Temporal Epipolar Regions
Supplementary Materials

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The following supplementary materials are split into two sections. In the first section we complete the proofs described in the main paper. The second section provides additional details relating to our experiments. In addition, we provide demonstrations of matching using TERs in the attached movie file.

1. Validity of TERs

In this section we formally define each TER type, and prove the valid orders for each. For each TER type, we present and prove the ordering of critical lines, and a single valid order in one of the sections. Once we have the order of critical lines and the valid order we can use observations **A1-A2** in order to find the remaining valid orders. As such, we find the 6 valid orders out of a possible 12 for each region. Note that the proofs below are all case based. To do so, we first parametrize the epipolar lines.

1.1. Parameterization

Without loss of generality, we set the coordinate system such that ℓ_k is parallel to the x-axis, at $y_k < 0$. The intersection of ℓ_i and ℓ_k is at the origin, that is $\gamma_{ij} = (0, 0)$. As such, ℓ_i and ℓ_j are parametrized by $y = m_i x$ and $y = m_j x$, respectively. Therefore, $\gamma_{ik} = (\frac{b_k}{m_i}, b_k)$ and $\gamma_{jk} = (\frac{b_k}{m_j}, b_k)$. Finally for $y > 0$, ℓ_j is left of ℓ_i, that is $\frac{y}{m_j} < \frac{y}{m_i}$. See Figs. 8-12.

Given a point $\alpha_u = (x_u, y_u)$, the critical lines are defined as follows:

$$c_i \;:\; y = m_i x + b_i$$

$$c_j \;:\; y = m_j x + b_j$$

$$c_k \;:\; y = y_u$$

$$c_{ij} :\; y = m_{ij} x, \text{ where } m_{ij} = \left(\frac{y_u}{x_u}\right)$$

$$c_{ik} :\; y = m_{ik} x + b_{ik}, \text{ where } m_{ik} = \left(\frac{y_k - y_u}{\frac{y_k}{m_i} - x_u}\right)$$

$$c_{jk} :\; y = m_{jk} x + b_{jk}, \text{ where } m_{jk} = \left(\frac{y_k - y_u}{\frac{y_k}{m_j} - x_u}\right)$$

To find the order of critical lines, we calculate the intersection of each critical line with the x-axis, given by the set $\beta = \{\beta_i, \beta_j, \beta_{ij}, \beta_{ik}, \beta jk\}$:

$$c_i \;:\; \beta_i = x_u - \frac{y_u}{m_i}$$

$$c_j \;:\; \beta_j = x_u - \frac{y_u}{m_j}$$

c_k : Parallel to x-axis

$$c_{ij} :\; \beta_{ij} = 0$$

$$c_{ik} :\; \beta_{ik} = \frac{(x_u - \frac{y_u}{m_i}) y_k}{(y_k - y_u)}$$

$$c_{jk} :\; \beta_{jk} = \frac{(x_u - \frac{y_u}{m_j}) y_k}{(y_k - y_u)}$$

In regions $R3$, $R4$, and $R5$, we need to consider the lines parallel to the epipolar lines (see Fig. 3). With that in mind, we also define the following:

$$\begin{aligned} \gamma_{ik} &= (\tfrac{y_k}{m_i}, y_k) \\ \hat{\ell}_j &= m_j x + (1 - \tfrac{m_j}{m_i}) y_k \\ \hat{\gamma}_{jk} &= (\tfrac{1}{m_i} - \tfrac{1}{m_j}) y_k \end{aligned}$$

Note that α_u is to the right of ℓ_i if $x_u > \frac{y_u}{m_i}$, otherwise α_u is the left of ℓ_i. Similarly, α_u is to the right of ℓ_j if $x_u > \frac{y_u}{m_j}$. Finally, α_u is to the right of $\hat{\ell}_j$ if $x_u > \frac{y_u}{m_j} - (\frac{1}{m_i} - \frac{1}{m_j}) y_k$.

1.2. The β ordering conditions

We next present the conditions that define the ordering of the set β along the x-axis. Since β depends on the location of $\alpha_u = (x_u, y_y)$ with respect to each of the lines. These conditions can be easily verified visually (using Figs. 8-12) or algebraically using the definitions for β and the line given above.

1. $\beta_i < \beta_j$
   This is true given our setup as for $y > 0$, $\frac{y}{m_j} < \frac{y}{m_i}$.

2. $\beta_i > 0$, that is, $x_u - \frac{y_u}{m_i} > 0$ *iff* $x_u > \frac{y_u}{m_i}$ (x_u is right of ℓ_i)

3. $\beta_j > 0$, that is, $x_u - \frac{y_u}{m_j} > 0$ *iff* $x_u > \frac{y_u}{m_j}$ (x_u is right of ℓ_j)

4. $\beta_{ik} > 0$, that is, $\frac{(x_u - \frac{y_u}{m_i})y_k}{(y_k - y_u)} > 0$

   iff one of these cases holds:

   (i) $x_u > \frac{y_u}{m_i}$ (x_u is right of ℓ_i) and $y_u > y_k$
   (ii) $x_u < \frac{y_u}{m_i}$ (x_u is left of ℓ_i) and $y_k > y_u$.

5. $\beta_{jk} > 0$, that is, $\frac{(x_u - \frac{y_u}{m_j})y_k}{(y_k - y_u)} > 0$

   iff one of these cases hold:

   (i) $x_u > \frac{y_u}{m_j}$ (x_u is right of ℓ_j) and $y_u > y_k$
   (ii) $x_u < \frac{y_u}{m_j}$ (x_u is left of ℓ_j) and $y_k > y_u$

6. $\beta_{ik} > \beta_i$ that is,

   $\frac{(x_u - \frac{y_u}{m_i})y_k}{(y_k - y_u)} > (x_u - \frac{y_u}{m_i})$

   Through algebraic manipulations we get:

   $$\frac{(x_u - \frac{y_u}{m_i})y_u}{(y_k - y_u)} > 0$$

   iff one of these cases holds:

   (i) $y_k < y_u < 0$ and $x_u > \frac{y_u}{m_i}$ (x_u is right of ℓ_i)
   (ii) $y_u > 0$ and $x_u < \frac{y_u}{m_i}$ (x_u is left of ℓ_i)
   (iii) $y_k > y_u$ and $x_u < \frac{y_u}{m_i}$ (x_u is left of ℓ_i)

7. $\beta_{jk} > \beta_j$ that is,

   $\frac{(x_u - \frac{y_u}{m_j})y_k}{(y_k - y_u)} > (x_u - \frac{y_u}{m_j})$

   Through algebraic manipulations we get:

   $$\frac{(x_u - \frac{y_u}{m_j})y_u}{(y_k - y_u)} > 0$$

   iff one of these cases holds:

   (i) $y_k < y_u < 0$ and $x_u > \frac{y_u}{m_j}$ (x_u is right of ℓ_j)
   (ii) $y_u > 0$ and $x_u < \frac{y_u}{m_j}$ (x_u is left of ℓ_j)
   (iii) $y_k > y_u$ and $x_u < \frac{y_u}{m_j}$ (x_u is left of ℓ_j)

8. $\beta_j > \beta_{ik}$ that is:

   $x_u - \frac{y_u}{m_j} > \frac{(x_u - \frac{y_u}{m_i})y_k}{(y_k - y_u)}$

   Through algebraic manipulations we get that:

   $$y_u \frac{\hat{\gamma}_{jk} - \beta_j}{(y_k - y_u)} \ > \ 0$$

   iff one of these cases holds:

   (i) $y_k < y_u < 0$ and $x_u < \frac{y_u}{m_j} - (\frac{1}{m_i} - \frac{1}{m_j})y_k$ (x_u is left of $\hat{\ell}_j$)
   (ii) $y_u > 0$ and $x_u > \frac{y_u}{m_j} - (\frac{1}{m_i} - \frac{1}{m_j})y_k$ (x_u is right of $\hat{\ell}_j$)
   (iii) $y_u < y_k$ and $x_u > \frac{y_u}{m_j} - (\frac{1}{m_i} - \frac{1}{m_j})y_k$ (x_u is right of $\hat{\ell}_j$)

9. $\beta_{ik} < \beta_{jk}$, that is:

   $\frac{(x_u - \frac{y_u}{m_i})y_k}{(y_k - y_u)} < \frac{(x_u - \frac{y_u}{m_j})y_k}{(y_k - y_u)}$

   Through algebraic manipulations we get that

   $$\frac{y_u(-\hat{\gamma}_{jk})}{y_k - y_u} \ < \ 0$$

   As $\frac{y}{m_j} < \frac{y}{m_i}$ when $y > 0$, $\hat{\gamma}_{jk}$ must be negative, therefore this rule holds *iff* either of these cases hold:

   (i) $y_u < y_k$
   (ii) $y_u > 0$

1.3. Geometric Observations

In addition to the conditions on β order, we use two basic geometric observations to prove the order in which a line passing through a section crosses each epipolar line and α_u.

1. Define the points p_{a1} and p_{a2} as two points on a line ℓ_a. Consider the line ℓ_b which intersects ℓ_a, at $\tilde{\ell}_a \times \tilde{\ell}_b$ (where $\tilde{x}$ is the homogenous coordinate of x). If p_{a1} and p_{a2} are on opposite sides of ℓ_b, the intersection of the two lines must be between p_{a1} and p_{a2} along ℓ_a.

2. Given a point p between two parallel lines ℓ_a and $\hat{\ell}_a$. Let ℓ_b be a line passing through p which is not parallel to ℓ_a. The point p must be between $\tilde{\ell}_a \times \tilde{\ell}_b$ and $\tilde{\hat{\ell}}_a \times \tilde{\ell}_b$.

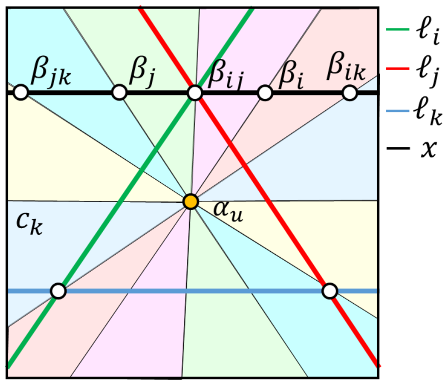

Figure 8: This figure shows the intersections of critical lines with the x-axis given α_u in $R1$.

1.4. Critical Line Orders for TERs

Here we formally define each region type and prove the order of critical lines for each region using the conditions on the order of β. The order of critical lines is defined by the order of the points in the set β. Note that as c_k is parallel to the x-axis and the order is cyclical, c_k can be either first or last in the order of critical lines for all regions. For reference, we provide in Table 2, the order of critical lines and valid orders for each of the 16 regions.

Region R1: *R1* is defined as the triangular region whose points are γ_{ij}, γ_{ik}, and γ_{jk} (Fig. 3). Formally:

$$R_1 = \{(x_u, y_u) \mid y_k < y_u < 0 \;\; \& \;\; \frac{y_u}{m_i} < x_u < \frac{y_u}{m_j}\}.$$

Critical line order for $R1$: $c_{jk}, c_j, c_{ij}, c_i, c_{ik}, c_k$
Rules which prove order:

(i) $\beta_{jk} < \beta_j$: Rule 7.

(ii) $\beta_j < \beta_{ij}$: Rule 3.

(iii) $\beta_{ij} < \beta_i$: Rule 2.

(iv) $\beta_i < \beta_{ik}$: Rule 6.

Region R2: $R2(i, j)$ is defined in the paper. It is the area comprised of ℓ_i and ℓ_j through which ℓ_k does not pass. Formally:

$$R_2(i, j) = \{(x_u, y_u) \mid y_u > 0 > y_k \;\; \& \;\; \frac{y_u}{m_j} < x_u < \frac{y_u}{m_i}\}.$$

Critical line order for $R_2(i, j)$: $c_i, c_{ik}, c_{ij}, c_{jk}, c_j, c_k$ Rules which prove order:

(i) $\beta_i < \beta_{ik}$: Rule 6.

(ii) $\beta_{ik} < \beta_{ij}$: Rule 4.

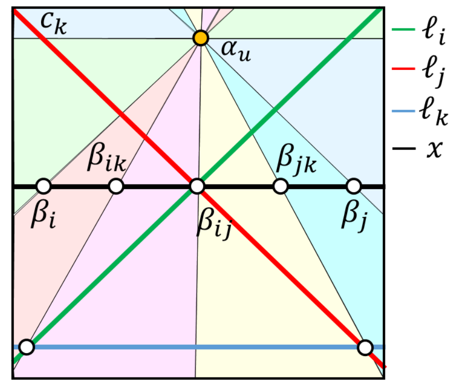

Figure 9: This figure shows the intersections of critical lines with the x-axis given α_u in $R2(i, j)$.

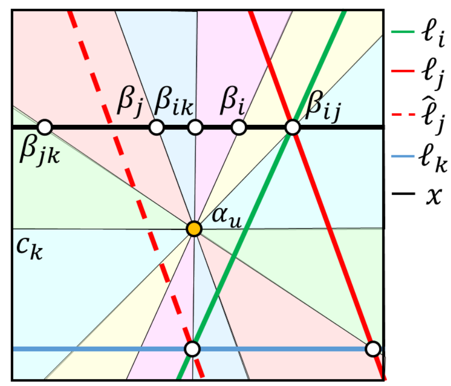

Figure 10: This figure shows the intersections of critical lines with the x-axis given α_u in $R3(i, \hat{j}, \hat{k})$.

(iii) $\beta_{ij} < \beta_{jk}$: Rule 5.

(iv) $\beta_{jk} < \beta_j$: Rule 7.

Region R3: $R3(i, \hat{j}, \hat{k})$ is defined by the triangle whose points are γ_{ij}, γ_{ik}, and $\hat{\gamma}_{jk} = (\hat{\ell}_j \times \hat{\ell}_k)$ (see Fig. 3). Note that the way in which we set up the lines, $\hat{\ell}_k$ is the x-axis. Formally:

$$R_3(i, \hat{j}, \hat{k}) = \{(x_u, y_u) \mid 0 > y_u > y_k \;\; \& \;\; \frac{y_u}{m_j} - (\frac{1}{m_i} - \frac{1}{m_j})y_k < x_u < \frac{y_u}{m_i}\}.$$

Critical line order for $R_3(i, \hat{j}, \hat{k})$: $c_{jk}, c_j, c_{ik}, c_i, c_{ij}, c_k$.
Rules which prove order:

(i) $\beta_{jk} < \beta_j$: Rule 7.

(ii) $\beta_j < \beta_{ik}$: Rule 8.

(iii) $\beta_{ik} < \beta_i$: Rule 6.

(iv) $\beta_i < \beta_{ij}$: Rule 2.

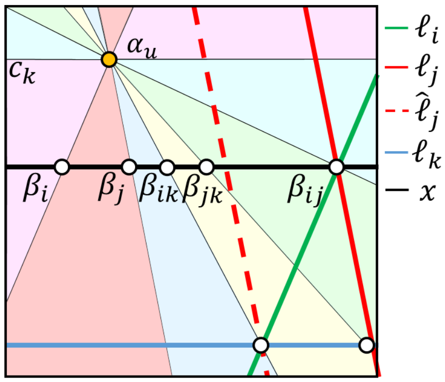

Figure 11: This figure shows the intersections of critical lines with the x-axis given α_u in $R4(\hat{j},\hat{k})$.

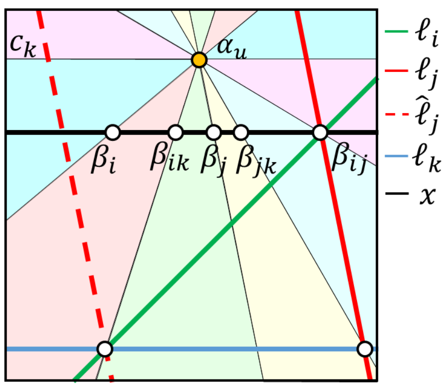

Figure 12: This figure shows the intersections of critical lines with the x-axis given α_u in $R5(j,\hat{j},\hat{k})$.

Region R4 : $R4(\hat{j},\hat{k})$ is defined as the area comprised of $\hat{\ell}_j$ and $\hat{\ell}_k$ through which no other line passes (see Fig. 3). Formally:

$$R4(\hat{j},\hat{k}) = \{(x_u,y_u)\,|\,y_u > 0 \ \&\ x_u < \frac{y_u}{m_j} - (\frac{1}{m_i} - \frac{1}{m_j})y_k\}.$$

Critical line order for $R_4(\hat{j},\hat{k})$: $c_i, c_j, c_{ik}, c_{jk}, c_{ij}, c_k$. Rules which prove order:

(i) $\beta_i < \beta_j$: Rule 1.

(ii) $\beta_j < \beta_{ik}$ Rule 8.

(iii) $\beta_{ik} < \beta_{jk}$: Rule 9.

(iv) $\beta_{jk} < \beta_{ij}$: Rule 5.

Region R5: $R5(j,\hat{j},\hat{k})$ is defined as the open area bordered by ℓ_j, $\hat{\ell}_j$, and $\hat{\ell}_k$ through which no other line passes (see Fig. 3). Formally:

$$R5(j,\hat{j},\hat{k}) = \{(x_u,y_u)\,|\,y_u > 0 \ \&$$

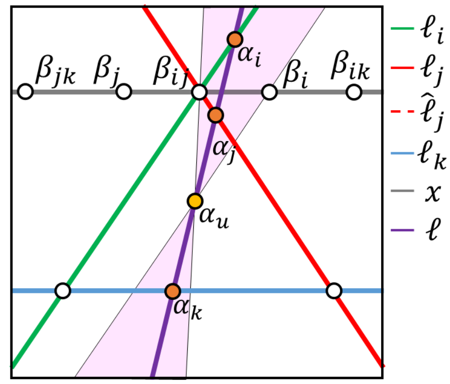

Figure 13: This is the same as Fig 8, only this time we highlight an an $\ell \in L(c_{ij}, c_i, \alpha_u)$ and its intersections.

$$\frac{y_u}{m_j} - (\frac{1}{m_i} - \frac{1}{m_j})y_k < x_u < \frac{y_u}{m_j}\}.$$

Critical line order for $R5(j,\hat{j},\hat{k})$:$c_i, c_{ik}, c_j, c_{jk}, c_{ij}, c_k$.

(i) $\beta_i < \beta_{ik}$: Rule 6.

(ii) $\beta_{ik} < \beta_j$: Rule 8.

(iii) $\beta_j < \beta_{jk}$: Rule 7.

(iv) $\beta_{jk} < \beta_{ij}$: Rule 5.

Note that all of the above orders are independent of the location of α_u in each region.

1.5. Valid Orders for TERs

We now prove the valid orders for each region. To that end, we make use of the basic geometric observations detailed above and the region definitions given. The following proofs can be easily visually verified using the corresponding Figs. 13-17, but are given here for completeness. We define α_i to be the intersection of a line ℓ which passes through α_u and the epipolar line ℓ_i, that is, $\tilde{\alpha}_i = \tilde{\ell} \times \tilde{\ell}_i$ where $\tilde{\alpha}_u \tilde{\ell} = 0$. The points α_j and α_k are defined similarly.

1.5.1 Region R1 (Fig. 13)

Let $L(c_{ij}, c_i, \alpha_u)$ be the set of lines passing through α_u in the section defined by the two neighboring critical lines c_{ij} and c_i. Consider an $\ell \in L(c_{ij}, c_i, \alpha_u)$ for $\alpha_u \in R1$, and define β_ℓ to be the x-intercept of ℓ.

Claim 1: The valid order given by $L(c_{ij}, c_i, \alpha_u)$ when $\alpha_u \in R1$ is (u,i,j,k) as given by the intersections of ℓ with α_u and each of the epipolar lines.

Proof of Claim 1:

1. As ℓ_k is parallel to the x-axis and $y_k < y_u < 0$, geometric observation (2) dictates that α_u must be between α_k and β_ℓ.

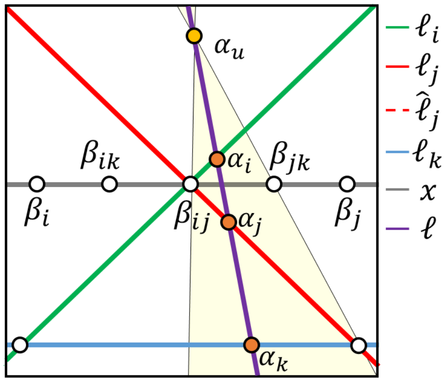

Figure 14: This is the same as Fig 9, only this time we highlight an $\ell \in L(c_{ij}, c_{jk}, \alpha_u)$ and its intersections.

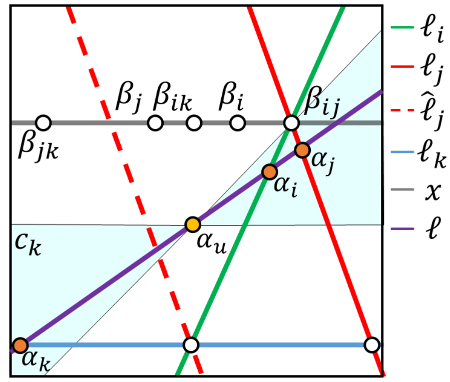

Figure 15: This is the same as Fig 10, only this time we highlight an ℓ connecting α_u and β_ℓ and its intersections.

2. By definition of this section, $\beta_\ell > 0$ (as $\beta_{ij} = 0$), and therefore, β_ℓ is right of ℓ_j. Furthermore, as the definition of $R1$ requires that any points within it be left of ℓ_j (as $x_u < \frac{y_u}{m_j}$), geometric observation (1) dictates that α_j be between α_u and β_ℓ.

3. β_ℓ is defined to be between β_{ij}, through which ℓ_i passes, and β_i, through which c_i passes. Note that c_i also passes through α_u and is parallel to ℓ_i. As such, geometric observation (2) dictates that β_ℓ must be between α_u and α_i

As such, the order of intersections along ℓ must be $\alpha_k < \alpha_u < \alpha_j < \alpha_i$ (up to direction).

1.5.2 Region $\mathbf{R2(i,j)}$ (Fig. 14)

Let $L(c_{ij}, c_{jk}, \alpha_u)$ be the set of lines passing through α_u in the section defined by the two neighboring critical lines c_{ij} and c_{jk}. Consider an $\ell \in L(c_{ij}, c_{jk}, \alpha_u)$ for $\alpha_u \in R2(i,j)$. As above, define β_ℓ to be the x-intercept of ℓ.

Claim 2: The valid order given by $L(c_{ij}, c_{jk}, \alpha_u)$ when $\alpha_u \in R2(i,j)$ is (u,i,j,k) as defined by the intersections of ℓ.

Proof of Claim 2:

1. By definition of the section, $\beta_\ell > 0$ and therefore must be to the right of ℓ_i (as ℓ_i intersects the origin). Furthermore, the definition of $R2(i,j)$ dictates that α_u be left of ℓ_i ($x_u < \frac{y_u}{m_i}$). Using geometric observation (1), we have that α_i is between α_u and β_ℓ.

2. Note that c_jk is defined as the line connecting α_u to the intersection of ℓ_k and ℓ_j. As such, α_k must be left of ℓ_j. Furthermore, as $\beta_\ell > 0$, it must also be to the right of ℓ_j. Therefore, using geometric observation (1), we have that α_j must be between α_k and β_ℓ.

As $y_u > 0$, $y_k < 0$ and β_ℓ is at the x-intercept, the order of intersections along ℓ must be $\alpha_u < \alpha_i < \alpha_j < \alpha_k$ (up to direction).

1.5.3 Region $\mathbf{R3(i,\hat{j},\hat{k})}$ (Fig. 15)

Consider an ℓ connecting α_u and $\beta_\ell = (x_\ell, 0)$ where $x_\ell > 0$. That is, ℓ passes through the section bordered by c_{ij} and c_k.

Claim 3: The valid order given by this section is (k,u,i,j) as defined by the intersections of ℓ.

Proof of Claim 3:

1. As $R3(i,\hat{j},\hat{k})$ defines $x_u < \frac{y_u}{m_i}$ (α_u left of ℓ_i) and $\beta_\ell > 0$ (β_ℓ right of ℓ_i), geometric observation (1) dictates that α_i is between α_u and β_ℓ.

2. Using the definition of $R3(i,\hat{j},\hat{k})$, we also have that $0 > y_u > y_k$, so α_u is between β_ℓ and α_k.

3. As the definition of our lines dictates that ℓ_j is left of ℓ_i when $y > 0$, and the two lines intersect at the origin, it follows that ℓ_j is right of ℓ_i when $y < 0$. As such α_j is between α_i and β_ℓ along ℓ.

We therefore have that the order of intersections along ℓ must be $\alpha_k < \alpha_u < \alpha_i < \alpha_j$ (up to direction).

1.5.4 Region $\mathbf{R4(\hat{j},\hat{k})}$ (Fig. 16)

Consider an ℓ connecting α_u and $\beta_\ell = (x_\ell, 0)$ where $x_\ell > 0$. That is, ℓ passes through the section bordered by c_ij and c_k.

Claim 4: The valid order given by this section is (u,j,i,k) as defined by the intersections of ℓ.

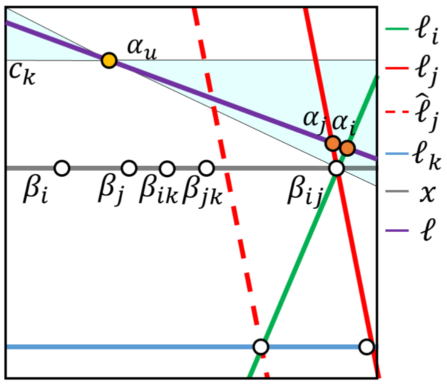

Figure 16: This is the same as Fig 11, only this time we highlight an ℓ connecting α_u and β_ℓ and its intersections. Note that due to space restrictions, we cannot clearly show all the intersections of ℓ. However it is clear that α_k is intersected to the right of the figure.

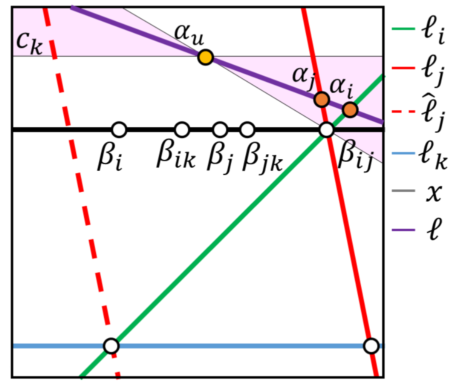

Figure 17: This is the same as Fig 12, only this time we highlight an ℓ connecting α_u and β_ℓ and its intersections. Note that due to space restrictions, we cannot clearly show all the intersections of ℓ. However it is clear that α_k is intersected to the right of the figure.

Proof of Claim 4:

1. As $R4(\hat{j},\hat{k})$ defines that $\alpha_u > 0$ and $y_k < 0$, geometric observation (1) dictates that β_ℓ is between α_u and α_k.

2. Note that $\hat{\ell}_j$ is parallel to ℓ_j and intersects γ_{ik} (the intersection point of ℓ_i and ℓ_k). As $y_k < 0$ and, as described above, when $y < 0$ ℓ_i is left of ℓ_j, γ_{ik} is left of ℓ_j. Therefore, $\hat{\ell}_j$ is left of ℓ_j. As $R4(\hat{j},\hat{k})$ defines that α_u is left of $\hat{\ell}_j$ ($x_u < \frac{y_u}{m_j} - (\frac{1}{m_i} - \frac{1}{m_j})y_k$), α_u must also be left of ℓ_j. As α_u is left of ℓ_j and β_ℓ is right of ℓ_j (as $\beta_\ell > 0$), geometric observation (1) dictates that α_j must be between α_u and β_ℓ.

3. As ℓ_j is left of ℓ_i for $y > 0$ and α_u is left of ℓ_j (as above), α_u must also be left of ℓ_i. Therefore, as β_ℓ is right of ℓ_i (as $\beta_\ell > 0$), geometric observation (1) is used to determine that α_i is between α_u and β_ℓ.

4. Given (2) and (3), we know that α_i and α_j must be above the x-axis. Therefore, as we know that ℓ_j is left of ℓ_i above $y > 0$, we know that α_i is between α_j and β_ℓ.

We therefore have that the order of intersections along ℓ must be $\alpha_u < \alpha_j < \alpha_i < \alpha_k$ (up to direction).

1.5.5 Region $\mathrm{R5}(\mathrm{j},\hat{\mathrm{j}},\hat{\mathrm{k}})$ (Fig. 17)

Consider an ℓ connecting α_u and $\beta_\ell = (x_\ell, 0)$ where $x_\ell > 0$. That is, ℓ passes through the section bordered by c_ij and c_k.

Claim 5: The valid order given by this section is (u, j, i, k) as defined by the intersections of ℓ.

Proof of Claim 5:

1. As $R5(j,\hat{j},\hat{k})$ defines that $\alpha_u > 0$ and $y_k < 0$, geometric observation (1) dictates that β_ℓ is between α_u and α_k.

2. As $R5(j,\hat{j},\hat{k})$ defines that $x_u < \frac{y_u}{m_j}$ (α_u is left of ℓ_j) and β_ℓ is right of ℓ_j (as $\beta_\ell > 0$), geometric observation (1) is used to determine that α_j is between α_u and β_ℓ.

3. As our lines are set up in such a way that ℓ_j is left of ℓ_i for $y > 0$ and α_u is left of ℓ_j (as above), α_u must also be left of ℓ_i. Therefore, as β_ℓ is right of ℓ_i (as $\beta_\ell > 0$), geometric observation (1) is used to determine that α_i is between α_u and β_ℓ.

4. Given (2) and (3), we know that α_i and α_j must be above the x-axis. Therefore, as we know that ℓ_j is left of ℓ_i above $y > 0$, we know that α_i is between α_j and β_ℓ.

As such, the order of intersections along ℓ must be $\alpha_u < \alpha_j < \alpha_i < \alpha_k$ (up to direction). Note that the order we showed to be valid for this region was also shown to make $R4(\hat{j},\hat{k})$ valid. However, as the order of critical lines between the two regions is different, not all orders will be shared.

Now that we have the order of the critical lines, and one valid order for each region, we can use observations **A1-A2** from Sec. 3.3, in order to find the remainder of the valid orders for each region (see Table 2).

In this section, we proved: (i) the critical line order using any α_u within each region and (ii) the valid orders for each region given any α_u in that region. Given these two claims, it follows that each point within a region has the same 6 valid orders.

#	Region	Critical Line Order	Valid Orders
1	$R1$	$c_{jk}, c_j, c_{ij}, c_i, c_{ik}, c_k$	$(i,u,j,k),(i,j,u,k),(i,u,k,j),(i,k,u,j),(k,i,u,j),(j,i,u,k)$
2	$R2(i,j)$	$c_i, c_{ik}, c_{ij}, c_{jk}, c_j, c_k$	$(u,j,k,i),(u,j,i,k),(u,i,j,k),(u,i,k,j),(j,u,i,k),(i,u,j,k)$
3	$R2(i,k)$	$c_j, c_i, c_{ij}, c_{ik}, c_{jk}, c_k$	$(u,k,j,i),(u,k,i,j),(u,i,k,j),(u,i,j,k),(k,u,i,j),(i,u,k,j)$
4	$R2(j,k)$	$c_{ik}, c_{jk}, c_{ij}, c_j, c_i, c_k$	$(u,k,i,j),(u,k,j,i),(u,j,k,i),(u,j,i,k),(k,u,j,i),(j,u,k,i)$
5	$R3(i,\hat{j},\hat{k})$	$c_{jk}, c_j, c_{ik}, c_i, c_{ij}, c_k$	$(u,i,j,k),(u,i,k,j),(j,u,k,i),(k,u,j,i),(j,u,i,k),(k,u,i,j)$
6	$R3(j,\hat{i},\hat{k})$	$c_{ij}, c_j, c_{jk}, c_i, c_{ik}, c_k$	$(u,j,i,k),(u,j,k,i),(i,u,k,j),(k,u,i,j),(i,u,j,k),(k,u,j,i)$
7	$R3(k,\hat{i},\hat{j})$	$c_{ik}, c_j, c_{ij}, c_i, c_{jk}, c_k$	$(u,k,i,j),(u,k,j,i),(i,u,j,k),(j,u,i,k),(i,u,k,j),(j,u,k,i)$
8	$R4(\hat{j},\hat{k})$	$c_i, c_j, c_{ik}, c_{jk}, c_{ij}, c_k$	$(u,i,j,k),(u,i,k,j),(u,j,i,k),(u,k,i,j),(i,j,u,k),(i,k,u,j)$
9	$R4(\hat{i},\hat{k})$	$c_{ij}, c_{ik}, c_{jk}, c_i, c_j, c_k$	$(u,j,i,k),(u,j,k,i),(u,i,j,k),(u,k,j,i),(j,i,u,k),(j,k,u,i)$
10	$R4(\hat{i},\hat{j})$	$c_j, c_{ik}, c_{ij}, c_{jk}, c_i, c_k$	$(u,k,i,j),(u,k,j,i),(u,i,k,j),(u,j,k,i),(k,i,u,j),(k,j,u,i)$
11	$R5(j,\hat{j},\hat{k})$	$c_i, c_{ik}, c_j, c_{jk}, c_{ij}, c_k$	$(u,i,j,k),(u,i,k,j),(u,j,i,k),(j,u,k,i),(k,u,j,i),(k,i,u,j)$
12	$R5(k,\hat{k},\hat{j})$	$c_{jk}, c_{ij}, c_j, c_i, c_{ij}, c_k$	$(u,i,k,j),(u,i,j,k),(u,k,i,j),(k,u,j,i),(j,u,k,i),(j,i,u,k)$
13	$R5(i,\hat{i},\hat{k})$	$c_{ij}, c_{ik}, c_i, c_{jk}, c_j, c_k$	$(u,j,i,k),(u,j,k,i),(u,i,j,k),(i,u,k,j),(k,u,i,j),(k,j,u,i)$
14	$R5(k,\hat{k},\hat{i})$	$c_{ij}, c_j, c_i, c_{jk}, c_{ik}, c_k$	$(u,j,k,i),(u,j,i,k),(u,k,j,i),(k,u,i,j),(i,u,k,j),(i,j,u,k)$
15	$R5(j,\hat{j},\hat{i})$	$c_{ik}, c_j, c_{ij}, c_{ik}, c_i, c_k$	$(u,k,j,i),(u,k,i,j),(u,j,k,i),(j,u,i,k),(i,u,j,k),(i,k,u,j)$
16	$R5(i,\hat{i},\hat{j})$	$c_j, c_{ik}, c_{ij}, c_i, c_{jk}, c_k$	$(u,k,i,j),(u,k,j,i),(u,i,k,j),(i,u,j,k),(j,u,i,k),(j,k,u,i)$

Table 2

2. Experimental Results

Here we present a summary of experiments from all datasets split into two tables. Table 3 shows the results of experiments in which three initial correspondences were given (Test 5, Sec. 5.2). Table 4 shows results of experiments in which only one point was given and initial correspondence was selected using nearest neighbors (Test 6, Sec. 5.2). Table 5 shows results relating to incorrect initializations and dead ends (Test 6, Sec. 5.2).

We present the results in a similar manner in Tables 3,4. Each dataset corresponds to a location, given by the letters a-i, and different image sets taken at each location at different times, given by the numbers 1-5. The number of images and the number of points to be matched vary between datasets. We tested our method a number of times on each dataset, each with a different combination of initial points summarized in the # combinations column. For each dataset we also present the percent of the tests in which dead ends were encountered. Note that, as described in test 5, Sec. 5.2, when encountering dead ends, we deduce that we do no better than matching without TERs. As such, the rest of the columns present results from the remainder of the data, which has no dead ends. We present the percent correct matching with and without the use of TERs over all data in each dataset. As we searched for matches for different points per dataset, improvements in matching may not have been consistent from point to point. In the "% Points Improved" column we present the percentage of the points which had improved matching when using TERs. We also present the difference in the percent of correct matching between matching with and without TERs. We present the results only for the points for which using TERs changed the matching results.

Figure 18: Four sample images from dataset a1.

We present here a number of images from each location as reference (Figs. 18- 26). Additional examples are shown in the video clip provided at: https://youtu.be/viH0jlL5gMA.

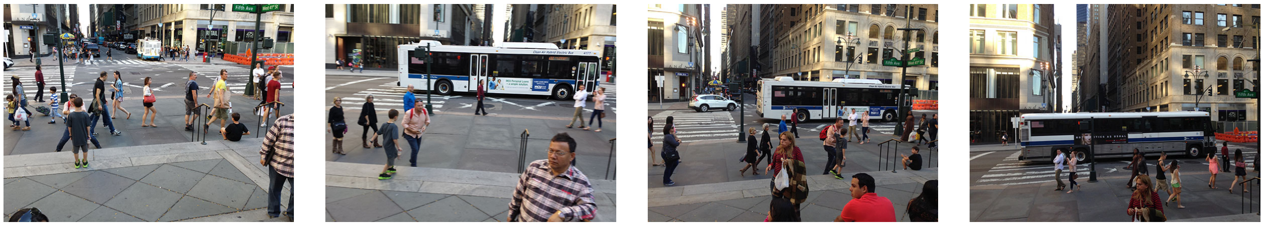

Figure 21: Four sample images from dataset d1.

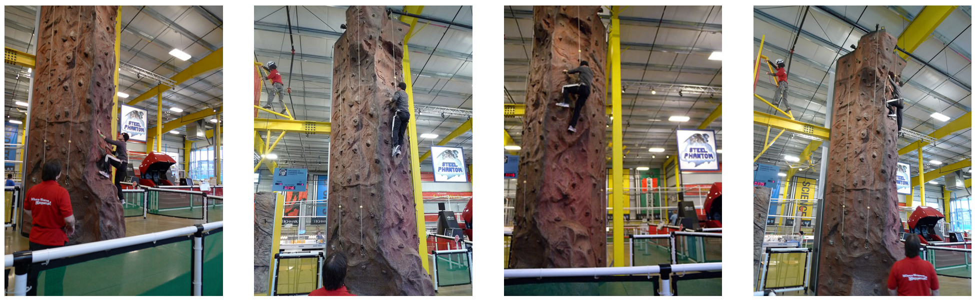

Figure 22: Four sample images from dataset e1.

Figure 19: Four sample images from dataset b2.

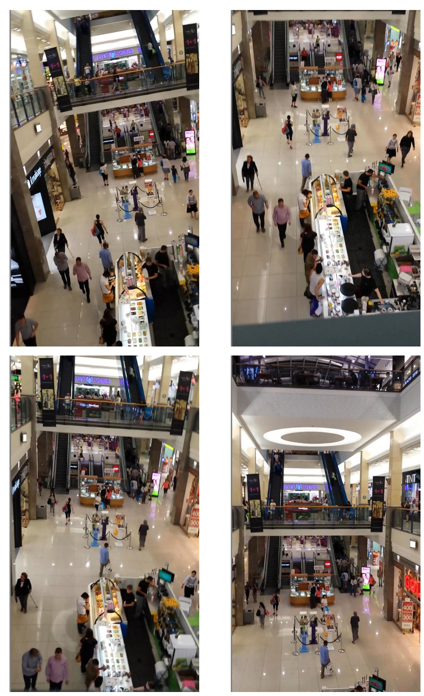

Figure 20: Four sample images from dataset c5.

Figure 23: Four sample images from dataset f1.

Figure 24: Four sample images from dataset g1.

Figure 25: Four sample images from dataset h1.

Figure 26: Four sample images from location i.

Dataset	# Images	# Points	# Combinations	% Dead End	% Matches Without TERs	% Matches With TER	% Points Improved	Average Diff. %
a1	5	4	40	0	22.5	22.5	0	
a2	5	5	50	0.4	13.6	15.3	60	4.6
a3	6	9	162	2.0	7.7	8.2	33	4.5
a4	10	2	185	31.9	19.9	19.9	0	
b1	7	1	35	22.9	29.6	45.4	100	15.7
b2	6	5	100	0.4	15.8	16.0	20	5.0
c1	6	3	60	9.4	20.6	26.5	100	5.9
c2	7	4	137	9.1	8.9	9.9	75	1.8
c3	6	4	80	0.9	14.0	15.2	100	1.2
c4	6	2	40	1.3	23.5	24.3	100	0.8
c5	8	5	269	3.7	12.5	13.8	100	1.3
d1	6	1	18	50.0	33.3	48.2	100	14.8
e1	6	4	80	4.1	4.8	6.7	25	29.6
f1	7	6	209	2.8	10.6	12.2	83	2.3
g1	8	4	203	21.3	11.7	12.9	75	2.2
h1	9	2	157	37.3	12.4	22.6	100	10.3
h2	6	4	74	5.4	10.8	12.5	50	7.1
i1	7	3	101	16.5	11.2	11.9	67	1.5

Table 3: This table presents the results of test 5, in which three initial correspondences are given manually. See details regarding columns in Sec. 2 above.

Dataset	# Images	# Points	# Combinations	% Dead End	% Matches Without TERs	% Matches With TER	% Points Improved	Average Diff. %
a1	5	4	20	2.5	7.3	6.9	25	-6.3
a2	5	5	25	1.6	3.8	3.6	20	-5.0
a3	6	9	52	4.3	2.9	3.3	22	6.7
a4	10	2	14	32.1	13.9	13.9	0	
b1	7	1	7	85.7	0	16.7	100	16.7
b2	6	5	30	4.0	6.7	6.9	20	3.3
c1	6	3	18	13.0	8.9	11.1	100	2.2
c2	7	4	24	20.8	0	1.6	25	16.7
c3	6	4	24	5.2	4.8	4.8	0	
c4	6	2	12	8.3	0	0	0	
c5	8	5	37	10.8	6.5	6.7	40	3.3
d1	6	1	6	66.7	10.0	30.0	100	20.0
e1	6	4	23	13.0	0.4	0.7	50	1.2
f1	7	6	42	7.9	5.3	5.7	67	0.7
g1	8	4	32	22.7	0	0	0	
h1	9	2	16	40.6	14.1	35.2	50	37.5
h2	6	4	24	12.5	3.9	3.9	0	
i1	7	3	21	17.5	4.3	7.1	33	25.0

Table 4: This table presents the results of test 6, in which nearest neighbors to a single given point are used to select the three initial correspondences.See details regarding columns in Sec. 2 above.

# Images	# Experiments With S_q Errors	# Dead Ends	# Dead Ends Given S_q Errors	% Dead Ends Given S_q Errors
5	19	4	4	21.05
6	116	73	59	50.86
7	60	57	48	80.00
8	42	49	34	80.95
9	29	39	28	96.55
10	9	9	9	100.00
12	19	24	19	100.00

Table 5: This table presents data from test 6 relating to incorrect matches selected for the initial set. For datasets containing different number of images, we present the number of experiments in which the initial set S_q was initialized incorrectly, the overall number of dead ends, the number of dead ends in experiments which had errors in the initial set, and the percentage of the time in which dead ends were reached given errors in the initial set.