Rolling Shutter Camera Relative Pose: Generalized Epipolar Geometry
Supplementary Material

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Abstract

In this supplementary material, we provide detailed derivations of other types of rolling shutter essential matrices as well as their linear algorithms.

1. Deriving the $7 \times 7$ rolling shutter essential matrix for uniform RS camera

Under the uniform rolling shutter camera model, the scanline coplanarity constraint can be expressed as:

\[ [u'_i, v'_i, 1] [t + u'_i d_2 - u_i R_{u_i, u'_i} d_1] x R_{u_i, u'_i} [u_i, v_i, 1]^T = 0, \]

where $R_{u_i, u'_i}$ defines the relative rotation while $t_{u_i, u'_i} = t + u'_i d_2 - u_i R_{u_i, u'_i} d_1$ defines the relative translation. Where $R$ defines the rotation between the central row of the second frame to the central row of the first row.

\[ [u'_i, v'_i, 1] [t] x R_{u_i, u'_i} \]
\[ - u_i R_{u_i, u'_i} [d_1] x \]
\[ - u'_i [d_2] x R_{u_i, u'_i} [u_i, v_i, 1]^T = 0, \]

Expanding this equation with the aid of the small rotation approximation results in

\[ R_{u_i, u'_i} = (I + u'_i [w_2] x) R_0 (I - u_i [w_1] x), \]

By defining the following auxiliary variables,

\[ E_0 = [t] x R, \]
\[ E_1 = R [d_1] x + [t] x R [w_1] x, \]
\[ E_2 = [d_2] x R + [t] x [w_2] x R, \]
\[ E_3 = R [w_1] x [d_1] x, \]
\[ E_4 = [d_2] x R [w_1] x + [w_2] x R [d_1] x + [t] x [w_2] x R [w_1] x, \]
\[ E_5 = [d_2] x [w_2] x R, \]
\[ E_6 = [w_2] x R [w_1] x [d_1] x, \]
\[ E_7 = [d_2] x [w_2] x R [w_1] x, \]

we arrive at,

\[ [u'_i, v'_i, 1] (E_0 - u_i E_1 + u'_i E_2 + u_i^2 E_3 \]
\[ - u_i u'_i E_4 + u'_i E_5 \]
\[ + u_i^2 u'_i E_6 - u_i u'_i^2 E_7) [u_i, v_i, 1]^T = 0. \]

Finally we obtain:

\[ [u_i^3, u_i^2 v_i, u_i v_i^2, u_i v_i, u_i^2 v_i, u_i v_i, v_i, 1]^T F [u_i^3, u_i^2 v_i, u_i v_i^2, u_i v_i, u_i^2 v_i, u_i v_i, v_i, 1]^T = 0, \]

where

\[ F = \begin{bmatrix} 0 & 0 & f_{13} & f_{14} & f_{15} & f_{16} & f_{17} \\ 0 & 0 & f_{23} & f_{24} & f_{25} & f_{26} & f_{27} \\ f_{31} & f_{32} & f_{33} & f_{34} & f_{35} & f_{36} & f_{37} \\ f_{41} & f_{42} & f_{43} & f_{44} & f_{45} & f_{46} & f_{47} \\ f_{51} & f_{52} & f_{53} & f_{54} & f_{55} & f_{56} & f_{57} \\ f_{61} & f_{62} & f_{63} & f_{64} & f_{65} & f_{66} & f_{67} \\ f_{71} & f_{72} & f_{73} & f_{74} & f_{75} & f_{76} & f_{77} \end{bmatrix}. \]

This gives a $7 \times 7$ uniform RS essential matrix $F$, whose elements are functions of the 18 unknowns \{($R, t, w_1, w_2, d_1, d_2$)\}. Also note the induced epipolar curves are cubic.

In total there are 45 homogeneous variables, thus minimum 44 points in general configuration are sufficient to solve this $7 \times 7$ RS essential matrix.

1.1. Detail of the linear 44-point solver

For the uniform rolling shutter relative pose problem, we first solve for the uniform rolling shutter essential matrix $F \in \mathbb{R}^{7 \times 7}$. Then from the 45 elements in $F$, recover the eight matrices $E_i, i = 0, \cdots, 7$. Finally, the relative pose $(R, t)$, rotational velocities $w_1, w_2$ and translational velocities $d_1, d_2$ are extracted from the eight matrices.

Due to its special structure, the uniform RS essential matrix $F$ consists of 45 homogeneous variables, \textit{i.e.}, 44 DoF. According to the uniform RS essential matrix Eq.-(10), by collecting 44 correspondences, we can solve for the uniform RS essential matrix $M$ linearly through the singular value decomposition (SVD).
1.2. Normalization

In solving the linear rolling shutter essential matrix $F$ through linear 20 point algorithm, it is important to implement a proper normalization.

Below we describe two approaches for performing such a normalization: 1) Normalizing the image coordinates data $(u_i, v_i)$ and $(u_i', v_i')$ in the way as described in [1]. 2) Under the linear rolling shutter relative pose formulation, the inputs are monomials $(u_i^2, u_i v_i, u_i, v_i, 1)$ and $(u_i', 2, u_i' v_i, u_i', v_i', 1)$, a better normalization should be defined on $(u_i^2, u_i v_i, u_i, v_i, 1)$ and $(u_i'^2, u_i' v_i', u_i', v_i', 1)$ rather than $(u_i, v_i)$ and $(u_i', v_i')$. Therefore, in this paper, we propose to normalize $(u_i^2, u_i v_i, u_i, v_i, 1)$ and $(u_i'^2, u_i' v_i', u_i', v_i', 1)$ in the way as in [1].

2. Details about recovering the atomic essential matrices from a $5 \times 5$ linear RS essential matrix

Once a $5 \times 5$ linear RS essential matrix $F$ is found, our next step is to recover the individual atomic essential matrices $E_0, E_1$ and $E_2$. In the main paper we derived 21 linear equations defined on the three essential matrices. Because these three essential matrices contain 27 elements, we need six extra constraints to solve for $E_0, E_1$ and $E_2$. To this end, we resort to the inherent constraints on the standard $3 \times 3$ essential matrices, e.g.

\[
\det(E) = 0 \quad \text{and} \quad 2EE^T E - \text{Tr}(EE^T) E = 0,
\]

since $E_0, E_1$ and $E_2$ are standard $3 \times 3$ essential matrices. Note that these non-linear constraints generally give rise to cubic (3-order) equations. Next we will show how to reduce them to quadratic ones.

2.1. Enforcing inherent constraints on the atomic essential matrices

Theorem 1. A real nonzero $3 \times 3$ matrix $E$ is a fundamental matrix if and only if it satisfies the equation:

\[
\det(E) = 0.
\]

Theorem 2. A real nonzero $3 \times 3$ matrix $E$ is an essential matrix if and only if it satisfies the equation:

\[
EE^T E - \frac{1}{2} \text{trace}(EE^T) E = 0.
\]

Theorem 3. If three essential matrices $E_0, E_1, E_2$ consist of a common rotation, i.e., $E_0 = [t] \times R, E_1 = [t_1] \times R, E_2 = [t_2] \times R$, the column reorganized matrices $F_1 = [E_0^3, E_1^3, E_2^3], F_2 = [E_0^3, E_1^3, E_2^3], F_3 = [E_0^3, E_1^3, E_2^3]$ are rank deficient.

\[
\det(F_1) = 0, \det(F_2) = 0, \det(F_3) = 0.
\]

Proof. According to the definition, $F_1 = [E_0^3, E_1^3, E_2^3] = [t \times R^3, t_1 \times R^3, t_2 \times R^3]$. Therefore, all the rows of $F_1$ are orthogonal to $R^3$, we must have rank($F_1$) = 2 or det($F_1$) = 0. Similarly, we have det($F_2$) = 0 and det($F_3$) = 0.

Note that $F_1, F_2$ and $F_3$ are not necessarily an essential matrix.

By collecting the rank deficient constraints on essential matrices $E_0, E_1$ and $E_2$ and column reorganized matrices $F_1, F_2, F_3$. In total, we have the following 6 rank constraints:

\[
\begin{align*}
\text{rank}(E_0) &= \text{rank}([t] \times R) = 2, \\
\text{rank}(E_1) &= \text{rank}([v_1] \times R) = 2, \\
\text{rank}(E_2) &= \text{rank}([v_2] \times R) = 2, \\
\text{rank}(F_1) &= \text{rank}([E_0^3, E_1^3, E_2^3]) = 2, \\
\text{rank}(F_2) &= \text{rank}([E_0^3, E_1^3, E_2^3]) = 2, \\
\text{rank}(F_3) &= \text{rank}([E_0^3, E_1^3, E_2^3]) = 2.
\end{align*}
\]

By enforcing the above six constraints together with the 21 linear equations, the atomic essential matrices $E_0, E_1$ and $E_2$ can be recovered. Besides the rank constraints, the cubic equations defined on the essential matrix also constrain $E_0, E_1$ and $E_2$. By exploiting the special structure of these essential matrices, we could reach the following method which involves quadratic equations only.

References