Surface Registration via Foliation

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Abstract

This work introduces a novel surface registration method based on foliation. A foliation decomposes the surface into a family of closed loops, such that the decomposition has local tensor product structure. By projecting each loop to a point, the surface is collapsed into a graph. Two homeomorphic surfaces with consistent foliations can be registered by first matching their foliation graphs, then matching the corresponding leaves.

This foliation based method is capable of handling surfaces with complicated topologies and large non-isometric deformations, rigorous with solid theoretic foundation, easy to implement, robust to compute. The result mapping is diffeomorphic. Our experimental results show the efficiency and efficacy of the proposed method.

1. Introduction

Surface registration plays a fundamental role in computer vision, it has a broad range of applications, such as 3D face recognition \cite{12}, shape retrieval \cite{14}, brain cortical surface registration \cite{22}, image registration \cite{31}, high resolution tracking of non-rigid motion \cite{53} and so on.

In the past decades, researchers have intensively investigated various surface registration approaches \cite{11,52}, such as ICP method \cite{7,4,15,43,51,21}, dense mapping method \cite{25,58,42,33}, graph matching method \cite{18,20,27,8,9,37}, parameterizatoin based method \cite{13,44,52,57,10,55,1,49}, functional based method \cite{41,24,34,45} and many other methods (e.g. \cite{7,12,36}). Although surface registration techniques have been advanced greatly, it still remains challenging to handle surfaces with complicated topologies and large non-isometric deformations.

In order to tackle these challenges, this work proposes a novel method to establish diffeomorphisms between surfaces with complicated topologies, \textit{surface registration via foliation}.

1.1. Proposed Approach

\textbf{Foliation} Fig. 1 shows the concept of foliation. Intuitively, a surface foliation decomposes the surface into a family of non-intersecting loops. Each loop is called a leaf. All the homotopic leaves form a cylinder, the whole surface is decomposed into cylinders $\{C_1, C_2, C_3\}$, different cylinders intersect at critical leaves (red leaves $\Gamma_1, \Gamma_2$), three cylinders meet at the singularities ($\{z_1, z_2\}$). We can shrink each leaf to a point to “project” the whole surface into a graph, where each cylinder is mapped to an arc, each critical leaf to a node. Furthermore, the number of leaves on each cylinder defines the length of the arc. The graph with the arc length represents the foliation.

Furthermore, the mapping from the foliation to the corresponding graph is a \textit{generalized harmonic map}. Inversely, under some mild conditions, given a graph with the arc length, the harmonic map from the surface to the graph induces a foliation.

\textbf{Registration Based on Foliation} Fig. 2 shows our registration algorithmic pipeline. Given two genus $g$ surfaces, we automatically compute $3g - 3$ cutting loops on...
the surface, which divide each surface into \(2g - 2\) pairs of pants (genus 0 surface with 3 boundaries), this process is called the pants decomposition. Then we construct the corresponding pants decomposition graph, where each node represents a pair of pants, each arc represents a cutting loop. The consistent pants decompositions induce the same graph. We assign the arc lengths of the graph, compute the harmonic map from each surface to the graph. The harmonic mappings produce consistent foliations.

Each point on the graph corresponds to a unique leaf on the source foliation, and a leaf on the target foliation. This gives the correspondence between the leaves, furthermore, the correspondences between the singularities and the cylinders. As shown in Fig. 2, the corresponding cylinders are rendered using the same color. By further adjusting the mapping between each pair of leaves, we can achieve a global diffeomorphism between the surface.

1.2. Contributions

In summary, the main contributions are as follows:

1. It introduces a novel algorithm to compute surface registration based on foliations. This is equivalent to dimension reduction, converts surface mapping to graph matching and circle matching.
2. The method is capable of dealing with surfaces with complicated topologies, and large non-isometric deformations. It can also satisfy the landmark constraints.
3. The method has solid theoretic foundation. The registration is guaranteed to be diffeomorphic, and the existence and uniqueness of the foliation is also guaranteed.
4. The method is intrinsic, it converts the Riemannian metric of the surface to a flat metric (Euclidean geometry) with cone singularities. Comparing to methods using Riemannian geometry, it is simple and easy to implement, robust to triangulations and geometric noises in practice.

To the best of our knowledge, it is the first work to introduce foliation theory for surface registration. The computation of general foliations is based on non-linear convex optimization, hence the convergence and the global optimality has theoretic guarantees.

The main drawbacks of the current method include the non-linearity of the algorithm, and it can not directly handle partial matching for surfaces with inconsistent boundaries.

2. Previous Works

Shape matching and registration is a well-studied field with several recent books and surveys [11, 32]. It is out of scope for this article to cover all existing shape matching and registration methods; we concentrate on the most related algorithms.

**Iterative Closest Points (ICP) Method** The ICP based methods find surface correspondences through an iterative procedure that starts with an initial correspondence and then repeatedly improves it by computing an aligning transformation from the correspondences and then updating the correspondences based on the transformation. These methods are most commonly used for aligning surfaces related by a rigid transformation [7], but have also been used for moderate non-rigid deformations [4, 15, 43, 51, 21]. Unfortunately, they do not guarantee that the final map is smooth or bijective (two points on one surface may map to the same point on another), and a good initial guess (which is sometimes difficult to obtain) is required to succeed in most cases.

**Dense Mapping Method** These methods represent a map between a pair of shapes as a point-to-point correspondence. Since it is infeasible to optimize over such correspondences directly, most methods aim to obtain a sparse set of point correspondences and extend them to dense mappings [23, 55, 42, 33]. Because sparse point correspondences are inherently discrete, common ways to enforce global consistency include preservation of various quantities between pairs or sets of points, including geodesic distances [11, 25], spectral quantities [28, 39, 40, 48, 42], or a combination of multiple geometric and topological tests [16, 6].

**Graph Matching Based Methods** Graph based methods [18, 20, 27, 8] are popular for surface registration. Reeb graphs based method can register high genus surfaces [9]. A Reeb graph describes the shape by storing the evolution of level sets of a given real-valued function associated to the shape. In general, the function is selected as the height function, which depends on the embedding of the surface. Our method is intrinsic, solely depends on the Riemannian metric. Furthermore, the level sets don’t form a foliation, the Reeb graph method generates more singularities and more decomposed patches, and is more complicated. Our pro-
posed foliation produces the least number of singularities and patches in theory, therefore is simpler.

Li et al. introduces the consistent pants decomposition method for registration in [37]. The surface is decomposed to pairs of pants, then the corresponding pair of pants are matched using harmonic mapping. In the process, each pair of pants are further decomposed into two topological hexagons. Therefore for a genus $g > 1$ surface, their method divides the surface into $4g - 4$ patches, ours produces $3g - 3$ cylinders. Hence foliation based method has lower complexity. We apply their automatic pants decomposition method in our pipeline.

Parameterization Based Method

Given a set of correct sparse correspondences (defined by a user or an algorithm), one can use a variety of methods to find a smooth map interpolating them. A common approach is to map both surfaces to a canonical domain where sparse feature points align to each other and then interpolate the map in that domain [3]. For example, [44] used a base coarse mesh (provided by a user) as such a domain. In their approach, the surface is cut into triangular patches defined by three geodesic curves, such that each geodesic curve is mapped to a triangle edge on a coarse base mesh. An automatic approach for creating the base domain has been developed [37, 38].

Conformal geometric methods based on the Euclidean metric have also been extensively studied [5, 57, 10, 55]. The methods in [11] and [2] uses conformal parameterization based method for surface registration with landmark constraints, the algorithms are independent of the choice of the cut graph. These methods mainly focus on genus zero surfaces. The method in Wang et al. [52] studied brain morphology with Teichmüller space coordinates where the hyperbolic conformal mapping was computed with the Yamabe flow method. Lui and Wen [38] also register high genus surfaces in their hyperbolic uniformization domain. Zeng [57] proposed a general surface registration method via the Klein model in hyperbolic space where they used the inverse distance curve flow method to compute the hyperbolic conformal mapping. Shi et al. proposed surface registration based on a hyperbolic harmonic map in [49]. These methods are based on hyperbolic harmonic maps and the registration is carried out on hyperbolic plane. The conformal factor goes to infinity near the boundary of the hyperbolic disk, therefore, the registration algorithms are error-prone. Our proposed method uses flat metric, the computation is simpler and stabler.

Functional Based Method

Functional based methods [41, 24, 34] are capable of handling surfaces with complicated topologies. Partial functional correspondence method [45] finds the partial matching between two surfaces with non-rigid, isometric deformations. It selects the corresponding subsets of the surfaces, and calculates the eigen functions of the Laplace-Beltrami operator, the corresponding points have the same values of all eigen functions. This method can handle complicated and different topology for partial matching. But this method requires the surfaces are with non-rigid but near-isometric deformation. The method can not handle surfaces with large non-isometric deformation, because the eigen functions will be very different. Our method is capable of processing shapes with large non-isometric deformations.

The current work proposes a registration method based on foliation. The existence and the uniqueness of the solution are with theoretic assurance, and the result mapping is guaranteed to be diffeomorphic. The method can handle surfaces with complicated topologies, and large non-isometric deformations. Furthermore, the algorithm is based on flat (Euclidean) metric with cone singularities, therefore simpler and more robust.

3. Theoretic Background

Our proposed method is based on fundamental concepts and theorems in conformal geometry. Here we briefly review the basic concepts. Detailed treatments can be found in [19, 50].

Riemann Surface and Conformal Mapping

Given a complex function $f : \mathbb{C} \to \mathbb{C}$, $f : x + iy \mapsto u(x, y) + iv(x, y)$, if $f$ satisfies the Cauchy-Riemann equation: $u_x = v_y$ and $u_y = -v_x$, then $f$ is called a holomorphic function. If $f$ is invertible, and $f^{-1}$ is also holomorphic, then $f$ is called a bi-holomorphic function. A surface is called a Riemann surface, if it is with a complex atlas $\mathcal{A}$, such that all chart transition functions are bi-holomorphic.

Holomorphic Quadratic Differential

Definition 3.1 (Holomorphic Quadratic Differentials)

Suppose $S$ is a Riemann surface. Let $\Phi$ be a complex differential form, such that on each local chart $\{z_\alpha\}$, $\Phi = \varphi_\alpha(z_\alpha) dz_\alpha^2$, where $\varphi_\alpha(z_\alpha)$ is a holomorphic function.

All holomorphic quadratic differentials form a linear space $[19]$. A point $z_\alpha \in S$ is called a zero of $\Phi$, if $\varphi(z_\alpha)$ vanishes. A holomorphic quadratic differential has $4g - 4$ zeros, as shown in Fig. 3. For any point away from zero, we can define a local coordinates

$$
\zeta(p) := \int_0^p \sqrt{\varphi(z)} \, dz.
$$

which is the so-called natural coordinates induced by $\Phi$. Suppose $\gamma \subset S$ is a curve, if $\zeta(\gamma)$ is a horizontal (vertical) line, then $\gamma$ is called a horizontal (vertical) trajectory. If $\gamma$ goes through the zeros, then it is a critical trajectory. If all the horizontal trajectories are finite, then $\Phi$ is called a Strebel differential.
Relation Between Foliation and Differential. Generally speaking, given a holomorphic quadratic differential $\Phi$, the horizontal trajectories form a foliation $F$; inversely, given a foliation $F$, there exists a differential $\Phi$, whose horizontal trajectories form a foliation $F$, such that there is an automorphism $\varphi$ of the surface, $\varphi$ maps $F$ to $F$.

Given a holomorphic quadratic differential $\Phi$, the natural coordinates in Eqn. 1 induces a flat metric with cone singularities, which is denoted as $|\Phi|$. Hubbard and Masur proved the following existence of a Strebel differential with prescribed type and heights.

**Theorem 3.1 (Hubbard and Masur [26])** Suppose $S$ is a compact Riemann surface with genus $g > 1$. Given non-intersecting simple loops $\Gamma = \{\gamma_1, \gamma_2, \cdots, \gamma_{3g-3}\}$, and positive numbers $\{h_1, h_2, \cdots, h_{3g-3}\}$, there exists a unique holomorphic quadratic differential $\Phi$, satisfying the following:

1. The critical graph of $\Phi$ partition the surface into $3g - 3$ cylinders, $\{C_1, C_2, \cdots, C_{3g-3}\}$, such that $\gamma_k$ is the generator of $C_k$.
2. The height of each cylinder $(C_k, |\Phi|)$ equals to $h_k$, $k = 1, 2, \cdots, 3g - 3$.

**4. Algorithm**

In this section, we explain every step in the algorithm pipeline in detail.

**4.1. Pants Decomposition**

In our current algorithm, the pants decomposition is carried out automatically using the algorithm in [37]. Given a genus $g > 1$ closed surface $S$, we automatically compute the $g$ handle loops, then find extra $2g - 3$ disjoint simple loops, to form the set of cutting loops $\{\gamma_1, \gamma_2, \cdots, \gamma_{3g-3}\}$. The cutting loops segment the surface into $2g - 2$ pairs of pants, $\{P_1, P_2, \cdots, P_{2g-2}\}$, to generate a pants decomposition of the surface. A pants decomposition can be represented as a graph $G$, the so-called pants decomposition graph, where each pair of pants is represented as a node, each simple loop is denoted by an edge. Suppose the simple loop $\gamma_i$ connecting two pairs of pants $P_j, P_k$, then the arc of $\gamma_i$ connects nodes of $P_j$ and $P_k$. Then we associate a positive real number $h_i$ for each simple loop $\gamma_i$, $1 \leq i \leq 3g - 3$, namely, the corresponding edge on the pants decomposition graph. We use $(G, h)$ denote the pants decomposition graph $G$ with the heights $h = (h_1, h_2, \cdots, h_{3g-3})$, and call it the weighted pants decomposition graph. In our current implementation, we adapt the uniform heights.

**4.2. Strebel Differential**

Given a weighted pants decomposition graph $(G, h)$, consider a map $f : (S, g) \rightarrow (G, h)$, where the length of edge $\gamma_i$ is $h_i$. We say a point $p \in S$ a regular point, if its image is not any node of $G$, otherwise a critical point. The set of all critical points is denoted as $\Gamma$. Then we can define the harmonic energy of the mapping $f$,

$$E(f) := \int_{S \setminus \Gamma} |\nabla_{\Phi} f|^2 dA_{\Phi}, \quad (2)$$

The critical point of the harmonic energy is called a harmonic map. Wolf [56] proved the existence and the uniqueness of the harmonic map. Furthermore, the harmonic map induces a holomorphic quadratic form $\Phi$ (Hopf differential)

$$\Phi = \langle f_z, f_{zz} \rangle dz^2.$$

By Theorem 26, the horizontal trajectories of $\Phi$ form a foliation $F$. The projection graph of $F$ is exactly $(G, h)$.

In practice, a surface is approximated by a triangular mesh $M = (V, E, F)$. We use $[v_i, v_j]$ to represent an edge connecting the vertices $v_i$ and $v_j$. The weighted graph $(G, h)$ is with a flat metric, the length of the edge representing the loop $\gamma_i$ is $h_i$. The harmonic energy of a map $f : M \rightarrow (G, h)$ is given by

$$E(f) := \frac{1}{2} \sum_{[v_i, v_j]} w_{ij} d(f(v_i), f(v_j))^2,$$

where $d(\cdot, \cdot)$ is the shortest distance between two points on the graph, $w_{ij}$ is the cotangent edge weight. Suppose two faces $[v_i, v_j, v_k]$ and $[v_j, v_i, v_k]$ share the edge $[v_i, v_j]$, then

$$w_{ij} = \cot \theta^{ij}_k + \cot \theta^{ji}_l,$$

where $\theta^{ij}_k$ represents the corner angle at the vertex $v_k$ in the face $[v_i, v_j, v_k]$.

We use the non-linear heat flow method to compute the harmonic map. First, we deform $\gamma_i$ to sweep a cylinder $C_i$, such that the union of all the cylinders cover the whole surface. Second, each cylinder $C_i$ is mapped to the edge $\gamma_i$, this constructs the initial map $f$. Third, we diffuse the map to reduce the harmonic energy. At each step, we move the image of vertex to the weighted geodesic center of the images of its neighbors. Suppose after the $k$-th iteration,
we have obtained the mapping \( f_k : M \to G \) already, the
vertices \( \{ v_j \} \)'s are adjacent to the vertex \( v_i \), the weighted
geodesic center of \( \{ f_k(v_j) \} \)'s is
\[
c_k(v_i) = \arg \min_{q \in G} \sum_{j=1}^n w_{ij} d(f_k(v_j), q)^2.
\]

The diffusion process moves the image of \( v_i \) to the weight-
ed geodesic center, \( f_{k+1}(v_i) \leftarrow c_k(v_i) \). The existence and
the uniqueness of the harmonic map is proven in Wolf’s work [56].

4.3. Registration

Given two surfaces \((S, g)\) and \((T, h)\), we would like to
find a diffeomorphism \( \varphi : S \to T \). We automatically
compute the pants decompositions on the surfaces using the
automatic algorithm in [37], with the corresponding pants
decomposition graphs \( G_S \) and \( G_T \). Then the two graph-
s are isomorphic, \( G_S \cong G_T \). We set the height vector
\( \mathbf{h}_S \) and \( \mathbf{h}_T \) respectively, then compute the harmonic maps
\( f_S : (S, g) \to (G_S, \mathbf{h}_S) \) and \( f_T : (T, h) \to (G_T, \mathbf{h}_T) \).
The harmonic maps induce the Strebel differentials \( \Phi_S \) and
\( \Phi_T \). The critical trajectories of \( \Phi_S \) and \( \Phi_T \) are
\( \Gamma_S \) and \( \Gamma_T \). The critical trajectories partition the surfaces into
\( 2g - 2 \) cylinders,
\[
S \setminus \Gamma_S = \bigcup_{k=1}^{2g-2} C^k_S, \quad T \setminus \Gamma_T = \bigcup_{k=1}^{2g-2} C^k_T
\]

The Strebel differentials \( \Phi_S \) and \( \Phi_T \) induce flat metrics,
\( |\Phi_S| \) and \( |\Phi_T| \) respectively.

The restriction of \( \Phi_S \) on each cylinder \( C^k_S \) is a harmonic
function, then we can define a harmonic 1-form
\[
\omega_k := d\Phi_S|_{C^k_S},
\]
then we use the Hodge star operator to act on \( \omega_k \) to get the
conjugate harmonic 1-form \( *\omega_k \), then pair them to form a
holomorphic 1-form \( \Omega_k := \omega_k + \sqrt{-1} *\omega_k \). Then the union
of all \( \Omega_k \)'s give the Strebel differential \( \Phi_S, \Omega_k \). \( \Omega_k \) gives the
flat metric \( |\Phi_S| \). We use the method in [30] to compute the
Hodge star operator and discrete holomorphic 1-form \( \Omega_k \).

For each pair of flat cylinders, we can construct a harmonic
map using conventional harmonic mapping method [54]
\[
\varphi_k : (C^k_S, |\Phi_S|) \to (C^k_T, |\Phi_T|),
\]
with consistent boundary conditions, such that the zero
points of \( \Phi_S \) on the boundaries of \( C^k_S \) are mapped to the
zero points of \( \Phi_T \) on the boundaries of \( C^k_T \).

All the harmonic maps \( \varphi_k \)'s are glued together to form a
global piecewise harmonic map \( \varphi : (S, |\phi_S|) \to (T, |\phi_T|) \).
By construction, the mapping \( \varphi \) is a homeomorphism. Fur-
thermore, the global map can be further diffused under the
metric \(|\phi_T| \) on the target surface to become a global har-
monic map. According to the harmonic map theorem [46],
the harmonic map exists and is unique, furthermore diffeo-
 morphic. Details can be found in Algorithm [1].

Figure 4. Registration between the eight surface and the amphora
surface. Fig. [a] and Fig. [b] illustrate the process of surface regis-
tration. Fig. [c] shows a pair of genus two surfaces, the eight
surface (a) and the amphora surface (c). The pants decom-
positions are automatically computed, the cutting loops are
\( \{ \gamma_1, \gamma_2, \gamma_3 \} \) (blue curves in frames (a) and (c)) on both sur-
faces, the pants decomposition graph is shown in (b) and
(d). The user specifies the height function, and compute the
harmonic maps from the surfaces to the graphs, which in-
duces foliations as shown in (a) and (c), the zero points are
\( \{ z_1, z_2 \} \), the critical trajectories are \( \{ \Gamma_1, \Gamma_2 \} \), the cylinders are
\( \{ C_1, C_2, C_3 \} \). The corresponding cylinders are regis-
tered using harmonic maps with consistent boundary condi-
tions, under the flat metrics induced by the Strebel differ-
entials. The piecewise harmonic maps are glued togeth-
er, and smoothed out. The resultant global diffeomorphism is
illustrated by consistent color map in frames (e) and (f),
the source point and the image point share the same col-
or. Frame (g) and (h) show the mapping using consistent
texture mapping within each cylinder. Each checker has
an alphabetic and numerical label, which shows the corre-
spondence and the distortions of the mapping. By visual
inspection, we can see the smoothness of the registration.

5. Experimental Results

In this section, we report our experimental results. We
have tested our algorithm on surfaces scanned from real life
[17] and reconstructed from medical images. Our algorithm
is implemented using generic C++, the numerical computa-
Algorithm 1 Surface Registration Algorithm Pipeline.

Require: Closed surfaces $S$ and $T$ with genus $g > 1$

Ensure: A diffeomorphism $\varphi : S \rightarrow T$

1. Construct the pants decomposition of $S$, with the graph $G_S$, assign the height function $h_S$.
2. Construct the pants decomposition of $T$, with the graph $G_T$, assign the height function $h_T$.
3. Compute the harmonic map $u_S : S \rightarrow G_S$, which induces a Strebel differential $\Phi_S$ with critical trajectory $\Gamma_S$.
4. Compute the harmonic map $u_T : T \rightarrow G_T$, which induces a Strebel differential $\Phi_T$ with critical trajectory $\Gamma_T$.
5. Slice $S$ along the critical trajectories $\Gamma_S$ to obtain cylindrical decomposition, $S \setminus \Gamma_S = \bigcup_{k=1}^{2g-2} C_k^S$.
6. Slice $T$ along the critical trajectories $\Gamma_T$ to obtain cylindrical decomposition, $T \setminus \Gamma_T = \bigcup_{k=1}^{2g-2} C_k^T$.

for all $k = 1, 2, \ldots, 2g-2$ do
7. Construct a harmonic maps $\varphi_k : (C_k^S, |\Phi_S|) \rightarrow (C_k^T, |\Phi_T|)$

end for
8. Glue all the local harmonic maps together to form the global map $\varphi : (S, |\Phi_S|) \rightarrow (T, |\Phi_T|)$.
9. Diffuse $\varphi$ to a harmonic map using non-linear heat diffusion method.

experiments demonstrate that our method can handle high genus surfaces.

5.1. Robustness Testing

In order to test the robustness of our foliation based method, we conducted an experiment as shown in Fig. 5. The input genus 2 mesh is with good quality triangulation, the foliation is calculated as shown in the top row. We modify the connectivity to reduce the triangulation quality to introduce skinny faces, but foliation is slightly affected as shown in the middle row. Then we add geometric noises to the shape by randomly moving every point along its normal, the perturbation distance is about 3% of the diagonal of the bounding box. The foliation of the bumpy surface looks very similar to original one. Because the computation of foliations is equivalent to solve an elliptic partial differential equation, the solution is stable and robust to the triangulation quality and geometric noise.

5.2. High Genus Testing

Fig. 6 and Fig. 7 show the registration result between the eight model and the girl sculpture. Fig. 6 frame (a) and (c) show the foliations, (b) and (d) illustrate the registration by consistent color encoding, which shows the smoothness of the mapping. Fig. 7 shows the correspondence by consistent texture mapping. Each checker on the texture has both alphabetic and numerical labels. By examining the labels of the checkers, one can verify the correspondence and visualize the distortions.

Similarly, Fig. 8 shows the registration result between the genus two surfaces, the amphora model and the star cup. Fig. 2 shows the mapping between genus 4 surfaces. These experiments demonstrate that our method can handle high genus surfaces.

5.3. Landmarks Constraints Testing

Fig. 9 shows registration of surfaces with feature points and boundaries. The 11 cat models from TOSCA datasets are with different poses, and with non-rigid but near-isometric deformations. There are 7 features points like the tips of the ears and the tail, 2 boundary landmark curves, which are treated as constraints for the registration. Each cat surface is decomposed into 15 cylinders, the corresponding cylinders are color-encoded similarly. By visual verification, we can see the correspondences among the leaves, and
the features and landmarks. Some cats models have self-intersections, which doesn’t affect our algorithm, since our method solely depends on the Riemannian metric, not the embedding.

Fig. 10 and 11 show another example for registering genus zero surfaces with landmark curves. The human cortical surface is with three consistent landmark Succi curves (red curves) in the top row in Fig. 10. Then we perform double covering technique [29] to covert each of them to a genus two closed surface and compute the foliation on the double cover, as shown in the bottom row in Fig. 11. The registration results between two cortical surfaces is illustrated by consistent color-encoding in Fig. 11.

We have perform the registration between each pair of cortical surfaces in a database containing 40 brains, and all 780 registrations are carried out automatically and successfully. This further demonstrates the robustness of our algorithm.

5.4. Efficiency Testing

We report the running time of our algorithm in this section. The time of computing the foliation are shown in Table 1. The timing for the registration in the algorithm pipeline are as follows: the mapping between the eight surface to ampora model is 2.32s; eight surface to the girl sculpture 30.188s; the star cup model to the amphora model 64.244s; the cortical surfaces 205.05s.

5.5. Comparison

We compare our method with the hyperbolic harmonic map algorithm introduced in [49] in terms of robustness, efficiency and accuracy.

The hyperbolic harmonic map algorithm depends on hyperbolic Ricci flow, which is sensitive to the meshing quality. It may fail on surfaces with low mesh qualities. In contrast, our method is based on harmonic mapping, which is more robust to the meshing quality and succeeds in all cases.

The Ricci flow method is highly non-linear, which is time-consuming with the total time 310s; our method is more efficient with the total time 275s on the same model.
Furthermore, we measure the registration quality by the curvature distortion metric. Basically, a mapping between two surfaces will pull back the Gaussian curvature and mean curvature on the target to the source. Then on the source surface, we compute the $L^2$ distance between the source curvature and the pulled-back curvature, which is the curvature distortion distance. If the registration is an isometric mapping, then the $L^2$ distance of Gaussian curvatures is 0; if the registration is a rigid motion in $\mathbb{R}^3$, then both Gaussian and mean curvature distances are 0. We compute the distribution of the curvature difference, and show the histograms in Fig. [12]. It is obvious that the curvature differences distribution of our method highly concentrates near the 0, the distribution of the hyperbolic harmonic mapping method in [49] is much more spread out. This demonstrates that our method achieves higher accuracy.

6. Conclusion

This work introduces a novel surface registration algorithm based on foliation theory. The surface is decomposed into a family of leaves (loops), the arrangement of which is represented as a graph. Surface registration is carried out by matching the graphs first, then match the leaves. This method can handle surfaces with complicated topologies and large non-isometric deformations; it has rigorous theoretic foundation to guarantee the existence and the uniqueness of the solution, the resulting mapping is diffeomorphic; the algorithm is simple to implement, robust to compute. Experimental results demonstrates the efficiency and efficacy of the proposed method.

On a surface, there are infinite many foliations. In theory, given a pair of homeomorphic surfaces, one can choose unique foliations on each of them, such that the mapping has the least distortions. In the future, we will explore how to find the optimal foliations to improve the mapping qualities.
References


