Derivation of Eqs. (37) and (38)

Starting with Eq. (36) of the paper,

$$\mathbf{\Delta s} \approx (\hat{\mathbf{1}} - \tilde{\mathbf{s}}\tilde{\mathbf{s}}^\top)\frac{\hat{\mathbf{n}}^{-1/2}\mathbf{\Delta}\tilde{\mathbf{v}}}{\|\hat{\mathbf{n}}^{-1/2}\tilde{\mathbf{v}}\|} \ , \tag{1}$$

where

$$\begin{aligned} \mathbf{\Delta}\tilde{\mathbf{v}} &\approx \hat{\hat{\mathbf{M}}}\,\hat{\hat{\mathbf{\Delta n}}}\,\tilde{\mathbf{v}} \ , && (2)\\ \hat{\hat{\mathbf{M}}} &= -\frac{1}{\tilde{\lambda}_1 - \tilde{\lambda}}\tilde{\mathbf{v}}_1\tilde{\mathbf{v}}_1^\top - \frac{1}{\tilde{\lambda}_2 - \tilde{\lambda}}\tilde{\mathbf{v}}_2\tilde{\mathbf{v}}_2^\top \ , && (3) \end{aligned}$$

the covariance matrix can be estimated:

$$\begin{aligned} &\langle \mathbf{\Delta s}\mathbf{\Delta s}^\top\rangle && (4)\\ &\approx \langle(\hat{\mathbf{1}} - \tilde{\mathbf{s}}\tilde{\mathbf{s}}^\top)\frac{\hat{\mathbf{n}}^{-1/2}\mathbf{\Delta}\tilde{\mathbf{v}}}{\|\hat{\mathbf{n}}^{-1/2}\tilde{\mathbf{v}}\|}\frac{\mathbf{\Delta}\tilde{\mathbf{v}}^\top\hat{\mathbf{n}}^{-1/2}}{\|\hat{\mathbf{n}}^{-1/2}\tilde{\mathbf{v}}\|}(\hat{\mathbf{1}} - \tilde{\mathbf{s}}\tilde{\mathbf{s}}^\top)\rangle && (5)\\ &= (\hat{\mathbf{1}} - \tilde{\mathbf{s}}\tilde{\mathbf{s}}^\top)\frac{\hat{\mathbf{n}}^{-1/2}\langle\mathbf{\Delta}\tilde{\mathbf{v}}\mathbf{\Delta}\tilde{\mathbf{v}}^\top\rangle\hat{\mathbf{n}}^{-1/2}}{\|\hat{\mathbf{n}}^{-1/2}\tilde{\mathbf{v}}\|^2}(\hat{\mathbf{1}} - \tilde{\mathbf{s}}\tilde{\mathbf{s}}^\top) && (6)\\ &= (\hat{\mathbf{1}} - \tilde{\mathbf{s}}\tilde{\mathbf{s}}^\top)\frac{\hat{\mathbf{n}}^{-1/2}\langle(\hat{\hat{\mathbf{M}}}\hat{\hat{\mathbf{\Delta n}}}\tilde{\mathbf{v}})(\hat{\hat{\mathbf{M}}}\hat{\hat{\mathbf{\Delta n}}}\tilde{\mathbf{v}})^\top\rangle\hat{\mathbf{n}}^{-1/2}}{\|\hat{\mathbf{n}}^{-1/2}\tilde{\mathbf{v}}\|^2}(\hat{\mathbf{1}} - \tilde{\mathbf{s}}\tilde{\mathbf{s}}^\top) && (7)\\ &= (\hat{\mathbf{1}} - \tilde{\mathbf{s}}\tilde{\mathbf{s}}^\top)\frac{\hat{\mathbf{n}}^{-1/2}\langle(\hat{\hat{\mathbf{M}}}\hat{\hat{\mathbf{\Delta n}}}\tilde{\mathbf{v}})(\tilde{\mathbf{v}}^\top\hat{\hat{\mathbf{\Delta n}}}^\top\hat{\hat{\mathbf{M}}}^\top)\rangle\hat{\mathbf{n}}^{-1/2}}{\|\hat{\mathbf{n}}^{-1/2}\tilde{\mathbf{v}}\|^2}(\hat{\mathbf{1}} - \tilde{\mathbf{s}}\tilde{\mathbf{s}}^\top) && (8)\\ &\approx (\hat{\mathbf{1}} - \tilde{\mathbf{s}}\tilde{\mathbf{s}}^\top)\hat{\mathbf{n}}^{-1/2}\hat{\hat{\mathbf{M}}}\frac{\langle\hat{\hat{\mathbf{\Delta n}}}\tilde{\mathbf{v}}\tilde{\mathbf{v}}^\top\hat{\hat{\mathbf{\Delta n}}}^\top\rangle}{\|\hat{\mathbf{n}}^{-1/2}\tilde{\mathbf{v}}\|^2}\hat{\hat{\mathbf{M}}}^\top\hat{\mathbf{n}}^{-1/2}(\hat{\mathbf{1}} - \tilde{\mathbf{s}}\tilde{\mathbf{s}}^\top) \ , && (9) \end{aligned}$$

where $\hat{\hat{\mathbf{M}}}$ was removed from the averaging in the final step.

Measured polarization vectors $\mathbf{p}_i$ deviate from error-free polarization vectors $\mathbf{p}_{i0}$ due to errors $\mathbf{\Delta p}_i$,

$$\mathbf{p}_i = \mathbf{p}_{0i} + \mathbf{\Delta p}_i \ , \tag{10}$$

and the measured polarization matrix $\hat{\mathbf{P}}$ can be expressed as the sum of an error-free matrix $\hat{\mathbf{P}}_0$ and error matrix $\hat{\mathbf{\Delta n}}$:

$$\begin{aligned} \hat{\mathbf{P}} &= \sum_i \tilde{w}_i(\mathbf{p}_{0i} + \mathbf{\Delta p}_i)(\mathbf{p}_{0i} + \mathbf{\Delta p}_i)^\top && (11)\\ &= \sum_i \tilde{w}_i\mathbf{p}_{0i}\mathbf{p}_{0i}^\top + \sum_i \tilde{w}_i(\mathbf{p}_{0i}\mathbf{\Delta p}_i^\top + \mathbf{\Delta p}_i(\mathbf{p}_{0i} + \mathbf{\Delta p}_i)^\top) && (12)\\ &= \hat{\mathbf{P}}_o + \sum_i \tilde{w}_i(\mathbf{p}_{0i}\mathbf{\Delta p}_i^\top + \mathbf{\Delta p}_i\mathbf{p}_i^\top) && (13)\\ &= \hat{\mathbf{P}}_o + \sigma^2\hat{\mathbf{\Delta n}} \ . && (14) \end{aligned}$$

Pre-whitening leads to the transformed error matrix $\hat{\hat{\boldsymbol{\Delta}}}\mathbf{n}$ defined as

$$\hat{\hat{\boldsymbol{\Delta}}}\mathbf{n} \quad = \quad \hat{\mathbf{n}}^{-1/2}\hat{\boldsymbol{\Delta}}\mathbf{n}\,\hat{\mathbf{n}}^{-1/2} \quad , \tag{15}$$

where $\hat{\mathbf{n}} = \sum_i \tilde{w}_i(\hat{\mathbf{1}} - \mathbf{e}_i\mathbf{e}_i^\top)$, see also page 4 of the paper.

Using these results leads to

$$\begin{aligned}
\frac{\hat{\hat{\boldsymbol{\Delta}}}\mathbf{n}\tilde{\mathbf{v}}}{\|\hat{\mathbf{n}}^{-1/2}\tilde{\mathbf{v}}\|} &= \frac{\hat{\mathbf{n}}^{-1/2}\hat{\boldsymbol{\Delta}}\mathbf{n}\,\hat{\mathbf{n}}^{-1/2}\tilde{\mathbf{v}}}{\|\hat{\mathbf{n}}^{-1/2}\tilde{\mathbf{v}}\|} && (16)\\
&= \hat{\mathbf{n}}^{-1/2}\hat{\boldsymbol{\Delta}}\mathbf{n}\frac{\hat{\mathbf{n}}^{-1/2}\tilde{\mathbf{v}}}{\|\hat{\mathbf{n}}^{-1/2}\tilde{\mathbf{v}}\|} && (17)\\
&= \hat{\mathbf{n}}^{-1/2}\hat{\boldsymbol{\Delta}}\mathbf{n}\,\tilde{\mathbf{s}} && (18)\\
&= \hat{\mathbf{n}}^{-1/2}\sum_i \tilde{w}_i(\boldsymbol{\Delta}\mathbf{p}_i\mathbf{p}_{0i}^\top + \mathbf{p}_i\boldsymbol{\Delta}\mathbf{p}_i^\top)\tilde{\mathbf{s}} && (19)\\
&= \hat{\mathbf{n}}^{-1/2}\sum_i \tilde{w}_i(\boldsymbol{\Delta}\mathbf{p}_i\mathbf{p}_{0i}^\top\tilde{\mathbf{s}} + \mathbf{p}_i\boldsymbol{\Delta}\mathbf{p}_i^\top\tilde{\mathbf{s}}) && (20)\\
&\approx \hat{\mathbf{n}}^{-1/2}\sum_i \tilde{w}_i(\boldsymbol{\Delta}\mathbf{p}_i 0 + \mathbf{p}_i\boldsymbol{\Delta}\mathbf{p}_i^\top\tilde{\mathbf{s}}) && (21)\\
&= \hat{\mathbf{n}}^{-1/2}\sum_i \tilde{w}_i\mathbf{p}_i\boldsymbol{\Delta}\mathbf{p}_i^\top\tilde{\mathbf{s}} \quad , && (22)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\langle\hat{\hat{\boldsymbol{\Delta}}}\mathbf{n}\tilde{\mathbf{v}}\tilde{\mathbf{v}}^\top\hat{\hat{\boldsymbol{\Delta}}}\mathbf{n}^\top\rangle}{\|\hat{\mathbf{n}}^{-1/2}\tilde{\mathbf{v}}\|^2} &= \hat{\mathbf{n}}^{-1/2}\langle\hat{\boldsymbol{\Delta}}\mathbf{n}\,\tilde{\mathbf{s}}\,\tilde{\mathbf{s}}^\top\hat{\boldsymbol{\Delta}}\mathbf{n}\rangle\hat{\mathbf{n}}^{-1/2} && (23)\\
&= \hat{\mathbf{n}}^{-1/2}\langle\sum_i \tilde{w}_i\mathbf{p}_i\boldsymbol{\Delta}\mathbf{p}_i^\top\tilde{\mathbf{s}}\sum_j \tilde{w}_j(\mathbf{p}_j\boldsymbol{\Delta}\mathbf{p}_j^\top\tilde{\mathbf{s}})^\top\rangle\hat{\mathbf{n}}^{-1/2} && (24)\\
&= \hat{\mathbf{n}}^{-1/2}\sum_i\sum_j \tilde{w}_i\tilde{w}_j\mathbf{p}_i\langle\boldsymbol{\Delta}\mathbf{p}_i^\top\tilde{\mathbf{s}}\tilde{\mathbf{s}}^\top\boldsymbol{\Delta}\mathbf{p}_j\rangle\mathbf{p}_j^\top\hat{\mathbf{n}}^{-1/2} && (25)\\
&= \hat{\mathbf{n}}^{-1/2}\Big(\sigma^2\sum_i w_i^2(1-(\tilde{\mathbf{s}}^\top\mathbf{e}_i)^2)\mathbf{p}_i\mathbf{p}_i^\top\Big)\hat{\mathbf{n}}^{-1/2} \quad , && (26)
\end{aligned}$$

where the transformation

$$\begin{aligned}
\langle\boldsymbol{\Delta}\mathbf{p}_i^\top\tilde{\mathbf{s}}\tilde{\mathbf{s}}^\top\boldsymbol{\Delta}\mathbf{p}_j\rangle &= \sigma^2\tilde{\mathbf{s}}^\top(\hat{\mathbf{1}} - \mathbf{e}_i\mathbf{e}_i^\top)\tilde{\mathbf{s}}\,\delta_{ij} && (27)\\
&= \sigma^2(1-(\tilde{\mathbf{s}}^\top\mathbf{e}_i)^2)\,\delta_{ij} && (28)
\end{aligned}$$

has been used. Thus,

$$\langle\hat{\boldsymbol{\Delta}}\mathbf{n}\,\tilde{\mathbf{s}}\,\tilde{\mathbf{s}}^\top\hat{\boldsymbol{\Delta}}\mathbf{n}\rangle \quad = \quad \sigma^2\sum_i w_i^2(1-(\tilde{\mathbf{s}}^\top\mathbf{e}_i)^2)\mathbf{p}_i\mathbf{p}_i^\top \quad , \tag{29}$$

and the covariance matrix is given by

$$
\begin{aligned}
&\langle \mathbf{\Delta s \Delta s}^\top \rangle &(30)\\
&\approx \quad (\hat{\mathbf{1}} - \tilde{\mathbf{s}}\tilde{\mathbf{s}}^\top)\hat{\mathbf{n}}^{-1/2}\hat{\tilde{\mathbf{M}}}\hat{\mathbf{n}}^{-1/2}\langle \hat{\mathbf{\Delta n}}\,\tilde{\mathbf{s}}\,\tilde{\mathbf{s}}^\top \hat{\mathbf{\Delta n}}\rangle \hat{\mathbf{n}}^{-1/2}\hat{\tilde{\mathbf{M}}}^\top \hat{\mathbf{n}}^{-1/2}(\hat{\mathbf{1}} - \tilde{\mathbf{s}}\tilde{\mathbf{s}}^\top) &(31)\\
&= \quad \hat{\mathbf{Q}}_{\tilde{\mathbf{s}}}\Big(\sigma^2 \sum_i \tilde{w}_i^2 (1 - (\tilde{\mathbf{s}}^\top \mathbf{e}_i)^2)\mathbf{p}_i \mathbf{p}_i^\top\Big)\hat{\mathbf{Q}}_{\tilde{\mathbf{s}}}^\top \quad , &(32)
\end{aligned}
$$

where $\hat{\mathbf{Q}}_{\tilde{\mathbf{s}}} = (\hat{\mathbf{1}} - \tilde{\mathbf{s}}\tilde{\mathbf{s}}^\top)\hat{\mathbf{n}}^{-1/2}\hat{\tilde{\mathbf{M}}}\hat{\mathbf{n}}^{-1/2}$.