Unsupervised Simultaneous Orthogonal Basis Clustering Feature Selection - Supplementary Material

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Algorithm 1: E, F updates algorithm

Input: $F_t$, $W_t$ and $B_t$; Parameter: $\gamma$
Initialization: $s = 0$ and $F'_s = F_t$
Repeat
1. $E'_s + 1 = V_E I_n,c U_E^T$ by (4) where
   $B^T W_t^T X_s + \gamma F'_s = U_E \Sigma_E V_E^T$;
2. Update $F'_s + 1 = \frac{E'_s + 1 + F'_s}{2}$ by (5);
3. $s = s + 1$;
4. Until $\|\Delta J_{EF}(E'_s, F'_s)\| \leq \epsilon$ or $s \leq S$;
Output: $E_{t+1} = E'_s, F_{t+1} = F'_s$

Algorithm 2: SOCFs

Input: Data matrix $X \in \mathbb{R}^{d \times n}$; Parameters: $\lambda, \gamma$
Initialization: $t = 0, D_t = I$ and $B_t, E_t$
Repeat
1. Update $E_{t+1}$ and $F_{t+1}$ by Algorithm 1;
2. Update $W_{t+1} = (XX^T + \lambda D_t)^{-1} X E_{t+1} B_{t+1}^T$ by (2);
3. Update $B_{t+1} = V_B I_{m,c} U_B^T$ by (3) where
   $E_{t+1} B_{t+1} = U_B \Sigma_B V_B^T$;
4. Update the $i$-th diagonal elements of the diagonal matrix $D_{t+1}$ with $\frac{1}{\|w_i\|^2}$;
5. $t = t + 1$;
6. Until $\|\Delta J(W_t, B_t, E_t, F_t)\| \leq \epsilon$ or $t \leq T$;
Output: Features are selected corresponding to the largest values of $\|w_i\|, i = 1 \ldots d$, which are sorted by descending order.

1. Preliminaries

1.1. The Reformulated Objective Function

$$\min_{W,B,E,F} \|W^T X - BE^T\|_F^2 + \lambda \|W\|_{2,1} + \gamma \|F - E\|_F^2.$$  
$s.t. B^T B = I, E^T E = I, F \geq 0.$

1.2. Update Rules

W update:

$$W = (XX^T + \lambda D)^{-1} X E B^T.$$  

B update:

$$B = V_B I_{m,c} U_B^T,$$  

where $U_B$ and $V_B$ are the left and right eigenvectors of $E^T X^T W$ computed by SVD, respectively.

E, F update:

$$E = V_E I_{n,c} U_E^T,$$  

where $U_E$ and $V_E$ are the left and right eigenvectors of $B^T W^T X + \gamma F^T$ computed by SVD, respectively.

$$F = \frac{1}{2} (E + |E|).$$

1.3. Algorithms

The optimization algorithm containing the E and F update rules is summarized in Algorithm 1. The overall proposed optimization algorithm of SOCFs is also presented in Algorithm 2.

2. Convergence Analysis

We prove the convergence of the proposed optimization algorithm with monotonic decrease at every iteration. We denote the objective function in problem (1) as $J(W, B, E, F)$ for convenience.

Theorem 1. $J^{(t)}_{EF}(E'_s, F'_s) \preceq J(W_t, B_t, E'_s, F'_s)$ monotonically decreases due to E, F updates in Algorithm 1.

Proof. For the F' update from by (5), we have

$$F'_{s+1} = \arg \min_{F' : F' \geq 0} \|F' - E'_s\|_F^2 = \arg \min_{F' : F' \geq 0} J^{(t)}_{EF}(E'_s, F') \Rightarrow J^{(t)}_{EF}(E'_s, F'_{s+1}) \leq J^{(t)}_{EF}(E'_s, F'_s).$$

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Similarly, for the $E'$ update by (4), we have
\[
E'_{s+1} = \arg\min_{E':E'} \|B_t E'^T - W_t^T X\|_F^2 + \gamma \|E' - F'_{s+1}\|_F^2
\]
\[
= \arg\min_{E':E'} J_{EF}^{(t)}(E', F'_{s+1})
\]
\[
\implies J_{EF}^{(t)}(E'_{s+1}, F'_{s+1}) \leq J_{EF}^{(t)}(E', F').
\]
By combining (6) and (7), we finally obtain
\[
J_{EF}^{(t)}(E'_{s+1}, F'_{s+1}) \leq J_{EF}^{(t)}(E'_s, F'_s).
\]
Thus $J_{EF}^{(t)}(E'_s, F'_s)$ monotonically decreases by the update rules (4) and (5) in Algorithm 1. We also notice that, since $J_{EF}^{(t)}(E'_s, F'_s)$ is convex in each variable, the algorithm must converge.

**Theorem 2.** $J(W_t, B_t, E_t, F_t)$ monotonically decreases due to the update rules in Algorithm 2.

**Proof.** For the $E$ and $F$ updates, $E_{t+1}$ and $F_{t+1}$ are updated at the same time by Algorithm 1, so that we have
\[
J(W_t, B_t, E_{t+1}, F_{t+1}) \leq J(W_t, B_t, E_t, F_t).
\]
(8)
For the $W$ update by (2), which follows the theorem in [1] closely, $W_{t+1}$ is also the solution of the following problem with fixed $D_t$ as
\[
W_{t+1} = \arg\min_W \|W_t^T X - B_t E_t^T\|_F^2 + \lambda \text{tr}(W_t^T D_t W_t).
\]
This implies that
\[
\|W_{t+1}^T X - B_t E_t^T\|_F^2 + \lambda \text{tr}(W_{t+1}^T D_t W_{t+1})
\]
\[
\leq \|W_t^T X - B_t E_t^T\|_F^2 + \lambda \text{tr}(W_t^T D_t W_t).
\]
(9)
And then according to the lemma in [1] with $u = w_{t+1}^i$, $u_t = w_t^i$ and summation over all rows, we have
\[
\sum_{i=1}^d \left(\|w_{t+1}^i\|_2 - \frac{\|w_{t+1}^i\|_2^2}{2\|w_t^i\|_2}\right) \leq \sum_{i=1}^d \left(\|w_t^i\|_2 - \frac{\|w_t^i\|_2^2}{2\|w_t^i\|_2}\right).
\]
We rewrite the inequality as
\[
\|W_{t+1}\|_{2,1} - \text{tr}(W_{t+1}^T D_t W_{t+1})
\]
\[
\leq \|W_t\|_{2,1} - \text{tr}(W_t^T D_t W_t).
\]
(10)
By combining (9) and (10), we finally obtain
\[
J(W_{t+1}, B_t, E_{t+1}, F_{t+1}) \leq J(W_t, B_t, E_{t+1}, F_{t+1}).
\]
(11)
For the $B$ update by (3), we have
\[
B_{t+1} = \arg\min_{B:B^T B=1} \|E_{t+1} B^T - X^T W_{t+1}\|_F^2
\]
\[
= \arg\min_{B:B^T B=1} J_B(W_{t+1}, B, E_{t+1}, F_{t+1}).
\]
(12)
This implies that
\[
J(W_{t+1}, B_{t+1}, E_{t+1}, F_{t+1}) \leq J(W_{t+1}, B_t, E_{t+1}, F_{t+1}).
\]
(13)
From (8), (11), and (13), each update rule monotonically decreases the objective function at every iteration. We also notice that, since $J(W_t, B_t, E_t, F_t)$ is convex in each variable, the algorithm with the update rules must converge.

**References**