HiPPI: Higher-Order Projected Power Iterations for Scalable Multi-Matching (Supplementary Material)

A. Proofs

A.1. Proof of Lemma 2

Proof. We show that conditions (i)-(iii) in Def. 1 are fulfilled. Let $i, j, \ell \in [k]$ be fixed. (i) We have that $X_i \mathbf{1}_d = \mathbf{1}_{m_i}$ and $X_i \in \mathbb{P}_{m_i d}$ imply that $X_i X_i^T = \mathbf{I}_{m_i}$, so that $X_{ii} = X_i X_i^T = \mathbf{I}_{m_i}$. (ii) Moreover, $X_{ij} = X_i X_j^T$ means that $X_{ij}^T = (X_i X_j^T)^T = X_j X_i^T = X_{ji}$. (iii) We have that $X_j \in \mathbb{P}_{m_j d}$ implies $X_j^T X_j \leq \mathbf{I}_d$. We can write

$$\begin{aligned} X_{ij} X_{j\ell} &= (X_i X_j^T)(X_j X_\ell^T) && (12)\\ &= X_i \underbrace{(X_j^T X_j)}_{\leq \mathbf{I}_d} X_\ell^T \leq X_i X_\ell^T = X_{i\ell}\,. \end{aligned}$$ ■

A.2. Proof of Lemma 4

Proof. With W being positive semidefinite, we can factorise it as $W = L^T L$. We have

$$\begin{aligned} 0 &\leq \|LUU^T L^T - LVV^T L^T\|^2 && (13)\\ &= \mathrm{tr}(LUU^T L^T LUU^T L^T) + \mathrm{tr}(LVV^T L^T LVV^T L^T)\\ &\quad - 2\,\mathrm{tr}(LUU^T L^T LVV^T L^T) && (14)\\ &= \mathrm{tr}(U^T WUU^T WU) - \mathrm{tr}(U^T WVV^T WU) && (15)\\ &\quad \underbrace{- \mathrm{tr}(U^T WVV^T WU) + \mathrm{tr}(V^T WVV^T WV)}_{\leq 0 \text{ by assumption}} \end{aligned}$$

$$\Rightarrow \mathrm{tr}(U^T WUU^T WU) - \mathrm{tr}(U^T WVV^T WU) \geq 0\,.$$ ■

A.3. Proof of Proposition 3

Proof. We first look at a simpler variant of Problem (9), where for a fixed Y we fix the inner term UU^T in f to YY^T, define $Z := \overline{W} YY^T \overline{W}$, and relax $\mathcal{U}$ to its convex hull $\mathcal{C} := \mathrm{conv}(\mathcal{U})$, so that we obtain the problem

$$\max_{U \in \mathcal{C}} \ \mathrm{tr}(U^T ZU) \ \Leftrightarrow \ \min_{U} \ \underbrace{\iota_{\mathcal{C}}(U)}_{g(U)} - \underbrace{\mathrm{tr}(U^T ZU)}_{h(U)}\,. \quad (16)$$

Here, $\iota_{\mathcal{C}}(U)$ is the (convex) indicator function of the set $\mathcal{C}$ and $\mathrm{tr}(U^T ZU)$ is a convex quadratic function. We can see that (16) is in the form of a difference of convex functions and can thereby be tackled based on DC programming, which for a given initial U_0 repeatedly applies the following update rules (see [29]):

$$\begin{aligned} V_t &= \nabla_U h(U_t) = 2ZU_t && (17)\\ U_{t+1} &= \arg\min_U \ g(U) - h(U_t) - \langle U - U_t, V_t \rangle && (18)\\ &= \arg\min_{U \in \mathcal{C}} \ -\langle U, V_t \rangle\,. && (19) \end{aligned}$$

Since partial permutation matrices form the vertices of their convex hull, see [32], and $\mathcal{U}$ (and $\mathcal{C}$) are formed by the Cartesian product of k partial permutation matrices (and their convex hull), the set $\mathcal{U}$ forms the extreme points of $\mathcal{C}$. As the maximum of a linear objective over a compact convex set is attained at its extreme points, we get

$$\begin{aligned} U_{t+1} &= \arg\min_{U \in \mathcal{U}} \ -\langle U, V_t \rangle && (20)\\ &= \arg\min_{U \in \mathcal{U}} \|V_t - U\|_F^2 = \mathrm{proj}_{\mathcal{U}}(V_t)\,, && (21) \end{aligned}$$

where for the latter we used that $\langle U, U \rangle = m$ (since any $U \in \mathcal{U}$ is a binary matrix that has exactly a single element in each row that is 1). As such, based on the descent properties of DC programming (i.e. the sequence $(g(U_t) - h(U_t))_{t=0,1,\ldots}$ is decreasing, cf. [29], and thus $(h(U_t))_{t=0,1,\ldots}$ is increasing), so far we have seen that when applying the update $U_{t+1} = \mathrm{proj}_{\mathcal{U}}(\overline{W} YY^T \overline{W} U_t)$ we get that

$$\mathrm{tr}(U_t^T \overline{W} YY^T \overline{W} U_t) \leq \mathrm{tr}(U_{t+1}^T \overline{W} YY^T \overline{W} U_{t+1})\,. \quad (22)$$

In particular, this also holds for the choice $Y := U_t$, i.e.

$$\begin{aligned} f(U_t) &= \mathrm{tr}(U_t^T \overline{W} U_t U_t^T \overline{W} U_t) && (23)\\ &\leq \mathrm{tr}(U_{t+1}^T \overline{W} U_t U_t^T \overline{W} U_{t+1})\,. && (24) \end{aligned}$$

Since A is positive semidefinite by assumption, $\overline{W} = W^T AW$ is also positive semidefinite, and therefore we can apply Lemma 4 to get

$$\begin{aligned} f(U_t) &\leq \mathrm{tr}(U_{t+1}^T \overline{W} U_t U_t^T \overline{W} U_{t+1}) && (25)\\ &\leq \mathrm{tr}(U_{t+1}^T \overline{W} U_{t+1} U_{t+1}^T \overline{W} U_{t+1}) = f(U_{t+1})\,. \end{aligned}$$ ■

A.4. Proof of Corollary 5

Proof. Since $\mathcal{U}$ is a finite set, $f(U)$ is bounded above for any $U \in \mathcal{U}$. Moreover, since for any $t \geq 0$ we have that $U_t \in \mathcal{U}$ (feasibility), the sequence $(f(U_t))_{t=0,1,\ldots}$ produced by Alg. 1 is bounded and increasing (Proposition 3), and hence convergent. Since the U_t are discrete, convergence implies that there exists a $t_0 \in \mathbb{N}$ such that for all $t \geq t_0$ it holds that $f(U_t) = f(U_{t_0})$. ■

B. Robustness Analysis

Here, we additionally present robustness evaluations. In Fig. 7 we use our method to solve multi-matching problems with a varying numbers of objects. We observe that with a larger number of images the overall result quality improves (since multi-matching problems with more objects contain more information that can be leveraged for establishing a better multi-matching).

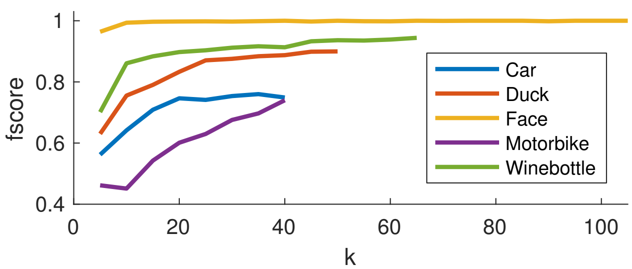

Figure 7. Robustness analysis of our method for different problem sizes k using the WILLOW dataset.

In Fig. 8 we compare our method and `QuickMatch` [48] with respect to the sensitivity to outliers, where it can be seen that our method is more robust. This can be explained by the fact that our method considers geometric consistency and thereby is able to disregard many spurious matchings.

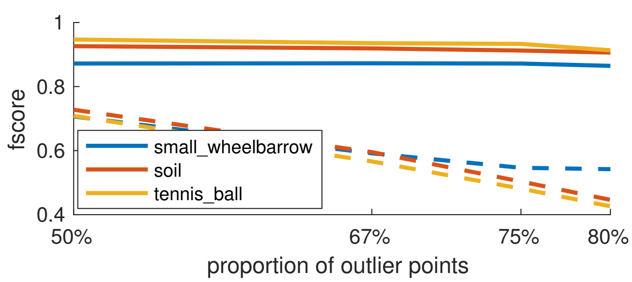

Figure 8. Sensitivity analysis of our method (solid lines) vs. `QuickMatch` [48] (dashed lines) w.r.t. different outlier proportions on three instances (colours) of the HiPPI dataset. Our method is more robust compared to `QuickMatch`. Outlier points were selected from the previously pruned points, as described in Sec. 4.1, so the largest proportion that we could evaluate is limited.