A Consensus-Based Framework for Distributed Bundle Adjustment - Supplementary Material

Theorem 4.1. *(Convergence) With f_1 and f_2 as in (17)-(18), let $\{(P^{(t)}, X^{(t)})\} \subset R^{3\times 4\times m} \times R^{3\times n}$ denote a sequence generated by algorithm 1. Assuming that local minimizers of (1) exists, are unique and that the scene depth d is bounded from below by $d = P_{i3}^{(t)} \begin{bmatrix} X_i^{(t)} \\ 1 \end{bmatrix} \geq d_{\min} > 0$, $i \in [1, m]$. Then there exists a $\mathbb{R} \ni \rho_{\min} > 0$ such that if $\rho^{(t)} > \rho_{\min}$ (with $t \geq T$ for some fixed T) then Algorithm 1 is guaranteed to converge and every limit point of $\{(P^{(t)}, X^{(t)})\}$ is a local minimizer of (1).*

Proof. This theorem follows from theorems A.1 and A.2. □

A. Proof of Theorem 4.1.

Let $e_{ij} : \mathcal{Q} \times \mathbb{R}^3 \mapsto \mathbb{R}^+$ denote the residual of 3D point j in image i,

$$\epsilon_{ij}(P_i, X_j) = w_{ij}||u_{ij} - \pi(P_i, X_j)||^2, \tag{27}$$

and $g_k : \mathcal{Q}^m \times \mathbb{R}^{3\times n}$

$$g_k(P, X) = \sum_{i\in c_k} \sum_{j=1}^{n} \epsilon_{ij}(P_i, X_j). \tag{28}$$

Define $P_k : \mathbb{R}^{3\times n} \mapsto \mathcal{Q}^{|c_k|}$ as

$$P_k(X) = \underset{P\in\mathcal{Q}^{|c_k|}}{\arg\min} \sum_{\substack{i\in c_k \\ 1\leq j\leq n}} w_{ij}||u_{ij} - \pi(P_i, X_j)||^2. \tag{29}$$

With $P_k(X) = \{P_{ik}(X)\}_{i\in c_k}$ and $P_{ik}(X) = \begin{bmatrix} p_{1ik}(X)^T \\ p_{2ik}(X)^T \\ p_{3ik}(X)^T \end{bmatrix}$ we can write

$$\bar{g}_k(X) = g_k(P_k(X), X) = \sum_{\substack{i\in c_k \\ 1\leq j\leq n}} \epsilon_{ij}(P_i(X), X_j) = \sum_{\substack{i\in c_k \\ 1\leq j\leq n}} w_{ij}||u_{ij} - \pi(P_i(X), X_j)||_2^2. \tag{30}$$

Lemma A.1. *If the scene depth of the resectioning problem (29) is bounded from below by $d_{\min}$,*

$$p_{3ik}(X)^T \begin{bmatrix} X_j \\ 1 \end{bmatrix}) \geq d_{\min}, \quad j = 1, ..., n, \tag{31}$$

and with $P_k(X)$ unique, then,
(i) the function $\bar{g}_k : \mathbb{R}^{3\times n} \mapsto \mathbb{R}$ is smooth,
(ii) $\bar{g}_k$ has a locally Lipschitz continuous gradient, that is, $\exists L_k \geq 0$ such that

$$||\nabla_X \bar{g}_k(Y_1) - \nabla_X \bar{g}_k(Y_2)|| \leq L_k||Y_1 - Y_2||, \tag{32}$$

(iii) the Hessian of $\bar{g}_k$ has eigenvalues that are bounded from below, $\exists \lambda_k < \infty$

$$\nabla_X^2 \bar{g}_k(Y) + \lambda_k I \succ 0, \tag{33}$$

for finite Y, Y_1, Y_2.

Proof. The error function $\epsilon_{ij}(P,X)$ (27) can be rewritten as

$$\epsilon_{ij}(P,X) = w_{ij}\frac{\left(p_1\left[\begin{smallmatrix}X\\1\end{smallmatrix}\right] - u_{ij}^x p_3\left[\begin{smallmatrix}X\\1\end{smallmatrix}\right]\right)^2 + \left(p_2\left[\begin{smallmatrix}X\\1\end{smallmatrix}\right] - u_{ij}^y p_3\left[\begin{smallmatrix}X\\1\end{smallmatrix}\right]\right)^2}{\left(p_3\left[\begin{smallmatrix}X\\1\end{smallmatrix}\right]\right)^2} = w_{ij}\frac{\alpha^2+\beta^2}{\gamma^2}, \tag{34}$$

with

$$\begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix} = \begin{bmatrix} p_1\left[\begin{smallmatrix}X\\1\end{smallmatrix}\right] - u_{ij}^x p_3\left[\begin{smallmatrix}X\\1\end{smallmatrix}\right]\\ p_2\left[\begin{smallmatrix}X\\1\end{smallmatrix}\right] - u_{ij}^y p_3\left[\begin{smallmatrix}X\\1\end{smallmatrix}\right]\\ p_3\left[\begin{smallmatrix}X\\1\end{smallmatrix}\right]\end{bmatrix}. \tag{35}$$

With the above expressions being linear in X and P respectively, (35) can then be written $\left[\begin{smallmatrix}\alpha & \beta & \gamma\end{smallmatrix}\right]^T = A_X \text{vec}(P) = A_P X + a_P$. In [2] it was shown that the gradient of (27) is given by

$$\frac{\partial \epsilon_{ij}}{\partial X} = w_{ij}\frac{2}{\gamma^2}A_P^T\begin{bmatrix}\alpha\\ \beta\\ -\frac{\alpha^2+\beta^2}{\gamma}\end{bmatrix}, \tag{36}$$

$$\frac{\partial \epsilon_{ij}}{\partial P} = w_{ij}\frac{2}{\gamma^2}A_X^T\begin{bmatrix}\alpha\\ \beta\\ -\frac{\alpha^2+\beta^2}{\gamma}\end{bmatrix}. \tag{37}$$

As $\gamma = (p_3^T\left[\begin{smallmatrix}X\\1\end{smallmatrix}\right]) \geq d_{\min}$ by assumption, it follows that the error function ϵ_{ij} is smooth.

From (29) we have that

$$\frac{\partial g_k}{\partial P}(P_k(X), X) = 0. \tag{38}$$

Differentiating (30) and using (38) yields

$$\nabla_X \bar{g}_k(X) = \frac{\partial g_k}{\partial P}(P_k(X),X)\frac{\partial P}{\partial X}(X) + \frac{\partial g_k}{\partial X}(P_k(X),X) = \frac{\partial g_k}{\partial X}(P_k(X),X) = \sum_{i\in c_k}\sum_{j=1}^{n}\frac{\partial \epsilon_{ij}}{\partial X}(P_i(X),X_j) \in C^\infty, \tag{39}$$

proving part (i) of the lemma. This result is a special instance of the *Envelope theorem*, see [1] for more. Extending this result to hold for more specific instances where $\mathcal{Q}$ contains additional constraints, such as Euclidean structure, is a straightforward application of more general forms of this theorem. For (ii), the second part of the lemma, local Lipschitz continuity follows directly from the smoothness of $\nabla_X \bar{g}_k$.

Differentiating (39) we obtain the Hessian of $\bar{g}_k$ as

$$\nabla_X^2 \bar{g}_k(X) = \sum_{i\in c_k}\sum_{j=1}^{n}\frac{\partial^2 \epsilon_{ij}}{\partial X^2}(P_i(X),X_j), \tag{40}$$

with

$$\frac{\partial^2 \epsilon_{ij}}{\partial X^2} = w_{ij}\frac{2}{\gamma}A_P^T\begin{bmatrix}1 & 0 & -\frac{2\alpha}{\gamma}\\ 0 & 1 & -\frac{2\beta}{\gamma}\\ -\frac{2\alpha}{\gamma} & -\frac{2\beta}{\gamma} & \frac{\alpha^2+\beta^2}{\gamma^2}\end{bmatrix}A_P. \tag{41}$$

It was shown in [2] that

$$\frac{\partial^2 \epsilon_{ij}}{\partial X^2} \succeq w_{ij}\frac{2}{d_{\min}^2}A_P^T\begin{bmatrix}\frac{1}{3} & 0 & 0\\ 0 & \frac{1}{3} & 0\\ 0 & 0 & -3\frac{\alpha^2+\beta^2}{\gamma^2}\end{bmatrix}A_P. \tag{42}$$

Now, for any $v \in \mathbb{R}^3$, $||v|| = 1$ we have

$$v^T \left(\frac{\partial^2 \epsilon_{ij}}{\partial X^2} \right) v \geq w_{ij} \frac{2}{d_{\text{min}}^2} v^T \left(A_P^T \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & -3\frac{\alpha^2+\beta^2}{\gamma^2} \end{bmatrix} A_P \right) v \geq -w_{ij} \frac{6}{d_{\text{min}}^2} \frac{\alpha^2 + \beta^2}{\gamma^2} ||A_P|| = \tag{43}$$

$$= -\frac{6}{d_{\text{min}}^2} \epsilon_{ij} ||A_P|| \geq -\lambda_{ij} > -\infty. \tag{44}$$

Then with $\lambda_k = \sum_{i \in c_k} \sum_{j=1}^n \lambda_{ij}$ statement (iii) follows. □

Using (30) we can write the iterations of algorithm 1 as

$$Z^{k(t+1)} = \text{prox}_{f_2/\rho^{(t)}}(2\bar{X}^{k(t)} - Z^{k(t)}) + Z^{k(t)} - \bar{X}^{k(t)} \tag{45}$$

$$\bar{X}^{k(t+1)} = \text{prox}_{\sum_k \bar{g}_k/\rho^{(t)}}(Z^{k(t+1)}) \tag{46}$$

$$X_j^{(t+1)} = \bar{X}_j^{k(t+1)}, \text{ for any } k \in [1, l] \text{ such that } \bar{w}_j^k = 1, \tag{47}$$

$$P^{(t+1)} = \bigcup_{k=1}^{l} P_k(\bar{X}^{k(t+1)}). \tag{48}$$

For convenience we introduce a further latent variable $Q \in \mathbb{R}^{3\times n}$ and visibility matrix $\bar{W} \in \mathbb{R}^{3\times n\times l}$ defined as, $\bar{W}^k = \left[\begin{smallmatrix}1\\1\\1\end{smallmatrix}\right] (\bar{w}^k)^T, k = 1, ..., l$. Then (45)-(48) can be written

$$\sum_{k=1}^{l} \bar{W}^k \circ Q^{(t+1)} = \text{prox} \left(\sum_{k=1}^{l} \bar{W}^k \circ (2\bar{X}^{k(t)} - Z^{k(t)}) \right), \tag{49}$$

$$Z^{k(t+1)} = Q^{(t+1)} + Z^{k(t)} - \bar{X}^{k(t)}, \tag{50}$$

$$\bar{X}^{k(t+1)} = \text{prox}_{\sum_k \bar{g}_k/\rho^{(t)}}(Z^{k(t+1)}), \tag{51}$$

$$X_j^{(t+1)} = \bar{X}_j^{k(t+1)}, \text{ for any } k \in [1, l] \text{ such that } \bar{w}_j^k = 1, \tag{52}$$

$$P^{(t+1)} = \bigcup_{k=1}^{l} P_k(\bar{X}^{k(t+1)}). \tag{53}$$

From the necessary conditions for optimality of (49) and (51) we have

$$0 = \bar{W} \circ \left(Q^{k(t+1)} - 2\bar{X}^{k(t)} + Z^{k(t)} \right), \tag{54}$$

$$0 = \nabla_X g_k(\bar{X}^{k(t+1)}) + \rho \bar{W} \circ \left(\bar{X}^{k(t+1)} - Z^{k(t+1)} \right), \tag{55}$$

with $\bar{W} = \sum_{k=1}^{l} \bar{W}^k$.

Next define the auxiliary function $F : \mathbb{R}^{l\times 3\times n} \times \mathbb{R}^{3\times n} \times \mathbb{R}^{l\times 3\times n} \mapsto \mathbb{R}^+$ as

$$F(\{\bar{X}^k\}, Q, \{Y^k\}) = \sum_{k=1}^{l} \left(g_k(\bar{X}^k) + \frac{\rho}{2} ||\bar{W}^k \circ (Q - \bar{X}^k + Y^k)||_F^2 - \frac{\rho}{2} ||\bar{W}^k \circ Y^k||_F^2 \right). \tag{56}$$

We now show that the change in function value of F between iterations is bounded.[1]

Lemma A.2. *Let $\{\bar{X}^{(t)}, Q^{(t)}, Z^{(t)})\}$ denote a sequence generated by (49)-(51). If the conditions of Lemma A.1 hold, we have*

$$F(\{\bar{X}^{k(t)}\}, Q^{(t)}, \{Z^{k(t)} - \bar{X}^{k(t)}\}) - F(\{\bar{X}^{k(t+1)}\}, Q^{k(t+1)}, \{Z^{k(t+1)} - \bar{X}^{k(t+1)}\}) \geq$$
$$\sum_{k=1}^{l} \left(\left(\frac{\rho - \lambda}{2} - \frac{L^2}{\rho} \right) || \left(\bar{X}^{k(t+1)} - \bar{X}^{k(t)} \right) ||_F^2 + \frac{\rho}{2} || \left(Q^{(t+1)} - Q^{(t)} \right) ||_F^2 \right). \tag{57}$$

[1] In order to simplify notation for the remainder of this appendix we assume that $\bar{W}_j^k = 1, \forall k, j$, extending this proof to general visibility matrices is entirely straightforward, however resulting in a substantially more cluttered notation.

With $\mathcal{Q} \in \mathbb{R}^{3\times 4}$, $L = \max_i L_i$ and $\lambda = \max_i \lambda_i$.

Proof.

$$F\left(\{\bar{X}^{k(t)}\}, Q^{(t)}, \{Z^{k(t)} - \bar{X}^{k(t)}\}\right) - F(\{\bar{X}^{k(t+1)}\}, Q^{(t+1)}, \{Z^{k(t+1)} - \bar{X}^{k(t+1)}\}) = \tag{58}$$
$$= F\left(\{\bar{X}^{k(t)}\}, Q^{(t)}, \{Z^{k(t)} - \bar{X}^{k(t)}\}\right) - F\left(\{\bar{X}^{k(t)}\}, Q^{(t+1)}, \{Z^{k(t)} - \bar{X}^{k(t)}\}\right) + \tag{59}$$
$$+ F\left(\{\bar{X}^{k(t)}\}, Q^{(t+1)}, \{Z^{k(t)} - \bar{X}^{k(t)}\}\right) - F\left(\{\bar{X}^{k(t+1)}\}, Q^{(t+1)}, \{Z^{k(t)} - \bar{X}^{k(t)}\}\right) + \tag{60}$$
$$+ F\left(\{\bar{X}^{k(t+1)}\}, Q^{(t+1)}, \{Z^{k(t)} - \bar{X}^{k(t)}\}\right) - F\left(\{\bar{X}^{k(t+1)}\}, Q^{(t+1)}, \{Z^{k(t+1)} - \bar{X}^{k(t+1)}\}\right) \geq \tag{61}$$
$$\geq \sum_{k=1}^{l} -\rho < Q^{(t+1)} - \bar{X}^{k(t)} + Z^{k(t)} - \bar{X}^{k(t)}, Q^{(t+1)} - Q^{(t)} > +\frac{\rho}{2}||Q^{k(t+1)} - Q^{(t)}||_F^2 + \tag{62}$$
$$-\rho < \nabla_{\bar{X}^k} F(\bar{X}^{k(t+1)}, Q^{(t+1)}, Z^{k(t)} - \bar{X}^{k(t)}), \bar{X}^{k(t+1)} - \bar{X}^{k(t)} > +\frac{\rho - \lambda}{2}||\bar{X}^{k(t+1)} - \bar{X}^{k(t)}||_F^2 + \tag{63}$$
$$-\rho < Z^{k(t+1)} - \bar{X}^{k(t+1)} - Z^{k(t)} + \bar{X}^{k(t)}, Q^{(t+1)} - \bar{X}^{k(t+1)} >= \tag{64}$$
$$= \sum_{k=1}^{l} -\rho < Q^{(t+1)} - (2\bar{X}^{k(t)} - Z^{k(t)}), Q^{(t+1)} - Q^{(t)} > +\frac{\rho}{2}||Q^{(t+1)} - Q^{(t)}||_F^2 + \tag{65}$$
$$-\rho < \nabla_X g_k(\bar{X}^{k(t+1)}) + \rho(Q^{(t+1)} - \bar{X}^{k(t+1)} + Z^{k(t)} - \bar{X}^{k(t)}), \bar{X}^{k(t+1)} - \bar{X}^{k(t)} > +\frac{\rho - \lambda}{2}||\bar{X}^{k(t+1)} - \bar{X}^{k(t)}||_F^2 + \tag{66}$$
$$-\rho||\{Z^{(t+1)} - \bar{X}^{k(t+1)} - (Z^{k(t)} - \bar{X}^{k(t)})\}||_F^2 = \tag{67}$$
$$= \sum_{k=1}^{l} \frac{\rho}{2}||Q^{(t+1)} - Q^{(t)}||_F^2 + \frac{\rho - \lambda}{2}||\bar{X}^{k(t+1)} - \bar{X}^{k(t)}||_F^2 - \frac{1}{\rho}||\nabla_X g_k(\bar{X}^{k(t+1)}) - \nabla_{\bar{X}^k} g_k(\bar{X}^{k(t)})||_F^2 \geq \tag{68}$$
$$\geq \sum_{k=1}^{l} \frac{\rho}{2}||Q^{(t+1)} - Q^{(t)}||_F^2 + \left(\frac{\rho - \lambda}{2} - \frac{L^2}{\rho}\right)||\bar{X}^{k(t+1)} - \bar{X}^{k(t)}||_F^2. \tag{69}$$

Here the first inequality follows from the strong convexity of F for $\rho > \lambda$. The simplification of (64) follows from (50), expression (65) becomes (68) through (54) and (66) is simplified using (50) and (54). The final inequality follows directly from applying Lemma A.1. □

Theorem A.1. *With conditions as in Lemma A.2 for all t. If*

$$\frac{\rho - \lambda}{2} - \frac{L^2}{\rho} > 0, \tag{70}$$

then the sequences $\{\bar{X}^{(t)}\}_{t=1}^{\infty}$, $\{Q^{(t)}\}_{t=1}^{\infty}$ and $\{Z^{(t)}\}_{t=1}^{\infty}$ are convergent. That is,

$$\lim_{t\to\infty} ||\bar{X}^{k(t+1)} - \bar{X}^{k(t)}||_F^2 = 0, \tag{71}$$
$$\lim_{t\to\infty} ||Q^{(t+1)} - Q^{(t)}||_F^2 = 0. \tag{72}$$
$$\lim_{t\to\infty} ||Z^{k(t+1)} - Z^{k(t)}||_F^2 = 0. \tag{73}$$

Proof. First we show that the sequence $F(\{\bar{X}^{k(t)}\}, Q^{(t)}, \{Z^{k(t)} - \bar{X}^{k(t)}\})$ is non-negative.

$$F(\{\bar{X}^{k(t)}\}, Q^{(t)}, \{Z^{k(t)} - \bar{X}^{k(t)}\}) = \tag{74}$$

$$= \sum_{k=1}^{l} \left(\bar{g}_k(\bar{X}^{k(t)}) + \frac{\rho}{2} ||Q^{(t)} - \bar{X}^{k(t)} + Z^{k(t)} - \bar{X}^{k(t)}||_F^2 - \frac{\rho}{2} ||Z^{k(t)} - \bar{X}^{k(t)}||_F^2 \right) = \tag{75}$$

$$= \sum_{k=1}^{l} \left(\bar{g}_k(\bar{X}^{k(t)}) + \frac{\rho}{2} ||Q^{(t)} - \bar{X}^{k(t)} + \frac{1}{\rho} \nabla_X \bar{g}_k(\bar{X}^{k(t)})||_F^2 - \frac{\rho}{2} ||\frac{1}{\rho} \nabla_X \bar{g}_k(\bar{X}^{k(t)})||_F^2 \right) = \tag{76}$$

$$= \sum_{k=1}^{l} \left(\bar{g}_k(\bar{X}^k) + < \nabla_X \bar{g}_k(\bar{X}^{k(t)}), Q^{(t)} - \bar{X}^k > + \frac{\rho}{2} ||Q^{k(t)} - \bar{X}^k||_F^2 \right) \geq \sum_{k=1}^{l} \bar{g}_k(\bar{X}^k) \geq 0. \tag{77}$$

The equality succeeding (75) follows from (55). The second-to-last inequality is a result of the strong convexity of (51) for $\rho > \lambda$, according to Lemma A.1 (iii). To see that (70) implies that $\rho > \lambda$ let ρ_1 and ρ_2 denote the two roots of $\rho^2 - \lambda\rho - 2L^2 = 0$. Since the discriminant of this quadratic equation, $\Delta = \lambda^2 + 8L^2$ is nonnegative, ρ_1, ρ_2 must be real. Choosing $\rho_1 \geq \rho_2$, from Vieta's formulas we have that

$$\rho_1\rho_2 = -2L^2 < 0, \tag{78}$$

$$\rho_1 + \rho_2 = \lambda > 0. \tag{79}$$

Since (78) implies that $\rho_1 > 0$ and $\rho_2 < 0$ then from (79) it follows that $\rho_1 > \rho_1 + \rho_2 = \lambda$.

Summing first the left hand side of (57) over all t from 1 to T yields the telescopic series,

$$\sum_{t=1}^{T} \left[F(\{\bar{X}^{k(t)}\}, Q^{(t)}, \{Z^{k(t)} - \bar{X}^{k(t)}\}) - F(\{\bar{X}^{k(t+1)}\}, Q^{(t+1)}, \{Z^{k(t+1)} - \bar{X}^{k(t+1)}\}) \right] = \tag{80}$$

$$F(\{\bar{X}^{k(1)}\}, Q^{(1)}, \{Z^{k(1)} - \bar{X}^{k(1)}\}) - F(\{\bar{X}^{k(T)}\}, Q^{(T)}, \{Z^{k(T)} - \bar{X}^{k(T)}\}). \tag{81}$$

By applying (77) and (81) to (57) and letting $T \to \infty$ we can bound the infinite sum of the right hand side of (57) as follows,

$$F(\{\bar{X}^{k(1)}\}, Q^{(1)}, \{Z^{k(1)} - \bar{X}^{k(1)}\}) \geq \tag{82}$$

$$\geq \sum_{t=1}^{\infty} \left[\sum_{k=1}^{l} \left(\left(\frac{\rho - \lambda}{2} - \frac{L^2}{\rho} \right) ||\bar{X}^{k(t+1)} - \bar{X}^{k(t)}||_F^2 + \frac{\rho}{2} ||Q^{(t+1)} - Q^{(t)}||_F^2 . \right) \right] \geq 0. \tag{83}$$

This then implies,

$$\lim_{t \to \infty} ||\bar{X}^{k(t+1)} - \bar{X}^{k(t)}||_F^2 = 0, \tag{84}$$

$$\lim_{t \to \infty} ||Q^{(t+1)} - Q^{(t)}||_F^2 = 0. \tag{85}$$

To show that $\left\{Z^{(t)}\right\}_{t=1}^{\infty}$ is also convergent we use (55) to write,

$$||Z^{k(t+1)} - Z^{k(t)}||_F^2 = ||\frac{1}{\rho} \left(\nabla_X \bar{g}_k(\bar{X}^{k(t+1)}) - \nabla_X \bar{g}_k(\bar{X}^{k(t)}) \right) + \left(\bar{X}^{k(t+1)} - \bar{X}^{k(t)} \right) ||_F^2 \leq \tag{86}$$

$$\frac{1}{\rho^2} ||\nabla_X \bar{g}_k(\bar{X}^{k(t+1)}) - \nabla_X \bar{g}_k(\bar{X}^{k(t)})||_F^2 + ||\bar{X}^{k(t+1)} - \bar{X}^{k(t)}||_F^2 \leq (\frac{L^2}{\rho^2} + 1) ||\bar{X}^{k(t+1)} - \bar{X}^{k(t)}||_F^2. \tag{87}$$

Letting $t \to \infty$ and using (84) we obtain

$$\lim_{t \to \infty} ||Z^{k(t+1)} - Z^{k(t)}||_F^2 = 0, \tag{88}$$

□

Theorem A.2. *With conditions as in Lemma A.2 for all t. Let $(P^{(*)}, X^{(*)})$ be a limit point of the sequence $\{(P^{(t)}, X^{(t)})\}$ generated by Algorithm 1. If $\rho > \rho_{\min}$ then $(P^{(*)}, X^{(*)})$ will be a local minima of (1).*

Proof. From (50) we have that

$$||Z^{k(t+1)} - Z^{k(t)}||_F^2 = ||Q^{(t+1)} - \bar{X}^{k(t)}||_F^2, \; \forall k. \tag{89}$$

Thus $\lim_{t\to\infty} \bar{X}^{k(t)} = Q^{(*)}$ and we have that $X^{(*)} = \bar{X}^{k(*)} = Q^{(*)}$,$\forall k$. Inserting this in (55), using (50) and (54), yields

$$0 = \nabla_X \bar{g}_k(X^{(*)}) + \rho \left(X^{(*)} - Z^{k(*)} \right) = \nabla_X \bar{g}_k(X^{(*)}) + \rho \left(X^{(*)} - (Q^{(*)} + Z^{k(*)} - X^{(*)}) \right) = \tag{90}$$

$$= \nabla_X \bar{g}_k(X^{(*)}) + \rho \left(X^{(*)} - (2X^{k(*)} - Z^{k(*)} + Z^{k(*)} - X^{(*)}) \right) = \nabla_X \bar{g}_k(X^{(*)}). \tag{91}$$

Finally, from (39) and (38) we have that

$$\nabla_X f(P^{(*)}, X^{(*)}) = \sum_{k=1}^{l} \nabla_X g_k(P^{(*)}, X^{(*)}) = \sum_{k=1}^{l} \nabla_X g_k(P(X^{(*)}), X^{(*)}) = \sum_{k=1}^{l} \nabla_X \bar{g}_k(X^{(*)}) = 0, \tag{92}$$

$$\nabla_P f(P^{(*)}, X^{(*)}) = \sum_{k=1}^{l} \nabla_P g_k(P(X^{(*)}), X^{(*)}) = 0. \tag{93}$$

Identifying (92) and (93) as the necessary conditions for local optimality of (1) completes the proof. □

Finally, a brief comment on certain critical configurations related to solving (29). The above results are based on the existence of $\frac{\partial P_k}{\partial X}$ for all $\{\bar{X}^{(t)}\}_{t=1}^{\infty}$. However, there are certain configurations $X \in \mathbb{R}^{3\times n}$ for which (29) does not have a unique solution and hence P_k is not guaranteed to have partial derivatives. Collapsing all the n entries in X to a single point is one example of such a configuration.

Here we handle such instances by adding a small regularizing term to (29) when an iterate $X^{(t)}$ does not permit a unique solution,

$$P_k^{\epsilon}(X) = \underset{P \in \mathcal{Q}^{|c_k|}}{\arg\min} \sum_{\substack{i \in c_k \\ 1 \le j \le n}} w_{ij} ||u_{ij} - \pi(P_i, X_j)||^2 + \epsilon ||P_i||_F^2. \tag{94}$$

With $\epsilon > 0$ it can be shown ([1]) that $\frac{\partial P_k^{\epsilon}}{\partial X}$ then exist for all $\bar{X}^{(t)}$ and bounded scene depths. By deriving the equivalent of theorems A.1 and A.2 for this reformulation and modifying algorithm 1 accordingly it holds that the function values of F will still decrease, even if such critical configurations are encountered. By our initial assumption on the uniqueness of local minimizers it can then be shown that, with ϵ sufficiently small, critical configurations will never occur after a finite number of iterations. Consequently $\frac{\partial P_k}{\partial X}$ will exist for all subsequent iterations and theorem 4.1 can then be applied directly.

References

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[2] C. Olsson, F. Kahl, and R. Hartley. Projective least-squares: Global solutions with local optimization. In *2009 IEEE Conference on Computer Vision and Pattern Recognition*, pages 1216–1223. IEEE, June 2009. 2