Noising versus Smoothing for Vertex Identification in Unknown Shapes

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***Abstract*—A method for identifying shape features of local nature on the shapes boundary, in a way that is facilitated by the presence of noise is presented. The boundary is seen as a real function. A study of a certain distance function reveals, almost counter-intuitively, that vertices can be defined and localized better in the presence of noise, thus the concept of *noising*, as opposed to smoothing, is conceived and presented. The method works on both smooth and noisy shapes, the presence of noise having an effect of *improving* on the results of the smoothed version. Experiments with noise and a comparison to state of the art validate the method.**

***Keywords*-Vertex;Shape;Noising;Vertex Localization;Noise Resistance;Shape Representation;Object Recognition;**

I. Introduction

Curvature is a local descriptor. For planar curves it is defined as the rate of change of the tangent to the curve per infinitesimal arc length. A *vertex* is a point on the curve where curvature attains local maximum or local minimum. In a noisy curve, the local nature of curvature restricts it in describing the high frequency Fourier components (hfFc) themselves rather than the underlying shape. The presence of hfFc is considered a problematic situation because, for one reason, there is no easy way to evaluate whether these represent noise or not. That would require solving the harder problem of recognizing the object. The hfFc might be defining for certain shapes, but might be just noise in others. For another reason, the calculation of critical local features on the boundary e.g. curvature, is dominated by the hfFc but again, it is unknown whether hfFc describes noise or not.

As a consequence, it seems inevitable to eliminate the hfFc from the boundary of all shapes, as a blind preposssessing step, at the expense of losing useful discriminating information, in cases where hfFc are not noise but represent actual shape features. Even worse, smoothing distorts the shape's metrics in an unpredictable manner, a highly undesirable effect whenever certain *morphometric* measurements are defining for classification. The inevitability of smoothing for local feature extraction is challenged in this paper.

II. Related work

Even though an enormous amount of work related to shape analysis has been performed in the last few decades, the presented herein method relates closer to approaches of describing shapes by real functions [1] and more specifically to attempts of estimating local shape features e.g. curvature in the presence of noise.The presented work is based on the *VAR descriptor* [2], that can be used to define curvature in a global sense. In [3], a framework for integral invariants is introduced and a resulting *localized* Local Area Integral Invariant (LAII) descriptor is defined. LAII is one of the few state of the art, low level methods, with complexity similar to the proposed hereinafter, thus we use it for comparison in estimating curvature/vertices in noise affected curves without smoothing. Other methods of calculating local features while resisting noise can be found in [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18].

III. Contribution

The main contribution lies in proposing a new method for vertex localization that is inspired by ([2]) but is simpler, faster and doesn't need curvature calculations. Bypassing curvature, the new method makes noise an enabling factor. A fundamental belief, according to which the negative effects of noise increase with the order of differentiation, is thus unexpectedly confronted in the manuscript while a new way of thinking emerges, that of *noise enabled* global vertex localizers.

To the best of our knowledge, there is no attempt in the past and current literature, to use noise as a means for extracting local shape features or higher order differentials. The method presented herein is in that respect, orthogonal to the popular differential approaches. The new concept of *noising*, as opposed to smoothing is conceived and presented. Under this new concept, a boundary can be preprocessed by adding noise in the form of hfFc and then analyzed according to the proposed hereinafter analysis for curvature calculation and vertex identification. Noising, is considered here as the process of adding hfFc to the existing boundary (noisy or not) in an additive manner that does not affect the initial boundary points. Vertices are then identified by treating the additional points as *neighborhood* for facilitating the global descriptors. In comparison to smoothing, a lossy operation that leaves no original boundary point untouched, *noising* emerges as a *strong alternative* for automatic local feature extraction in unknown curves. As was also mentioned in the Related Work section, the proposed method is based on theoretical findings established in [2] but the concern there is for shape matching. The concept of *noising* and its ability to improve local feature extraction, the method for localizing vertices and estimating curvature, the relevant experiments with smooth and noisy shapes and the such comparison to LAII in these tasks, is presented here for the first time.

Since the presented method is valid in both smooth and noisy boundaries, it conveys a *generality*, appreciated especially in comparison to the methods that rely on smoothing. Indeed the latter are valid only after a smooth boundary is guaranteed and they carry no credibility in the presence of noise. The presented method therefore, not only challenges, in an almost counterintuitive manner, the inevitability of smoothing for local boundary feature extraction, but also serves a *unification* purpose: that of of extracting local features from all shapes regardless of noise without jeopardizing useful shape information. A comparison to LAII[3] in curvature calculation/vertex identification in the presence of noise, reveals the advantages of the proposed approach.

IV. The Proposed Method

The method is based on the relationship between the *relative location* of the points on a closed curve and the curvature at those points. By *relative location*, we mean location with respect to the rest of the curve. This relationship is important because location, as opposed to curvature, is not affected by noise. For completeness, we briefly introduce here the VAR descriptor and its relation to location and curvature [2]:

A. The VAR descriptor

Let $(0, \lambda] \subset \mathbb{R}$ and $\boldsymbol{\alpha} : (0, \lambda] \to \mathbb{R}^2$ a continuous one to one[1], at least C^3, *planar curve of non zero length* λ in $\mathbb{R}^2$, parametrized with respect to the arc length s. Throughout this paper we consider the following additional *regularity assumptions* regarding $\boldsymbol{\alpha}$:

1) $\boldsymbol{\alpha} : S^1 \to \mathbb{R}^2$ a *closed* curve in $\mathbb{R}^2$ thus equivalent to a continuous mapping of the unit circle S^1 into the real plane $\mathbb{R}^2$.
2) $\boldsymbol{\alpha}(s) \in C^3(S^1, \mathbb{R}^2)$, the closed curve $\boldsymbol{\alpha}$ is at least three times differentiable at every point in the unit circle ensuring that the curve is sufficiently smooth at all the points.

For every point on the curve we consider the sum of its distances to all the other points. In the discrete case it is a summation but in the continuous case it is an integral. We proceed with the continuous case since the discrete is just a simplification.

[1]Equivalent to non self intersecting.

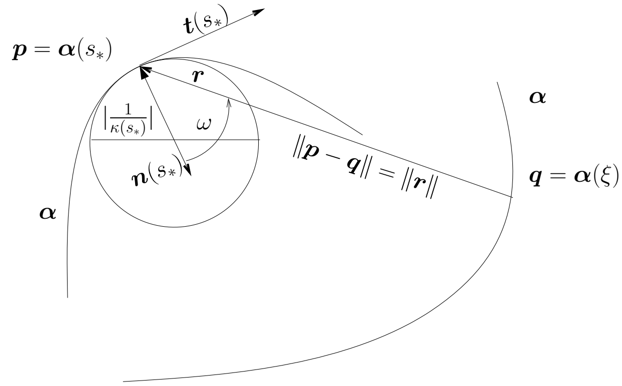

Figure 2. Local bindings of the view functions. Two portions of the same boundary, the osculating circle and the Frenet frame at $\boldsymbol{\alpha}(s_*)$.

Definition 1: Let $\boldsymbol{\alpha}$ be a closed curve of length λ as above and ϕ_α be a distance function defined on $[0, \lambda]$ and taking values in $\mathbb{R}$ as follows:

$$\phi_\alpha : [0, \lambda] \to \mathbb{R}\colon s \mapsto \phi_\alpha(s) := \int_0^\lambda \|\boldsymbol{\alpha}(s) - \boldsymbol{\alpha}(\xi)\| \, d\xi \tag{1}$$

$\phi_\alpha(s)$ is called the *VAR descriptor* and can be interpreted as modeling a notion of *total distance* between the curve point $\boldsymbol{\alpha}(s)$ and the rest of the curve.

Now let $s_*, \in (0, \lambda]$ such that the normal to the curve at $\boldsymbol{\alpha}(s_*)$ is considered explicitly and $\xi \in (0, \lambda]$ with $\xi \neq s_*$ signifying a random point $\boldsymbol{\alpha}(\xi)$ on the curve. As shown in Fig. (2), we denote with $\boldsymbol{r}(s_*, \xi) \equiv \boldsymbol{r}$ the vector $\boldsymbol{\alpha}(s_*) - \boldsymbol{\alpha}(\xi)$ and $\omega(s_*, \xi) \equiv \omega$ the angle from the *normal* to the curve at s_* to $-\boldsymbol{r}(s_*, \xi)$ measured counter-clockwise. In the form of a Theorem, we gather results from [2]. Dots represent derivatives always with respect to s.

***Theorem** 1:* Let $\boldsymbol{\alpha} \in C^3((0, \lambda], \mathbb{R}^2)$ a closed planar curve of nonzero length λ, at least 3 times differentiable as a function defined on the unit circle. If $\phi_\alpha(s)$ the total distance function (VAR descriptor), $\kappa(s)$ the curvature function and s_*, ξ, $\boldsymbol{r}$ and ω as above, then:

1)
$$\dot{\phi}_\alpha(s_*) = -\int_0^\lambda sin(\omega) d\xi \Big|_{s=s_*} \tag{2}$$

2)
$$\ddot{\phi}_\alpha(s_*) = \kappa(s_*) A(s_*) + B(s_*) \tag{3}$$

where $A(s_*) = \int_0^\lambda cos(\omega) d\xi \Big|_{s=s_*}$ and $B(s_*) = \int_0^\lambda \frac{cos^2(\omega)}{\|\boldsymbol{r}\|} d\xi \Big|_{s=s_*}$ global shape descriptors measured at $\boldsymbol{\alpha}(s_*)$.

3) If in addition, $\phi_\alpha(s_*)$ a local extremum of $\phi_\alpha(s)$. Then $\kappa(s_*) \neq 0$ and $A(s_*) \neq 0$ and

$$\kappa(s_*) = \frac{\ddot{\phi}_\alpha(s_*) - B(s_*)}{A(s_*)} \tag{4}$$

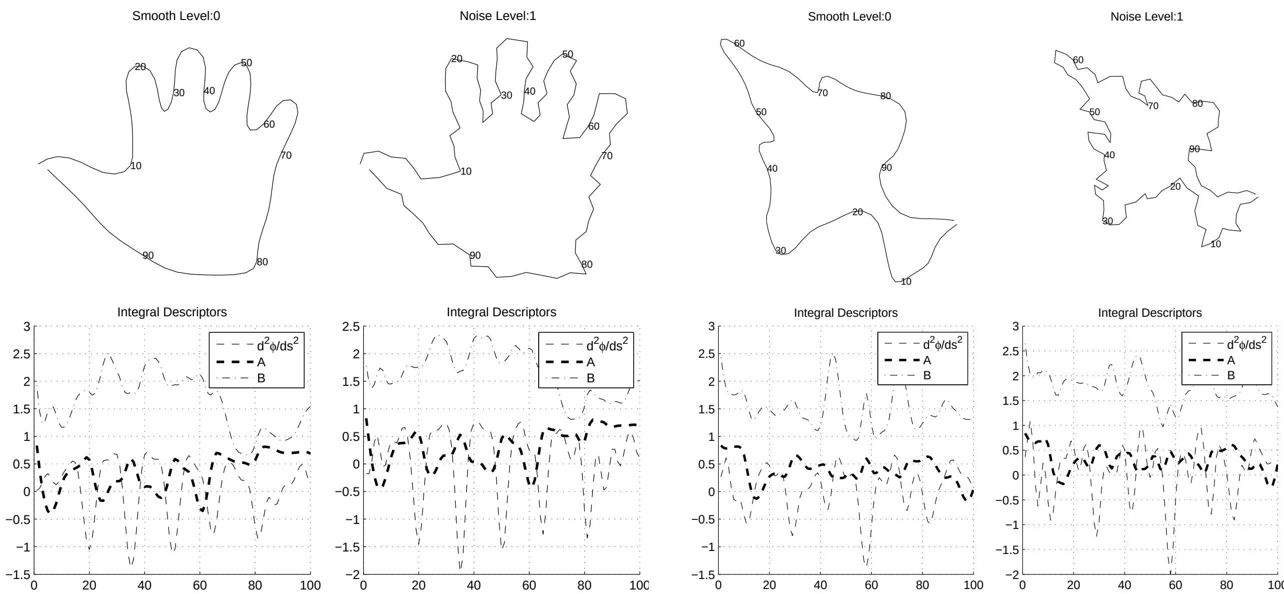

Figure 1. The integral descriptors A, B and $\ddot{\phi}$, in noisy and smoothed versions of Kimia silhouettes. Each shape is sampled by 100 equal spaced points marked in steps of 10 in the Figure and depicted in the x-axis of the corresponding plots. Notice there is no significant distortion due to noise. Curvature can be defined through these descriptors in a global manner.

As we can see from the Theorem, the integral descriptor A has a similar interpretation with $\dot{\phi}$. They both quantify a notion of displacement of the whole curve with respect to the normal at s_*. Indeed, if we consider a point $\boldsymbol{\alpha}(\xi)$ that traverses the curve, angle ω measures the angular displacement of this point with respect to the normal at s_*. Thus the integral A (and $\dot{\phi}$) can be thought of measuring the *total angular displacement* of the whole curve with respect to the normal at s_* and it is important to notice that they are related to the tangent at their point of calculation. The integral descriptor B has a similar interpretation.

B. Identifying Vertices and the Concept of Noising

In this section we proceed with observations regarding equation(4) and a new method for vertex localization.We observe that equation (4) defines curvature through global descriptors. All quantities on the right hand side of equation (4) are integrals defined on the whole of the shape and as such they don't change significantly with noise. The above observations lead to the conclusion that this definition of curvature is *stronger than the differential one*. In fact, noise not only is not affecting significantly this definition of curvature but, as we will later show, it also improves the identification of vertices, giving rise to the concept of *noising* as opposed to *smoothing*.

This result is counterintuitive since vertices are of a higher differential order than curvature, thus even more sensitive to noise with traditional methods. An explanation of this paradox emerges if we look closer into the interrelation of A with ϕ. From the Theorem (item 1) we get that the zero crossings of $\dot{\phi}$, (non trivial local extrema of ϕ) are related to the local extrema of A. According to equation (4), this is an indication that at high curvatures, the local extrema of κ are *collocated* with the local exrema of $\ddot{\phi}$ and ϕ, the later through A by means of equation (2). A thorough mathematical analysis of this exact relation involves examining the location of zero crossings of linear combinations of analytical functions, a non trivial task that would dominate the current manuscript, we thus confine ourselves here to the supporting experiments. In fact, the experiments indicate that at certain points (where the curve is not equally displaced around the normal), the induction of further noise around these points has the effect of *correcting* the curve's displacement, thus facilitating extrema curvature measurements by means of the distance function $\ddot{\phi}$.

As an illustration of this phenomenon, one can imagine traveling along a closed boundary being restricted to look only in the direction normal to the boundary at the current position. In such a scenario, a noisy journey would mean traveling along a noisy boundary and would provide richer views, since the fixed direction of one's gaze would be compensated by the diversity in the directions of movement. That would increase the ability to understand the whole shape compared to a smooth journey traveled along a smooth boundary.

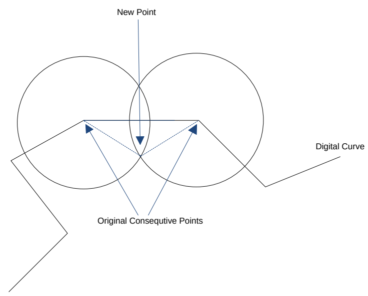

Figure 3. The method of noising as opposed to smoothing is illustrated here. New points are added to the contour with the purpose to enrich the tangent directions around the original points. This will facilitate the proposed method in identifying vertices.

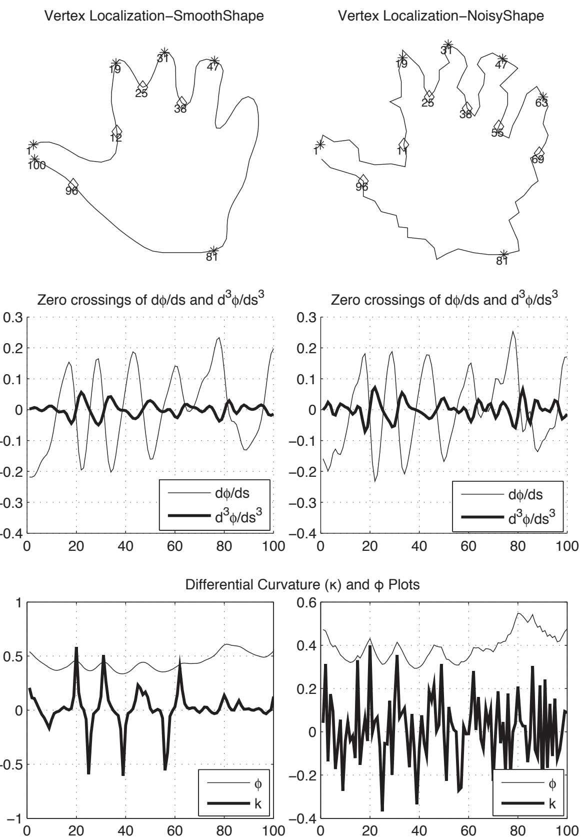

Figure 4. Vertex localization for smooth and noise versions of the same Kimia Silhouette using co- localization of ϕ and $\ddot{\phi}$ extreme points. Stars and diamonds are curvature's local maxima and minima respectively. Points are marked on the shapes for every 10th point in a total of 100 points per shape and are also assumed as the x axis in all plots. The co-localization of zero crossings that appear in the second row of plots are validated against ϕ and curvature plots appearing in the last row. We notice that more points are correctly identified in the noisy version. We also notice that the proposed method produces correct results even though the differential curvature descriptor has collapsed in the noisy case.

In Fig. (1) we show the integral descriptors A, B and $\ddot{\phi}$ involved in equation (4) for smoothed and noisy versions of Kimia silhouettes [19]. We notice there is no significant distortion due to noise. After the above observations one can conclude at this point that Theorem (1) actually describes a method for localizing vertices of significant curvature. Indeed, at the points of local maxima of ϕ, both κ and A are non zero, thus the curvature there can be calculated by equation (4). The local maxima of ϕ can be easily identified in the *zero crossings* of $\dot{\phi}$ but also the extreme points of the second derivative $\ddot{\phi}$ are easily identifiable as well, in the zero crossings of $\dddot{\phi}$ (this is the reason we require differentiability of at least third order for ϕ). Now we recall that an extreme point of ϕ declares extreme location on the boundary, whereas an extreme point of $\ddot{\phi}$ declares extreme curvature on the boundary, as equation (4) suggests thus a method of identifying points of extreme location and curvature in the *collocation* of $\dot{\phi}$ and $\dddot{\phi}$ zero crossings, is in place. Under this method noise would have no effect since it doesn't affect location significantly.

The new concept of *noising*, as opposed to smoothing is now described. Under this concept a boundary can be preprocessed by adding noise in the form of hfFc.

Noising can be performed in an additive manner to the existing boundary, therefore not affecting the initial boundary points. In the discrete case of a digital curve, for each pair of consecutive points on the initial boundary a *new point* will be added at the intersection of the circles centered at the original points and having equal radii drawn from a normal distribution with zero mean and variance equal to the distance between the two original points (Fig.3).

The curvature and other calculations can then be estimated by treating the additional points as *neighborhood* as this facilitates the global descriptors. For some points that are close of being extreme on the boundary, increasing noise may affect the location of their neighborhood points, not much but enough to make the initial points appear as extreme. In that case the initial points will also be correctly considered as vertices, if they so qualify as extreme curvature points as well. At the same time points can appear as extremes, due to the induced noise around them, but they will not be selected as vertices unless the rate of change of their total distance to the rest of the curve is also achieving an extreme.

In comparison to smoothing, a lossy operation that leaves no original boundary point untouched, noising emerges as a *strong alternative* for automatic local feature extraction in unknown curves. It is important to notice that according to this method vertices are detected directly, by the collocation of the zero crossings as above, without having to calculate the actual curvature. In other words, the method does not

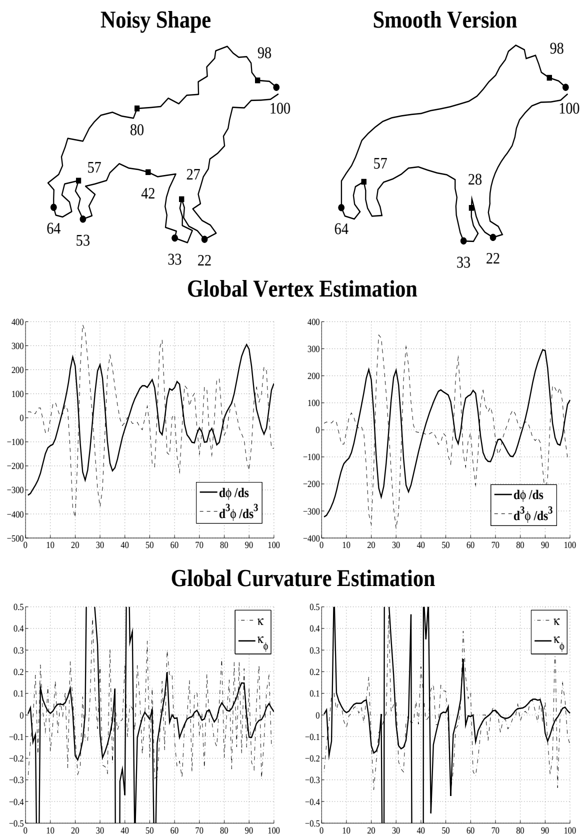

Figure 5. Curvature estimation and vertex localization according to the proposed method (based on equation (4)). At the top row of plots, solid circles and squares are curvature local minima and maxima respectively, identified by the proposed method for smooth and noisy versions of the same KIMIA silhouette. At the middle row of plots, the global descriptors involved in the localization of vertices according to the proposed method, namely $\dot{\phi}$ and $\dddot{\phi}$, are shown for the same silhouettes. At the bottom row of plots we see the differential curvature (κ) and the curvature estimator according to the proposed method (κ_{ϕ}), valid only at points where $A \neq 0$, for both noisy and smooth shapes. Notice that more vertices are correctly identified in the noisy case and the robustness of the proposed curvature estimator where the differential curvature has collapsed. In comparison to LAII method in Fig.(6), the proposed globally defined vertices are more intuitive and less affected by local formations.

rely on a smooth estimation of curvature around vertices but it is a *native* method for calculating vertices directly, albeit them being of a higher differential order than curvature.

Another issue worth noticing, is that the above procedure of noising can be applied recursively to form neighborhoods of increasing differential order *around* the initial curve points, resulting in an analogous concept to that of incremental smoothing. According to the previous analysis, further noising will only improve the results of identifying more vertices, given that the variance of the normal distribution from which the radii are drawn is not increasing with the degree of noising. In the experimental section that follows, the above method for curvature calculation and vertex localization is validated. A comparison to LAII in these tasks, for smooth and noisy versions of shapes is also presented.

V. Experimental Validation

In this section we validate the above observations with experiments. We show that in the proposed method, vertex localization is indeed improved by adding noise. A comparison to LAII in both vertex localization and curvature estimation in smooth and noisy shapes is another experiment revealing the advantages of the proposed approach.

A. Vertex Localization and the Effect of Noise

In Fig.(4) a noisy silhouette is compared to its smoothed version. There, we see that the noise is actually improving vertex identification by introducing new vertices, not being identified in the smooth version. The reason this happens as we explained before, is due to the nature of the integral descriptors involved in the proposed method of localizing vertices. In a smoothed curve, a point of maximum curvature may or may not appear as extreme point on the boundary, depending on the location of the point with respect to the rest of the curve. Point No. 63 e.g. is not identified in the smoothed version since $\dot{\phi}$, even though it is close, it does not actually achieve zero crossing at a neighborhood of 63. Applying noise in the neighborhood of 63 leads to greater diversity in the tangent directions around 63 and in the noisy version we see $\dot{\phi}$ finally achieving a *zero crossing* there and 63 correctly being identified as a vertex. This effect of noise is valid only for points that are close of being extremes on the boundary and is therefore location dependent and has no effect for points that are not well located globally, e.g. point 15 where we see that, due to its specific location on the shape, no matter how much noise we apply to its neighborhood, it will never be identified as a vertex.

B. Comparison to LAII for Vertex Localization and Curvature Estimation

A comparison to LAII [3] for estimating curvature but also for localizing vertices in noisy shapes is presented in this experiment. LAII is a low level descriptor of similar complexity to the proposed method, that generalizes the concept of curvature over the noisy segments of curves. LAII was chosen as the most robust out of the local methods, to demonstrate inherent disadvantages local methods have in vertex identification under noise. Under LAII a circle of certain radius is used, centered at each point, and the curvature is calculated as the ratio of the area of this circle that lies in the interior of the closed contour. In the case of zero curvature, e.g. a noisy straight line, half of the disk will

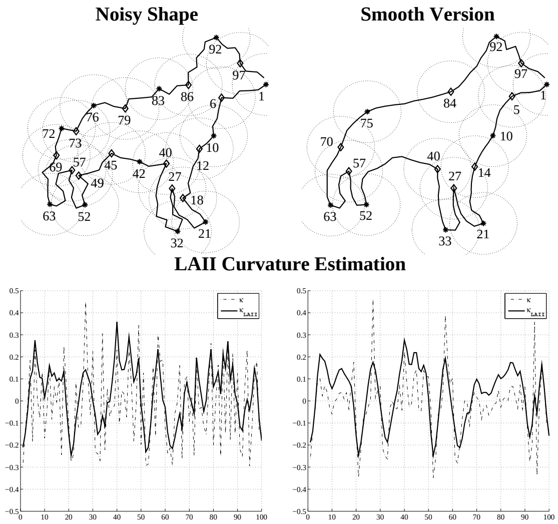

Figure 6. Curvature estimation and vertex localization by means of the LAII method[3]. At the top row plots, stars and diamonds are curvature local minima and maxima respectively, identified by the LAII method for smooth and noisy versions of the same KIMIA silhouette. Compare this with the circles and squares identified by the proposed method in Fig. (5). At the bottom row of plots we see the differential curvature (κ) and the curvature estimator according to LAII(κ_{LAII}). LAII shows high resistance to noise but the results in both curvature estimation and vertex localization are misled by local formations on the boundary.

lie in the interior of the shape, whereas in the case of infinite curvature this portion will tend to zero or to one depending on the sign of the curvature at this point. LAII therefore uses an integral (area of the circle) to estimate curvature but it is essentially a local descriptor.

Our implementation of LAII is as follows: Starting from a binary image of the shape to be encoded, first we extract the boundary. The boundary is then discretized by sampling 100 equally spaced points on it. Then using a circular kernel (constructed as a binary image of a circle of radius 15, as is suggested in [3]) we convolve the filter with the shape image only at the boundary points. The values of the convolution at each of the boundary points are the values of the LAII estimated curvature at these points. For vertex localization we pick the LAII points of local minima and maxima, marked with stars and diamonds accordingly in Fig. (6) .

We compare this implementation of LAII to the proposed method of calculating curvature from equation (4), whereas vertex identification is performed as in the previous section by examining the co-localization of the local extrema of the total distance function ϕ with those of its second derivative $\ddot{\phi}$. For both methods the same extracted contours were used. Many advantages of the proposed method are apparent in this experiment. A comparison of the curvature estimators κ_ϕ and κ_{LAII} in Fig.(5) and Fig.(6) respectively, reveals better accuracy for the proposed method κ_ϕ. The comparison is performed against the differential curvature which, as we see in the figures, collapses in the presence of noise. Also the localization of vertices, marked in the top row of plots as filled squares and circles for the proposed method in Fig.(5), is context dependent and thus more *meaningful* than the respective stars and diamonds marked as such by the LAII method in Fig.(6). While the LAII localization of vertices improves by smoothing (as is expected from all local methods), it follows blindly the local boundary formations without considering context. The amount of smoothing in relation to shapes alterations and the effects on the location of the identified vertices are thus still unsolved problems for the LAII method(and all local methods in that respect). The middle row plots in Fig.(5) reveal the behavior of the zero crossing components in the proposed localization of vertices.

VI. Discussion

One could argue that the problem of identifying *meaningful* vertices, (as opposed to rely solely on the formal mathematical definition) can be resolved only after having recognized the object. Indeed, especially in the presence of noise, the identification of *meaningful* vertices seems the result of an implicit comparison: The given noisy shape is registered against an ideal smooth shape, representative of the recognized class. However, the such ideal shape cannot be the result either of smoothing or of any other low level method that performs on an unrecognized object. In this context, the proposed method is attempting to bring in perceptual characteristics to this low level task by combining a robust vertex estimator $\ddot{\phi}$ and a global position estimator ϕ. The non trivial[2] local extrema of the former are identified in the *zero crossings* of $\dddot{\phi}$ and is already a robust vertex estimator, since it is based on the distance to the rest of the curve and not on local to the curve differentials. In the experiments and also according to the theory in the manuscript, one could follow that $\ddot{\phi}$ can be used as a vertex estimator by itself and vertices could be defined directly on its local extrema (zero crossings of $\dddot{\phi}$) accordingly. However, although robust in comparison to the differential alternatives, vertices identified this way would be according to the formal mathematical definition, lacking any attempt of differentiation with respect to their importance. This is where ϕ plays an important role.

The introduction of ϕ in the process of identifying vertices, provides an additional capability, that of acquiring a *global view* of the locations of the vertex points, introducing the concept of the *remote points*, as *undisputed* points of curvature, points of extreme curvature that due to their location also at the extremes of the shape (zero crossings

[2]Of non constant curvature.

of $\dot{\phi}$) should be characterized as perceptually important [3]. It could be noticed here that extreme location implies extreme curvature in a way that $\dot{\phi}$ can be seen as a *Global* allocator of meaningful vertices. One could conclude by contemplating the important implications in simplicity, implementation diversity and performance of the proposed method, given that it relies on the derivatives of a single distance function ϕ that is robust to noise, natively RSTM (RST + Mirroring) invariant and of low complexity in its calculation.

VII. Conclusion

A global definition of curvature, through integrals of global descriptors, instead of differentials, is used to describe a method of vertex localization and curvature estimation in planar shapes. The method conveys generality by giving valid results on both smooth and noisy shapes. Furthermore, the results of the method are improved in the presence of noise, thus the concept of *noising* emerges as a strong alternative to smoothing. This counterintuitive result was indicated by the theoretical findings and was validated here by experiments. A comparison to LAII, a state of the art noise-resistant method of estimating curvature, reveals the advantages of the proposed approach.

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