GRSA: Generalized Range Swap Algorithm for the Efficient Optimization of MRFs

1. Proof of Theorem 1

Before proving Theorem 1, we first give the following lemmas and the definition of submodular set.

**Lemma 1.** For $b_1, b_2 > 0$, the following conclusion holds.

\[
\frac{a_1}{b_1} \geq \frac{a_2}{b_2} \iff \frac{a_1 + a_2}{b_1 + b_2} \geq \frac{a_2}{b_2}.
\]

The proof is straightforward and we omit it.

**Lemma 2.** Assuming that function $g(x)$ is convex on $[a, b]$ and there are three points $x_1, x_2, x_3 \in [a, b]$ satisfying $x_1 > x > x_2$, there is

\[
\frac{g(x_1) - g(x)}{x_1 - x} \geq \frac{g(x_1) - g(x_2)}{x_1 - x_2} \geq \frac{g(x) - g(x_2)}{x - x_2}.
\]

**Proof.** Since $x_1 > x > x_2$, there exists $\lambda \in (0, 1)$ satisfying $x = (1 - \lambda)x_1 + \lambda x_2$. Then by the definition of convex function, there is $(1 - \lambda)g(x_1) + \lambda g(x_2) \geq g(x)$ and thus

\[ (1 - \lambda)(g(x_1) - g(x)) \geq \lambda(g(x) - g(x_2)) \]

Considering that $x_1 > x_2$ and $0 < \lambda < 1$, we can divide $\lambda(1 - \lambda)(x_1 - x_2)$ on both sides of (3) and obtain

\[
\frac{g(x_1) - g(x)}{x_1 - x} \geq \frac{g(x) - g(x_2)}{x - x_2}, \quad \frac{g(x_1) - g(x_2)}{x_1 - x_2} \geq \frac{g(x) - g(x_2)}{x - x_2}.
\]

At last, the conclusion (2) can be proved by applying Lemma 1 to (5).

**Lemma 3.** Assuming that $g(x)$ is convex on $[a, b]$ and there are four points $x_1, x_2, x_3, x_4 \in [a, b]$ satisfying $x_1 > x_3$, there is $x_1 \geq x_2 > x_4$, there is

\[
\frac{g(x_1) - g(x_3)}{x_1 - x_3} \geq \frac{g(x_2) - g(x_4)}{x_2 - x_4}.
\]

**Proof.** 1. If $x_2 = x_3$, the conclusion is straightforward by Lemma 2.

2. If $x_2 > x_3$ and $x_1 \geq x_2 > x_3 \geq x_4$. We first consider the case where $x_1 > x_2 > x_3 > x_4$ and can use Lemma 2 to obtain (6)

\[
\frac{g(x_1) - g(x_3)}{x_1 - x_3} \geq \frac{g(x_2) - g(x_3)}{x_2 - x_3} \geq \frac{g(x_2) - g(x_4)}{x_2 - x_4}.
\]

For the case where $x_1 = x_2 > x_3 > x_4$, the right half of the above inequality holds and we can obtain (6) by replacing $x_2$ with $x_1$. The case where $x_1 > x_2 > x_3 = x_4$ can be obtained in a similar way while the case where $x_1 = x_2 > x_3 = x_4$ is straightforward.

3. If $x_2 < x_3$, there is $x_1 > x_3 > x_2 > x_4$. Using Lemma 2, we can obtain (6)

\[
\frac{g(x_1) - g(x_3)}{x_1 - x_3} \geq \frac{g(x_3) - g(x_2)}{x_3 - x_2} \geq \frac{g(x_2) - g(x_4)}{x_2 - x_4}
\]

Therefore, the proof of Lemma 3 is completed.

**Lemma 4.** Given a function $g(x)$ ($x = |a - \beta|$) on domain $X = [0, c]$, assume $g(x)$ is locally convex on interval $X_0 = [a, b]$ ($0 < a < b < c$), and it satisfies $\alpha \{g(a + 1) - g(a)\} \geq g(a) - g(0)$. Then we have

\[
\frac{g(x_1) - g(x_3)}{x_1 - x_3} \geq \frac{g(x_2) - g(0)}{x_2} \quad \text{where } x_1, x_2, x_3 \in X_0 \text{ where } x_3 < x_1 \text{ and } x_2 < x_1.
\]

**Proof.** Since $x_1 > x_3 \geq a$ and $x_1 \in N$, we have $x_1 \geq a + 1 > a$. Then considering that $x_1 > x_3 \geq a$, we can use Lemma 3 to obtain

\[
\frac{g(x_1) - g(x_3)}{x_1 - x_3} \geq \frac{g(a + 1) - g(a)}{a + 1 - a} \geq \frac{g(a) - g(0)}{a}.
\]

where the second inequality comes from $\alpha \{g(a + 1) - g(a)\} \geq g(a) - g(0)$.

If $x_2 = a$, the conclusion is obtained from (8). Otherwise, there is $x_1 > x_2 > a$ and $x_1 > x_3 \geq a$. Using Lemma 3, we obtain

\[
\frac{g(x_1) - g(x_3)}{x_1 - x_3} \geq \frac{g(x_2) - g(a)}{x_2 - a}.
\]
Combining (8) and (9), we can obtain
\[
\frac{g(x_1) - g(x_3)}{x_1 - x_3} \geq \max \left\{ \frac{g(x_2) - g(a)}{x_2 - a}, \frac{(g(a) - g(0))}{a} \right\}
\]
\[
\geq \frac{g(x_2) - g(a) + g(a) - g(0)}{x_2 - a + a}
\]
\[
= \frac{g(x_2) - g(0)}{x_2},
\]
where the second inequality is due to Lemma 1 and this completes the proof. \(\square\)

**Definition 1.** Given a pairwise potential \(\theta(\alpha, \beta)\), we call \(\mathcal{L}_s\) a submodular set, if it satisfies
\[
\theta(l_{i+1}, l_j) - \theta(l_{i+1}, l_{j+1}) - \theta(l_i, l_j) + \theta(l_i, l_{j+1}) \geq 0
\]
for any pair of labels \(l_i, l_j \in \mathcal{L}_s\). \(1 \leq i, j < m\).

**Theorem 1.** Given a pairwise function \(\theta(\alpha, \beta) = g(x) = |\alpha - \beta|\) on domain \(X = [0, c]\), assume there is an interval \(X_s = [a, b]\) \((0 \leq a < b \leq c)\) satisfying: (i) \(g(x)\) is locally convex on \([a, b]\), and (ii) \(g(a+1) - g(a) \geq g(a) - g(0)\).
Then \(\mathcal{L}_s = \{l_1, \ldots, l_m\}\) is a submodular subset, if \(|l_i - l_j| \in [a, b]\) for any pair of labels \(l_i, l_j \in \mathcal{L}_s\).

**Proof.** Since \(\theta(\alpha, \beta)\) is symmetric and satisfies \(\theta(\alpha, \beta) = \theta(\beta, \alpha)\), we only consider \(l_i, l_{i+1}, l_j, l_{j+1} \in \mathcal{L}_s\) where \(i \geq j\). Let
\[
x_i = l_{i+1} - l_i,
x_j = l_j - l_i,
x_3 = l_{i+1} - l_{j+1},
x_4 = l_j - l_{j+1}
\]
For \(i \geq j\), we have \(x_i > x_j\), and \(x_1 - x_2 = x_3 - x_4\). We can define
\[
\lambda = \frac{x_3 - x_4}{x_1 - x_4} = \frac{x_1 - x_2}{x_1 - x_4}, \quad (0 < \lambda < 1)
\]
then, we get
\[
x_3 = \lambda x_1 + (1 - \lambda) x_4,
x_2 = \lambda x_2 + (1 - \lambda) x_1.
\]
If \(a = 0\), i.e. \(X_s = [0, b]\) we have \(x_1, x_2, x_3, x_4 \in X_s\) according to the assumption in Theorem 1. Since \(d(x)\) is convex on \(X_s\), with Eq. (12) we obtain
\[
g(x_1) \leq \lambda g(x_1) + (1 - \lambda) g(x_4),
g(x_2) \leq \lambda g(x_2) + (1 - \lambda) g(x_1)
\]
Summing the two equations in Eq. (13), we can get
\[
g(x_2) + g(x_3) \leq g(x_1) + g(x_4)
\]
and \(\theta(l_{i+1}, l_j) - \theta(l_{i+1}, l_{j+1}) - (\theta(l_i, l_j) + \theta(l_i, l_{j+1}) \geq 0\) is satisfied for any pair of labels \(l_i, l_j \in \mathcal{L}_s\).
Therefore, \(\theta(l_{i+1}, l_j) - \theta(l_{i+1}, l_{j+1}) - \theta(l_i, l_j) + \theta(l_i, l_{j+1}) \geq 0\) is satisfied for any pair of labels \(l_i, l_j \in \mathcal{L}_s\). The proof is completed. \(\square\)

**Corollary (Theorem 1).** Assuming the interval \([a, b]\) is a candidate interval, then \(\{\alpha, \alpha + x_1, \alpha + x_1 + x_2, \ldots, \alpha + x_1 + \cdots + x_m\} \subseteq \mathcal{L}\) is a submodular set, if \(x_1, \ldots, x_m \in [a, b]\) and \(x_1 + \cdots + x_m \leq b\).

**Proof.** Let \(\mathcal{L}_s = \{\alpha, \alpha + x_1, \alpha + x_1 + x_2, \ldots, \alpha + x_1 + \cdots + x_m\}\). We consider a pair of labels \(\alpha_1\) and \(\alpha_2\), which can be any pair of labels chosen in \(\mathcal{L}_s\). According to the definition, there always exist \(p, q\) \((1 \leq p, q \leq m)\) such that
\[
|\alpha_1 - \alpha_2| = x_p + x_{p+1} + \cdots + x_q.
\]
Since $x_i \in [a, b]$ for $\forall i \in [p, q]$, we have $|\alpha_1 - \alpha_2| \geq a$.
Since $x_1 + \cdots + x_m \leq b$, we have $x_p + x_{p+1} + \cdots + x_q \leq b$.
Thus, we have $|\alpha_1 - \alpha_2| \in [a, b]$ for any pair of labels $\alpha_1, \alpha_2 \in L_s$.
Thus, $L_s$ is a submodular set according to Theorem 1.