Efficient Algorithms for Moral Lineage Tracing
– Supplementary Material –

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Transformation Costs for GLA

Here, we detail on the calculation of the change of objective for the transformations applied in GLA (*c.f.* Sec. 3.1). We start with setParent. Setting a as parent of b will change the objective by

$$\Delta^{\text{set}}_{ab} = c_{ab} - c^+_b - \mathbb{1}\left(|\text{children(a)}| = 0\right) c^-_a \quad , \tag{1}$$

where $c_{ab} = -\sum_{e\in E_{ab}} c_e$, $c^+_a = \sum_{v\in V_a} c^+_v$ and $\mathbb{1}(\ldots)$ is the indicator function. It accounts for the activated arc ab, the fact that b no longer marks the birth of a new cell and, if a did not have a child previously, it takes the vanishing termination cost into account. A similar reasoning applies to changeParent. When we change the parent of b from a' to a, we get the following transformation cost:

$$\begin{aligned}\Delta^{\text{change}}_{a'b\rightarrow ab} =& c_{ab} - \mathbb{1}\left(|\text{children(a)}| = 0\right) c^-_a \\ & - c_{a'b} + \mathbb{1}\left(|\text{children(a}')| = 1\right) c^-_{a'} \quad , \end{aligned} \tag{2}$$

where we have to consider the possibility that a' could form a terminus after the transform. Finally, for a merge of two components a and b of the same frame, we calculate:

$$\begin{aligned}\Delta^{\text{merge}}_{ab} =& \; c_{ab} - \Delta^{\text{birth}}_{ab} - \Delta^{\text{term}}_{ab} \\ & + \sum_{ad\in\mathcal{A}:d\in\text{children(b)}} c_{ad} \quad + \sum_{bd\in\mathcal{A}:d\in\text{children(a)}} c_{bd} \; , \end{aligned} \tag{3}$$

where the last two sums account for arcs to active children, which will be contracted into active arcs with the merge, and therefore change their state and affect the objective. Birth $\Delta^{\text{birth}}_{ab}$ and termination costs $\Delta^{\text{term}}_{ab}$ depend on the current parents and children, that is:

$$\Delta^{\text{term}}_{ab} = \begin{cases} c^-_a & \text{if hasChild(b)} \wedge \neg\text{hasChild(a)} \;, \\ c^-_b & \text{if hasChild(a)} \wedge \neg\text{hasChild(b)} \;, \\ 0 & \text{otherwise} \;, \end{cases} \tag{4}$$

and

$$\Delta^{\text{birth}}_{ab} = \begin{cases} c^+_a + c_{pa} & \text{if hasParent(b)} \wedge \neg\text{hasParent(a)} \;, \\ c^+_b + c_{pb} & \text{if hasParent(a)} \wedge \neg\text{hasParent(b)} \;, \\ 0 & \text{otherwise} \;, \end{cases} \tag{5}$$

with pa and pb being the arc from the parent of b or a, respectively. Note that the merge is not feasible if a and b have distinct parents.

Minimum Cost Branching Coefficients

We derive the weights for the minimum cost branching problem (MCBP) used in our KLB heuristic (Sec. 3.2). Given a fixed intra-frame partitioning and the corresponding $\mathcal{G} = (\mathcal{V}, \mathcal{A})$, we note that all edges E_{ab} of an arc from component a to component b must have the same state (otherwise, space-time constraints would be violated). We can thus represent them with a set of binary arc indicator variables y_{ab} satisfying $\forall e \in E_{ab} : 1 - y_{ab} = x_e$. Similiarly, birth and termination indicator variables x^+ and x^- can be grouped with respect to their component, *i.e.* $\forall v \in V_a : y^+_a = x^+_v$ (and analogous for y^- and x^-), since all nodes v within a cell must have the same state. Substituting these branching variables into (4), leads to:

$$\begin{aligned} & \sum_{e\in E} c_e x_e + \sum_{v\in V} c^+_v x^+_v + \sum_{v\in V} c^-_v x^-_v \\ =& \sum_{e\in\bigcup_{t\in\mathcal{T}} E_t} c_e x_e + \sum_{ab\in\mathcal{A}} \sum_{e\in E_{ab}} c_e (1 - y_{ab}) \\ & + \sum_{a\in\mathcal{V}} y^+_a \underbrace{\sum_{v\in V_a} c^+_v}_{c^+_a} + \sum_{a\in\mathcal{V}} y^-_a \underbrace{\sum_{v\in V_a} c^-_v}_{c^-_a} \\ =& \sum_{e\in\bigcup_{t\in\mathcal{T}} E_t} c_e x_e + \sum_{e\in\bigcup_{t\in\mathcal{T}} E_{t,t+1}} c_e + \sum_{ab\in\mathcal{A}} y_{ab} \underbrace{\left(- \sum_{e\in E_{ab}} c_e \right)}_{c_{ab}} \end{aligned}$$

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$$+\sum_{a\in\mathcal{V}} y_a^+ c_a^+ + \sum_{a\in\mathcal{V}} y_a^- c_a^-$$
$$=\sum_{e\in\bigcup_{t\in\mathcal{T}} E_t} c_e x_e \;+ \sum_{e\in\bigcup_{t\in\mathcal{T}} E_{t,t+1}} c_e$$
$$+\sum_{ab\in\mathcal{A}} c_{ab} y_{ab} + \sum_{a\in\mathcal{V}} y_a^+ c_a^+ + \sum_{a\in\mathcal{V}} y_a^- c_a^- \quad ,$$

where the first sum only depends on the fixed intra-frame partitioning, the second term is constant and the remaining three terms correspond to the objective of the MCBP, where we identify the coefficients c_{ab}, c_a^+ and c_a^-. Whenever the arcs selected by y form a branching (which at most bifurcates), then the corresponding x satisfy morality (and bifurcation) constraints.

Proofs for Section 4

Proof of Lemma 4. We first show that any $x \in \{0,1\}^E$ satisfying all of (1) – (3) also satisfies (13) and (14) by contraposition. First, assume $x \in \{0,1\}^E$ violates an inequality of (13) for some $t \in \mathcal{T}$, $\{v,w\} \in E_t \cup E_{t,t+1}$ and chordless vw-path P. We distinguish the following cases: If $\{v,w\} \in E_t$ and P is a path in G_t, then the inequality is included in (1). If $\{v,w\} \in E_{t,t+1}$, then the inequality is included in (2). It remains to consider the case that $\{v,w\} \in E_t$ and P is not entirely contained in G_t. Let $\{v_t, v_{t+1}\}, \{w_t, w_{t+1}\} \in E_{t,t+1}$ with $v_t, w_t \in V_t$ be the first and the last inter frame edges in P, respectively. Furthermore, let $P_{v_{t+1}w_{t+1}}$ be the subpath of P between those edges. Now, either there is a $v_t w_t$-cut S in G_t such that $x_S = 1$ or there is a $v_t w_t$-path P' in G_t such that $x_{P'} = 0$. It is clear that P' can be extended to a vw-path of edges labeled 0, because $x_P = 0$. This yields either an inequality of (3) corresponding to $S, \{v_t, v_{t+1}\}, \{w_t, w_{t+1}\}$ and $P_{v_{t+1}w_{t+1}}$ or an inequality of (1) corresponding to $\{v,w\} \cup P'$ that is violated by x.

Next, suppose $x \in \{0,1\}^E$ violates an inequality of (14) for some $t \in \mathcal{T}$, $\{v', w'\} \in E_t$, a $v'w'$-cut S in G_t and a chordless $v'w'$-path P in G_t^+. Then $x_S = 1$ and $x_P = 0$. Clearly, x violates the inequality of (3) corresponding to $S, \{v_t, v_{t+1}\}, \{w_t, w_{t+1}\}$ and $P_{v_{t+1}w_{t+1}}$, where $\{v_t, v_{t+1}\}, \{w_t, w_{t+1}\}$ and $P_{v_{t+1}w_{t+1}}$ are defined similar to the last paragraph.

For the converse, we show that if $x \in \{0,1\}^E$ satisfies the inequalities (13) and (14), then it also satisfies (1) – (3). Any cycle in G_t^+ which is not chordless can be split into two cycles contained in G_t, G_t^+ or G_{t+1} which share exactly one edge. Therefore, any inequality of (1) – (2) is implied by a combination of inequalities from (13). This is a standard argument for multicut polytopes, cf., for instance, [1]. Moreover, for any $\{v_t, w_t\} \in E_t$ and any $v_t w_t$-cut S in G_t it holds that $\{v_t, w_t\} \in S$. Thus, reapplying the previous argument and the simple fact that

$$1 - \sum_{e\in S}(1 - x_e) \le 1 - (1 - x_{v_t w_t}) = x_{v_t w_t},$$

we conclude that the inequalities (3) are implied by a combination of inequalities from (13) and (14).

Proof of Lemma 5. We show the claim only for birth constraints since the proof for termination constraints is analogous. Let $x \in X'_G$ and $x^+, x^- \in \{0,1\}^V$. Apparently, if (15) is satisfied, then

$$\sum_{e\in S\setminus E(V_t\setminus\{v\}, V_{t+1}(v))} (1 - x_e) \le \sum_{e\in S}(1 - x_e)$$

implies that (6) also holds. Conversely, suppose (15) is violated. Then there exists some $t \in \mathcal{T}$ and $v \in V_{t+1}$, $S \in V_t v\text{-cuts}(G_t^+)$ such that $x_v^+ = 0$ and $x_e = 1$ for all $e \in S \setminus E\big(V_t(v), V_{t+1} \setminus \{v\}\big)$. Assume (6) is not violated, then there is a path P in G_t^+ from some node in V_t to v with $x_P = 0$. Then P must have non-empty intersection with $E\big(V_t(v), V_{t+1} \setminus \{v\}\big)$. Let $u \in V_t(v)$ and $v' \in V_{t+1} \setminus \{v\}$ be such that $\{u, v'\} \in P$. Since $x_{uv} = 1$ it follows that x violates the inequality

$$x_{uv} \le \sum_{e\in P_{uv}} x_e$$

of (2) where P_{uv} is the subpath of P from u to v. This is a contradiction to $x \in X'_G$.

Additional Results

We report additional, detailled results in terms of runtime, bounds, objective for feasible solutions, and derived gaps obtained on the two additional instances *Flywing-wide I* and *II* in Table 1. In Fig. 1, we present a more detailled analysis of the effect of locality parameter d_{MCBP} of our KLB heuristic.

References

[1] A. Horňáková, J.-H. Lange, and B. Andres. Analysis and optimzation of graph decompositions by lifted multicuts. In *ICML*, 2017. 2

Table 1. Detailed quantitative comparison of algorithms for the MLTP on the two additional instances *Flywing-wide I* and *II*. BestGap is calculated using the tightest bound of any algorithm.

Method	Time / s	objBest	objBound	BestGap
		Flywing-wide I		
GLA	0.72	-89895.00		0.0293
KLB-d=10	104.09	-91316.14		0.0133
KLB-d=inf	477.50	-91316.14		0.0133
ILP (ours)	10000.80	-91774.40	-92528.30	0.0082
		Flywing-wide II		
GLA	3.43	-167029.00		0.0214
KLB-d=10	3359.34	-168998.95		0.0095
KLB-d=inf	9129.41	-168998.95		0.0095
ILP (ours)	10245.80	-168862.00	-170606.00	0.0103

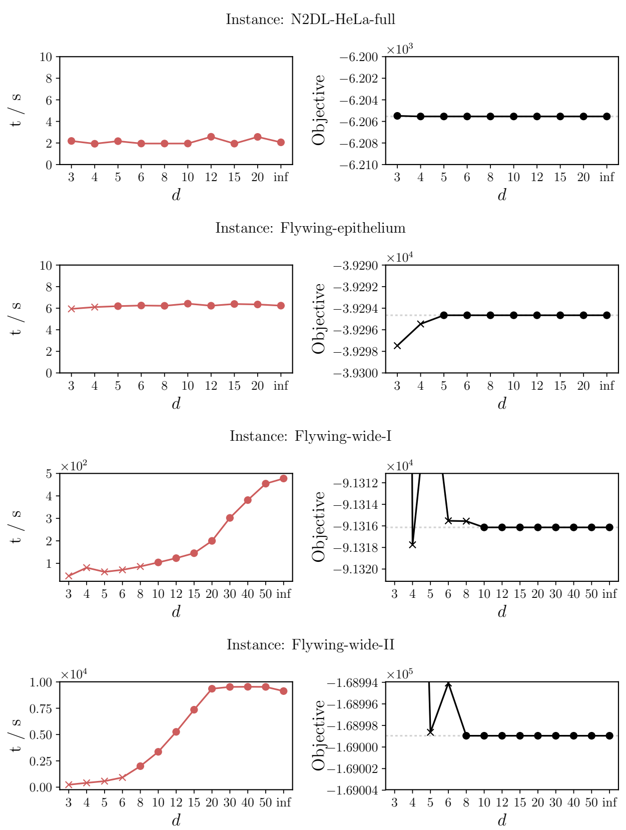

Figure 1. Comparison of varying d_{MCBP} within KLB in terms of runtime (**left**) and obtained objective (**right**). Parametrizations that were found to (sometimes) misjudge the change of objective due to a too restricted locality are marked with $\times$, while the others are depicted as $\bullet$. For the latter parametrizations, we observe that all obtain the same objective value on all instances. However, their runtime varies considerably for the larger two Flywing-wide instances.