mathcal{A}_I^{\mathbf{P},1} \\ j\in\mathcal{A}_I^{\mathbf{P},*}\cup\mathcal{A}_I^{\mathbf{P},1}}} \mathcal{E}_{ij}||\mathbf{A}_{ij} - \mathbf{B}_{\tau(i)\tau(j)}||^2 - \sum_{\substack{i\in\mathcal{A}_I^{\mathbf{P},*}\cup\mathcal{A}_I^{\mathbf{P},1} \\ j\in\mathcal{A}_I^{\mathbf{P},*}\cup\mathcal{A}_I^{\mathbf{P},1}}} \mathcal{E}_{ij}||\mathbf{A}_{ij} - \mathbf{B}_{\tau^*(i)\tau^*(j)}||^2 \tag{29}$$

$$+[\sum_{\substack{i\in\mathcal{A}_O^{\mathbf{P},1} \\ j\in\mathscr{A}_I/\mathcal{A}_I^{\mathbf{P},0}}} + \sum_{\substack{i\in\mathscr{A}_I/\in\mathcal{A}_I^{\mathbf{P},0} \\ j\in\mathcal{A}_O^{\mathbf{P},1}}} + \sum_{\substack{i\in\mathcal{A}_O^{\mathbf{P},1} \\ j\in\mathcal{A}_O^{\mathbf{P},1}}}]\mathcal{E}_{ij}||\mathbf{A}_{ij} - \mathbf{B}_{\tau(i)\tau(j)}||^2 - [\sum_{\substack{i\in\mathcal{A}_I^{\mathbf{P},0} \\ j\in\mathscr{A}_I/\mathcal{A}_I^{\mathbf{P},0}}} + \sum_{\substack{i\in\mathscr{A}_I/\mathcal{A}_I^{\mathbf{P},0} \\ j\in\mathcal{A}_I^{\mathbf{P},0}}} + \sum_{\substack{i\in\mathcal{A}_I^{\mathbf{P},0} \\ j\in\mathcal{A}_I^{\mathbf{P},0}}}]\mathcal{E}_{ij}||\mathbf{A}_{ij} - \mathbf{B}_{\tau^*(i)\tau^*(j)}||^2. \tag{30}$$

We can see that Eq. (28) ≥ 0, Eq. (29) ≥ 0, Eq. (30) ≥ 0 due to the proposition 2, 3 and 4 in our manuscript. Note that, the equation holds if and only if $\mathbf{P} = \mathbf{P}^*$. □

Similarly, we can prove that $F_{p_2}(\mathbf{P}) \geq F_{p_2}(\mathbf{P}^*)$. Finally, we have proved that $F(\mathbf{P}) \geq F(\mathbf{P}^*)$ because $F = \lambda_1 F_u + \lambda_2(F_{p_1} + F_{p_2})$ with $\lambda_1, \lambda_2 \geq 0$. □

There is one more interesting conclusion we can summarize here. For general graph matching on simple graphs without outliers or with outliers arising in only one graph (*e.g.*, $\mathcal{G}'$), we can see that the index sets satisfy $\mathcal{A}_I^{\mathbf{P},0} = \varnothing, \mathcal{A}_O^{\mathbf{P},0} = \varnothing, \mathcal{A}_O^{\mathbf{P},1} = \varnothing$. Therefore, it naturally holds that Eq. (28) $= 0$, Eq. (30) $= 0$. And Eq. (29) can be simplified as

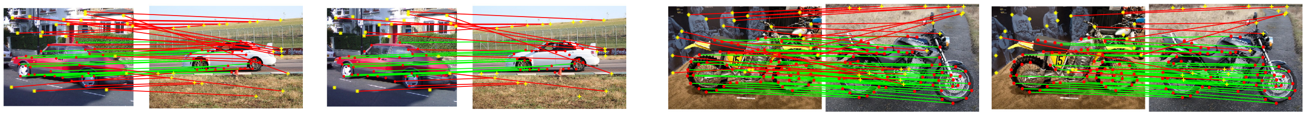
$$(29) = \sum_{\substack{i\in\mathcal{A}_I^{\mathbf{P},1}, j\in\mathcal{A}_I^{\mathbf{P},*}\cup\mathcal{A}_I^{\mathbf{P},1} \\ \text{or } i\in\mathcal{A}_I^{\mathbf{P},*}\cup\mathcal{A}_I^{\mathbf{P},1}, j\in\mathcal{A}_I^{\mathbf{P},1}}} \mathcal{E}_{ij}\left(||\mathbf{A}_{ij} - \mathbf{B}_{\tau(i)\tau(j)}||^2 - ||\mathbf{A}_{ij} - \mathbf{B}_{\tau^*(i)\tau^*(j)}||^2\right). \tag{31}$$

For simple graphs, it usually satisfies that the discrepancy $||\mathbf{A}_{ij} - \mathbf{B}_{\tau^*(i)\tau^*(j)}||$ between $\mathbf{A}_{ij}$ and the ideally matched one $\mathbf{B}_{\tau^*(i)\tau^*(j)}$ is smaller than $||\mathbf{A}_{ij} - \mathbf{B}_{\tau(i)\tau(j)}||$, which directly ensures that Eq. (31)≥ 0. It means that, for simple graphs, the choices for $\mathcal{E}, \mathbf{A}, \mathbf{B}$ can be more flexible. Furthermore, it demonstrates why the GM methods suitable for simple graphs can not be directly applied on complicated graphs consisting of numerous outliers arising in both graphs.

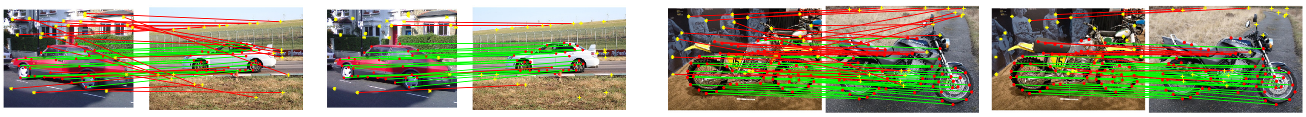
2. More comparison results

In this section, we present more results of the graph matching methods on the three datasets used in our manuscript.

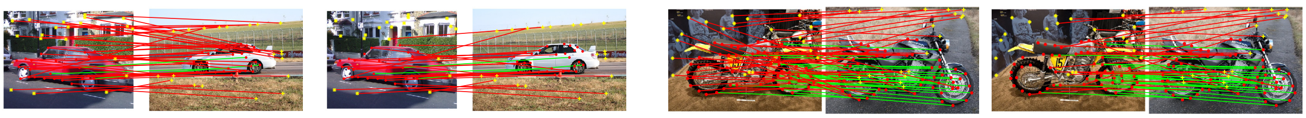
2.1. More results on PASCAL dataset



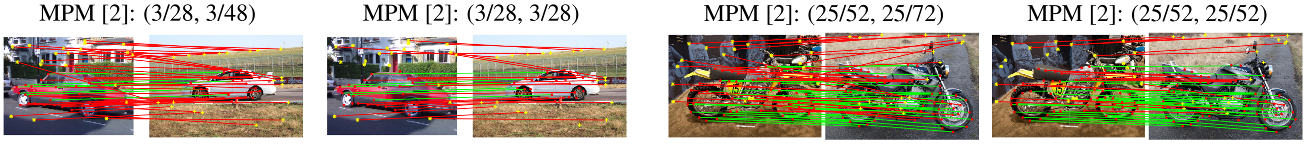
GA [3]: (17/28, 17/48) GA [3]: (13/28, 13/28) GA [3]: (46/52, 46/72) GA [3]: (41/52, 41/52)



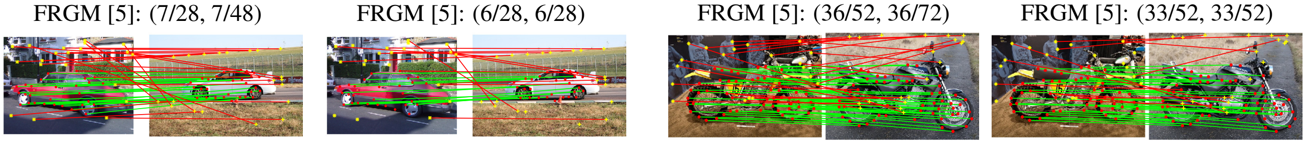
RRWM [1]: (24/28, 24/48) RRWM [1]: (24/28, 24/28) RRWM [1]: (40/52, 40/72) RRWM [1]: (38/52, 38/52)



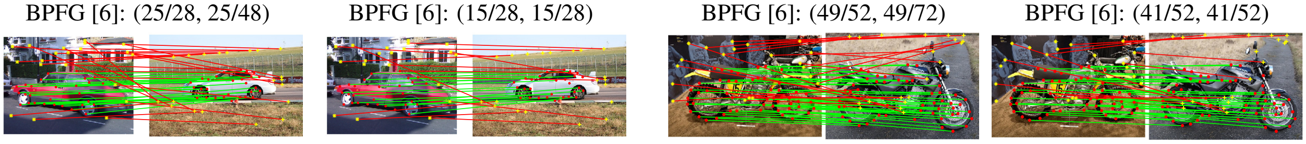
MPM [2]: (3/28, 3/48) MPM [2]: (3/28, 3/28) MPM [2]: (25/52, 25/72) MPM [2]: (25/52, 25/52)



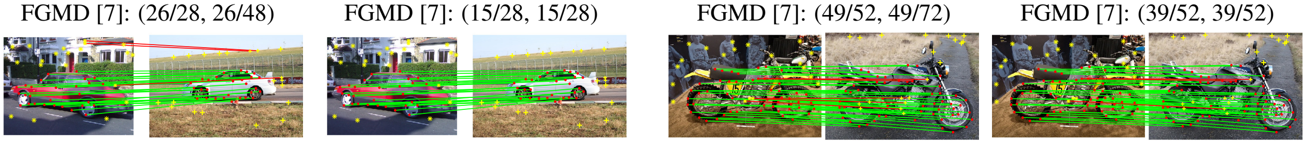
FRGM [5]: (7/28, 7/48) FRGM [5]: (6/28, 6/28) FRGM [5]: (36/52, 36/72) FRGM [5]: (33/52, 33/52)



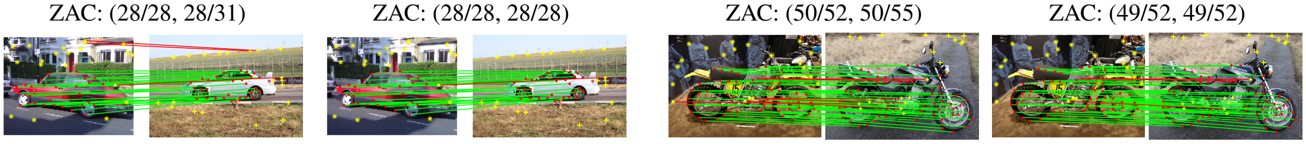
BPFG [6]: (25/28, 25/48) BPFG [6]: (15/28, 15/28) BPFG [6]: (49/52, 49/72) BPFG [6]: (41/52, 41/52)



FGMD [7]: (26/28, 26/48) FGMD [7]: (15/28, 15/28) FGMD [7]: (49/52, 49/72) FGMD [7]: (39/52, 39/52)



ZAC: (28/28, 28/31) ZAC: (28/28, 28/28) ZAC: (50/52, 50/55) ZAC: (49/52, 49/52)

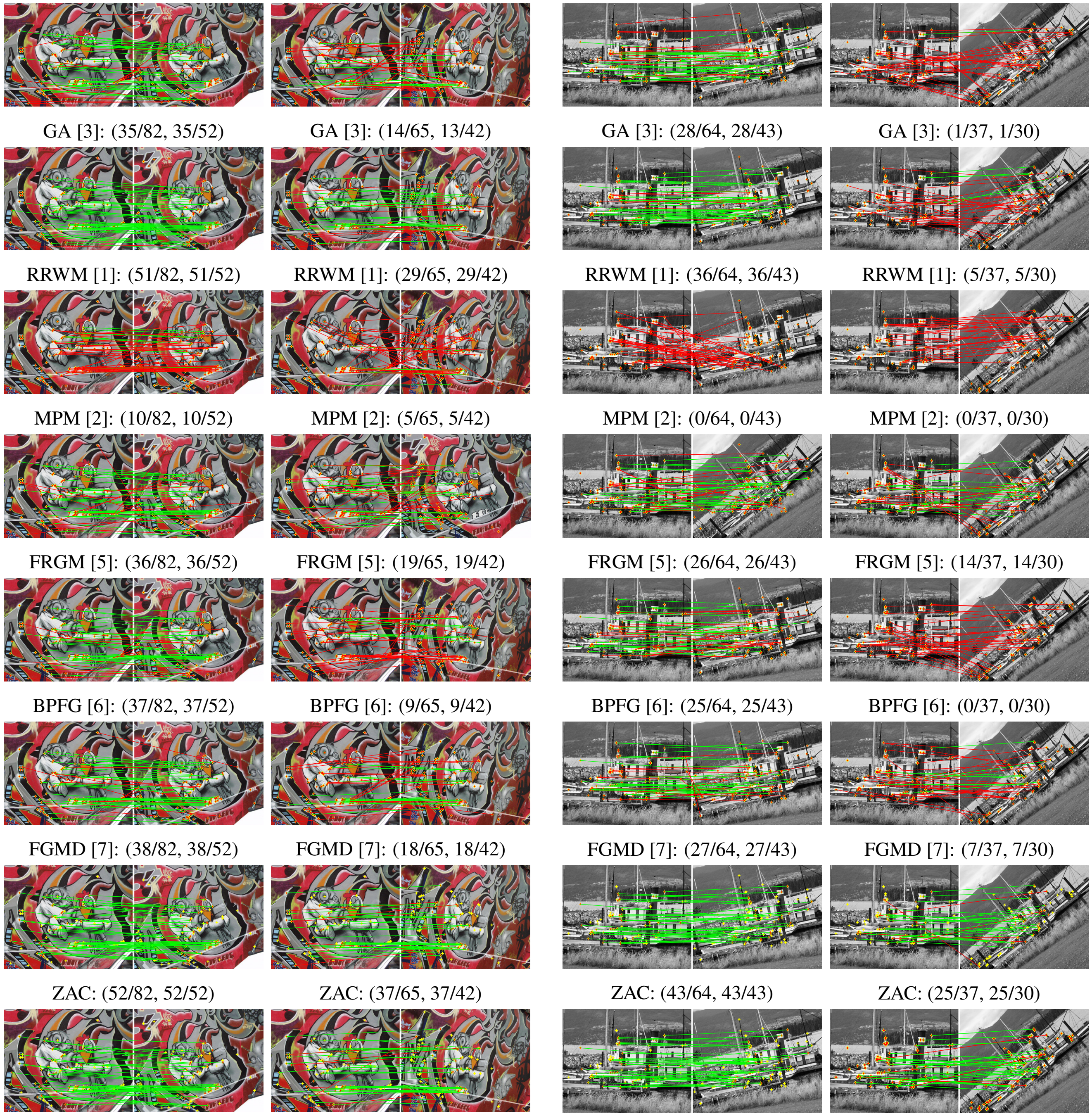


ZACR: (28/28, 28/31) ZACR: (27/28, 27/28) ZACR: (50/52, 50/56) ZACR: (49/52, 49/52)

(a) Original matchings (b) Top $k = 28$ matchings (c) Original matchings (d) Top $k = 52$ matchings

Figure 3: Examples of graph matching with inliers (red dots) and outliers (yellow signs) on the car images (left two columns) and motorbike images (right two columns) in PASCAL dataset. For each method, the recall and precision are computed with their original matchings and top $k = \#inliers$ matchings, respectively.

2.2. More results on VGG dataset



GA [3]: (35/82, 35/52)
RRWM [1]: (51/82, 51/52)
MPM [2]: (10/82, 10/52)
FRGM [5]: (36/82, 36/52)
BPFG [6]: (37/82, 37/52)
FGMD [7]: (38/82, 38/52)
ZAC: (52/82, 52/52)
ZACR: (52/82, 52/52)

(a) Top $k = 52$ matchings

GA [3]: (14/65, 13/42)
RRWM [1]: (29/65, 29/42)
MPM [2]: (5/65, 5/42)
FRGM [5]: (19/65, 19/42)
BPFG [6]: (9/65, 9/42)
FGMD [7]: (18/65, 18/42)
ZAC: (37/65, 37/42)
ZACR: (36/65, 36/42)

(b) Top $k = 42$ matchings

GA [3]: (28/64, 28/43)
RRWM [1]: (36/64, 36/43)
MPM [2]: (0/64, 0/43)
FRGM [5]: (26/64, 26/43)
BPFG [6]: (25/64, 25/43)
FGMD [7]: (27/64, 27/43)
ZAC: (43/64, 43/43)
ZACR: (43/64, 43/43)

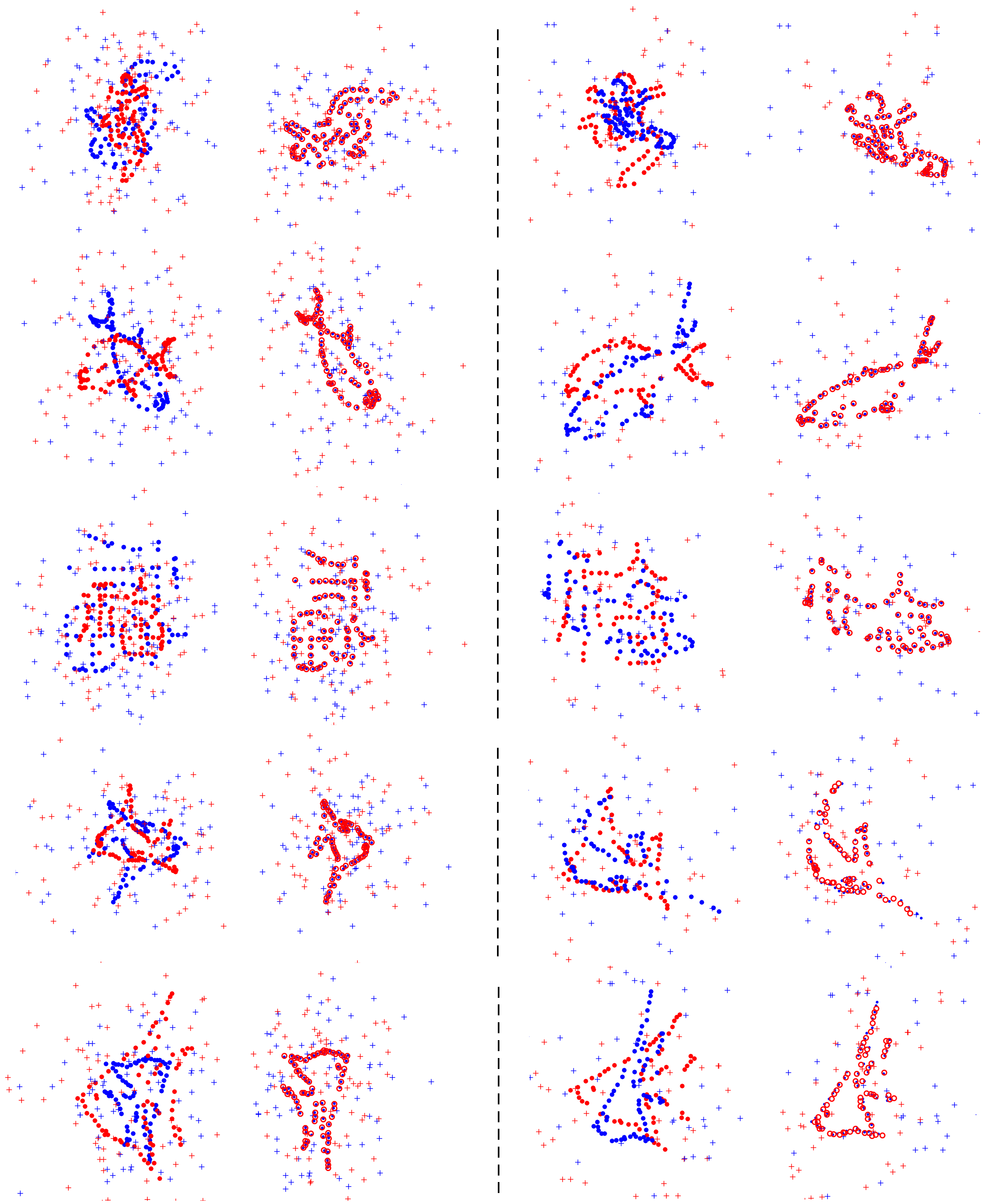
(c) Top $k = 43$ matchings

GA [3]: (1/37, 1/30)
RRWM [1]: (5/37, 5/30)
MPM [2]: (0/37, 0/30)
FRGM [5]: (14/37, 14/30)
BPFG [6]: (0/37, 0/30)
FGMD [7]: (7/37, 7/30)
ZAC: (25/37, 25/30)
ZACR: (25/37, 25/30)

(d) Top $k = 30$ matchings

Figure 4: Examples of graph matching under varying viewpoint (left two columns) and zoom+rotation (right two columns) on the “graf” images and “boat” images in VGG dataset. For each method, the recall and precision are computed with their top $k = \lfloor \{0.5, 0.4\} \times \min\{m, n\} \rfloor$ matchings, respectively.

2.3. More results of deformable graph matching



(a) Rigid deformation

(b) Non-rigid deformation

Figure 5: Examples of deformable graph matching by our method ZAC on widely used shape templates (also used in [5, 4], *etc*.). The graph pairs are disturbed by geometric deformations, noises, missing points and outliers.

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