Online Robust Image Alignment via Subspace Learning from Gradient Orientations: Supplementary Material

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1. Proof of Lemma

Lemma 1. *Let $\mathbf{A} = \mathbf{R}_{(:,1:n)}$, $\mathbf{B} = \mathbf{R}_{(:,n+1:n+m)}$ be the low-rank estimated data matrices and $\mathbf{C} = [\mathbf{A}\ \mathbf{B}]$ be a concatenated matrix of all estimated data. Let scatter matrix be the outer product of the centered data matrix. The means and scatter matrices of $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are $\bar{\mathbf{R}}_A, \bar{\mathbf{R}}_B, \bar{\mathbf{R}}_C$ and $\mathbf{S}_A, \mathbf{S}_B, \mathbf{S}_C$, respectively. It can be shown that*

$$
\begin{aligned}
\bar{\mathbf{R}}_C &= \frac{n}{n+m}\bar{\mathbf{R}}_A + \frac{m}{n+m}\bar{\mathbf{R}}_B, \\
\mathbf{S}_C &= \mathbf{S}_A + \sum_{i=n+1}^{n+m}(\mathbf{R}_i - \bar{\mathbf{R}}_C)(\mathbf{R}_i - \bar{\mathbf{R}}_C)^T \\
&\quad + \frac{nm^2}{(n+m)^2}(\bar{\mathbf{R}}_A - \bar{\mathbf{R}}_B)(\bar{\mathbf{R}}_A - \bar{\mathbf{R}}_B)^T.
\end{aligned} \tag{1}
$$

Proof of Lemma 1: By the definition of mean, we have

$$
\begin{aligned}
\bar{\mathbf{R}}_C &= \frac{1}{n+m}\{\sum_{i=1}^{n}\mathbf{R}_i + \sum_{i=n+1}^{n+m}\mathbf{R}_i\} \\
&= \frac{1}{n+m}\{n\bar{\mathbf{R}}_A + m\bar{\mathbf{B}}_B\} \\
&= \frac{n}{n+m}\bar{\mathbf{R}}_A + \frac{m}{n+m}\bar{\mathbf{R}}_B
\end{aligned} \tag{2}
$$

Then we have $\bar{\mathbf{R}}_A - \bar{\mathbf{R}}_C = \frac{m}{n+m}(\bar{\mathbf{R}}_A - \bar{\mathbf{R}}_B)$ and the scatter matrix of previous n samples with shift mean $\bar{\mathbf{R}}_C$ can be calculated with

$$
\begin{aligned}
&\sum_{i=1}^{n}(\mathbf{R}_i - \bar{\mathbf{R}}_C)(\mathbf{R}_i - \bar{\mathbf{R}}_C)^T \\
&= \sum_{i=1}^{n}(\mathbf{R}_i - \bar{\mathbf{R}}_A + \bar{\mathbf{R}}_A - \bar{\mathbf{R}}_C)(\mathbf{R}_i - \bar{\mathbf{R}}_A + \bar{\mathbf{R}}_A - \bar{\mathbf{R}}_C)^T \\
&= \mathcal{S}_A + n(\bar{\mathbf{R}}_A - \bar{\mathbf{R}}_C)(\bar{\mathbf{R}}_A - \bar{\mathbf{R}}_C)^T \\
&= \mathcal{S}_A + \frac{nm^2}{(n+m)^2}(\bar{\mathbf{R}}_A - \bar{\mathbf{R}}_B)(\bar{\mathbf{R}}_A - \bar{\mathbf{R}}_B)^T
\end{aligned} \tag{3}
$$

Therefore, we have

$$
\begin{aligned}
\mathcal{S}_C =& \sum_{i=1}^{n}(\mathbf{R}_i - \bar{\mathbf{R}}_C)(\mathbf{R}_i - \bar{\mathbf{R}}_C)^T \\
&+ \sum_{i=n+1}^{n+m}(\mathbf{R}_i - \bar{\mathbf{R}}_C)(\mathbf{R}_i - \bar{\mathbf{R}}_C)^T \\
=& \mathcal{S}_A + \sum_{i=n+1}^{n+m}(\mathbf{R}_i - \bar{\mathbf{R}}_C)(\mathbf{R}_i - \bar{\mathbf{R}}_C)^T \\
&+ \frac{nm^2}{(n+m)^2}(\bar{\mathbf{R}}_A - \bar{\mathbf{R}}_B)(\bar{\mathbf{R}}_A - \bar{\mathbf{R}}_B)^T.
\end{aligned} \tag{4}
$$

Lemma 2. *Let $\mathbf{Q} = \begin{bmatrix} \mathbf{\Sigma} & \mathbf{U}^T\hat{\mathbf{B}} \\ \mathbf{0} & \tilde{\mathbf{B}}^T\hat{\mathbf{B}} \end{bmatrix}$, $\tilde{\mathbf{U}}\tilde{\mathbf{\Sigma}}\tilde{\mathbf{V}}^T$ denotes the thin SVD of $\mathbf{Q}$, the update of basis matrix and eigenvalues can be calculated with $\mathbf{U}^* = [\mathbf{U}\ \tilde{\mathbf{B}}]\tilde{\mathbf{U}}$ and $\mathbf{\Sigma}^* = \tilde{\mathbf{\Sigma}}$.*

Proof of Lemma 2: Since $\mathbf{A} = \mathbf{U\Sigma V}^T$, $\tilde{\mathbf{B}}$ denotes the component of $\hat{\mathbf{B}}$ orthonormal to $\mathbf{U}$. Then the concatenation of $\mathbf{A}$ and $\hat{\mathbf{B}}$ can be decomposed as

$$
\begin{aligned}
\begin{bmatrix}\mathbf{A}\ \hat{\mathbf{B}}\end{bmatrix} =& \mathbf{U}^*\mathbf{\Sigma}^*\mathbf{V}^{*T} \\
=& [\mathbf{U}\ \tilde{\mathbf{B}}]\begin{bmatrix} \mathbf{\Sigma} & \mathbf{U}^T\hat{\mathbf{B}} \\ \mathbf{0} & \tilde{\mathbf{B}}^T\hat{\mathbf{B}} \end{bmatrix}\begin{bmatrix} \mathbf{V}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}
\end{aligned} \tag{5}
$$

Since the incremental SVD only interests in computing $\mathbf{U}^*$,$\mathbf{\Sigma}^*$ and $\mathbf{V}^*$ with the newly observed data, to construct

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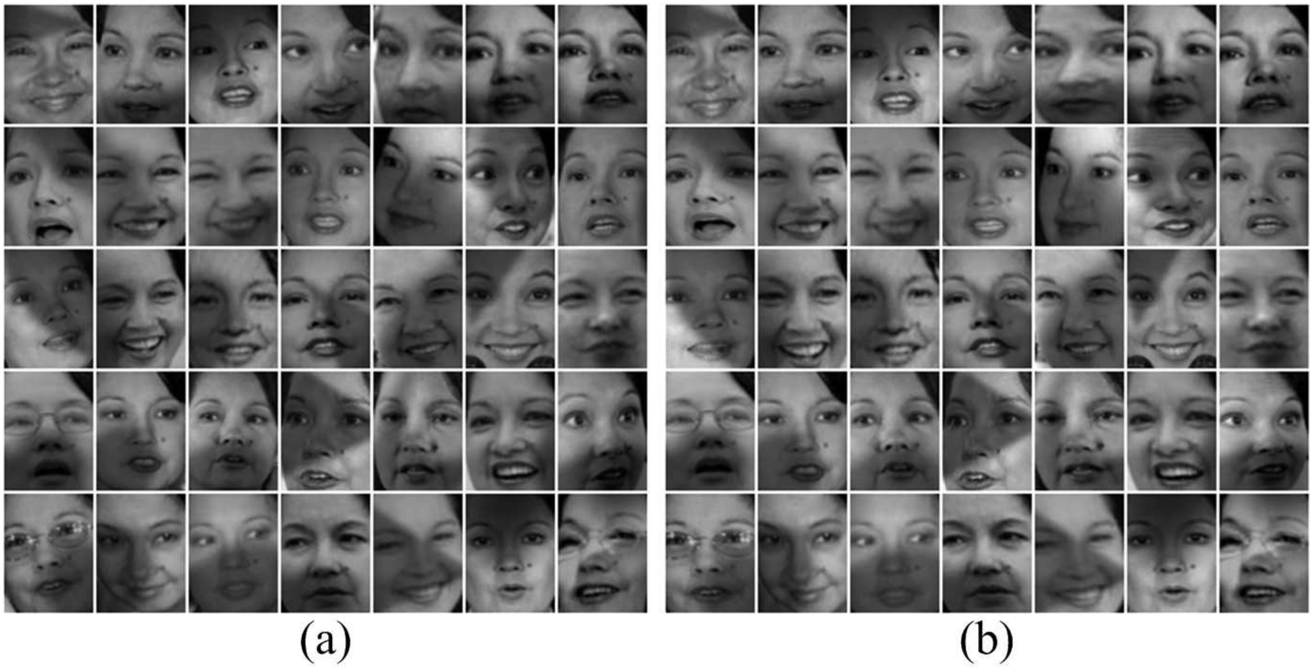

Figure 1: Alignment results of Gloria Macapagal Arroyo's facial images. (a) Unaligned images all with artificially added shadows. (b) Well-aligned images by our method despite illumination variations.

the diagonal matrix $\mathbf{\Sigma}^*$, we again employ thin SVD on $\begin{bmatrix} \mathbf{\Sigma} & \mathbf{U}^T\hat{\mathbf{B}} \\ \mathbf{0} & \tilde{\mathbf{B}}^T\hat{\mathbf{B}} \end{bmatrix}$ and decompose it as $\tilde{\mathbf{U}}\tilde{\mathbf{\Sigma}}\tilde{\mathbf{V}}^T$, then

$$\begin{bmatrix}\mathbf{A}\ \hat{\mathbf{B}}\end{bmatrix} = \left([\mathbf{U}\ \tilde{\mathbf{B}}]\tilde{\mathbf{U}}\right)\ \tilde{\mathbf{\Sigma}}\ \left(\tilde{\mathbf{V}}^T \begin{bmatrix} \mathbf{V}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}\right) \qquad (6)$$

Thus $\mathbf{U}^* = \left([\mathbf{U}\ \tilde{\mathbf{B}}]\tilde{\mathbf{U}}\right)$ and $\mathbf{\Sigma}^* = \tilde{\mathbf{\Sigma}}$.

2. Occlusion and illumination variation

To further demonstrate the advantage of our method in aligning images despite severe intensity distortions, we show the alignment results of Gloria Macapagal Arroyo's facial images all with artificially added shadows in Fig. 1. This further verifies the robustness of our method.