Consistency of silhouettes and their duals

Supplementary material

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Contents

This supplementary material clarifies some technical aspects of convex duality in the projective setting, and completes two proofs that were only sketched in the main body of the paper.

1. Convex cones and polarity

Before discussing convexity in projective spaces we need to recall some basic definitions about *convex cones*. More details can be found in [1, 2]

A set C in a real vector space V is a *cone* if $x \in C$ implies $\lambda x \in C$ for all $\lambda \geq 0$. A cone C is *pointed* if it contains no line, *i.e.*, if $x \in C$ and $-x \in C$ then $x = 0$. Finally, a cone C is *solid* if it has a non-empty interior.

For any cone $C \subseteq V$, we can define the *polar cone* $C^\circ \subseteq V^*$ (we denote by V^* the space of linear functionals on V):

$$C^\circ = \{\varphi \in V^* \,|\, \phi(x) \leq 0, \ \ \forall x \in C\} \tag{1}$$

We should note that many authors only consider $V = \mathbb{R}^N$ and identify $\mathbb{R}^N$ and $(\mathbb{R}^N)^*$ by using the standard scalar product; for our purposes it is useful to keep the two spaces distinct.

For any cone C, the polar cone C° is closed and convex. Moreover, the following properties hold (see [1]):

- If C is solid, then C° is pointed.
- If the closure of C is pointed, then C° is solid.
- $C^{\circ\circ}$ is the closure of the convex hull of C. In particular, if C is convex and closed, $C^{\circ\circ} = C$.

2. Convex sets in projective space

Let us first recall that an *affine chart* in projective space $\mathbb{P}^n$ is simply the complement of any hyperplane $A = \mathbb{P}^n \setminus H$: in practice, A may be seen as a copy of affine space in $\mathbb{P}^n$. In fact, any affine chart can described as $A = \{[x] \in \mathbb{P}^n \,|\, \varphi(x) = 1\}$ for some $\varphi \in (\mathbb{R}^{n+1})^*$ (this is the complement of the set $H = \{[x] \,|\, \varphi(x) = 0\}$), so this provides an identification with the affine hyperplane $\{x \in \mathbb{R}^{n+1} \,|\, \varphi(x) = 1\}$ in $\mathbb{R}^{n+1}$. In general, it will be convenient to say that a set $T \subseteq \mathbb{P}^n$ is *finite* if its closure is contained in some affine chart (*i.e.*, if the closure of T does not intersect all hyperplanes).

For any set $S \subseteq \mathbb{R}^{N+1}$, we can consider the projectivization $\mathbb{P}S = \{[x] \in \mathbb{P}^n \,|\, x \in S\} \subseteq \mathbb{P}^n$, consisting of all classes of vectors up to scale that contain elements of S. Conversely, if $T \subseteq \mathbb{P}^n$ is *connected* and *finite*, there are exactly two *pointed* cones C_1, C_2 such that $\mathbb{P}C_i = T$, and these are such that $C_1 = -C_2$. We will say that C_i is a cone *over* T. We now give the following definition:

Definition 1. *A closed finite set $K \subseteq \mathbb{P}^n$ is* convex *if there exists a closed pointed convex cone $C \subseteq \mathbb{R}^{n+1}$ such that $K = \mathbb{P}C$.*

As stated in the paper, the following fact holds.

Proposition 1. *A finite set $K \subseteq \mathbb{P}^n$ is convex if and only if there exists a affine chart where K is compact and convex in the affine sense.*

Proof. If C is a pointed convex cone in $\mathbb{R}^{n+1}$, there exists $\varphi \in (\mathbb{R}^{n+1})^*$ such that $\varphi(x) < 0$ for all $x \in C \setminus \{0\}$. Thus, if $K = \mathbb{P}C$, K is contained in the affine chart $\{[x] \in \mathbb{P}^n \,|\, \varphi(x) = 1\}$, and in this chart it is compact and convex as an affine set. Conversely, if K is compact and convex in $\{[x] \in \mathbb{P}^n \,|\, \varphi(x) = 1\}$, then $C = \{x \in \mathbb{R}^{n+1} \,|\, [x] \in K, \varphi(x) \geq 0\}$ is a closed pointed convex cone in $\mathbb{R}^{n+1}$ such that $\mathbb{P}C = K$. $\square$

For any finite and connected set $T \subseteq \mathbb{P}^n$, we can define the *convex hull* in $\mathbb{P}^n$ of T as $\mathbb{P}Conv(C)$ where $C \subseteq \mathbb{R}^{n+1}$ is one of the two pointed cones over T (the projectivization makes the choice of C irrelevant).

Finally, we can define the *dual* $T^\circ \subseteq (\mathbb{P}^n)^*$ of a finite set $T \subseteq \mathbb{P}^n$ as $\mathbb{P}C^\circ$ for a pointed cone over C over S (again, the choice is irrelevant). Note that since Definition 1 requires a convex set to be the projectivization of *pointed* convex cones, it is possible that the dual of a convex set $K \subseteq \mathbb{P}^n$

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may not be convex: indeed, if $C \subseteq \mathbb{R}^{n+1}$ is a convex cone with empty interior, then C° is not pointed.

The next result shows that the convex dual of a set $T \subseteq \mathbb{P}^n$ coincides with the closure of the hyperplanes that do not meet T in $\mathbb{P}^n$:

Proposition 2. *Let $T \subseteq \mathbb{P}^n$ be a finite connected set and let C be a pointed cone over T. A hyperplane $[\varphi] \in (\mathbb{P}^n)^*$ belongs T° if and only if $\varphi(x) \geq 0$ for all $x \in C$ or $\varphi(x) \leq 0$ for all $x \in C$.*

Proof. By definition $T^\circ = \mathbb{P}C^\circ$ and $C^\circ = \{\varphi \in (\mathbb{R}^n)^* \mid \varphi(x) \leq 0, \forall x \in C\}$: it is clear now that if $[\varphi] \in \mathbb{P}C^\circ$ then necessarily $\varphi \in C^\circ$ or $-\varphi \in C^\circ$. $\square$

3. Proofs

We now use polarity for convex cones to revisit the proofs from Section 3.2 of the paper. For any finite connected set $S \subseteq \mathbb{P}^n$, we denote by $\hat{S}$ an arbitrary pointed cone over S. We also indicate with $\hat{\mathcal{M}}$ a linear projection map $\hat{\mathcal{M}} : \mathbb{R}^4 \to \mathbb{R}^3$ associated with a perspective projection $\mathcal{M}$ on $\mathbb{P}^3$.

Proposition 3. *Let $\mathcal{M}$ be a perspective projection with center $\boldsymbol{c}$, and let $K \subseteq \mathbb{P}^3 \setminus \{\boldsymbol{c}\}$ be a convex set. Then $\mathcal{M}(K) = L$ is equivalent to*

$$\mathcal{M}^*(L^\circ) = K^\circ \cap \boldsymbol{c}^*. \tag{2}$$

Proof. It is enough to show the analogous property for cones in $\mathbb{R}^4$, namely that $\hat{\mathcal{M}}(\hat{K}) = \hat{L}$ is equivalent to $\hat{\mathcal{M}}^*(\hat{L}^\circ) = \hat{K}^\circ \cap \hat{\boldsymbol{c}}^\perp$, where $\hat{\mathcal{M}}^* : (\mathbb{R}^3)^* \to (\mathbb{R}^4)^*$ is the dual map associated with $\hat{\mathcal{M}}$ (*i.e.*, the map $\varphi \mapsto \varphi \circ \hat{\mathcal{M}}$ for $\varphi \in (\mathbb{R}^3)^*$), and $\hat{\boldsymbol{c}}^\perp$ denotes the hyperplane in $(\mathbb{R}^4)^*$ of all linear functionals that vanish on the null-space of $\hat{\mathcal{M}}$. Indeed, we have that:

$$\begin{aligned}
&\hat{L} = \{\hat{\mathcal{M}}(x) \mid x \in \hat{K}\} \\
&\Leftrightarrow \hat{L}^\circ = \{\varphi \in (\mathbb{R}^3)^* \mid \varphi(\hat{\mathcal{M}}(x)) \leq 0, \ \ \forall x \in \hat{K}\} \\
&\Leftrightarrow \hat{L}^\circ = \{\psi \in (\mathbb{R}^4)^* \mid \psi(x) \leq 0, \ \ \forall x \in \hat{K}\} \cap Im(\hat{\mathcal{M}}^*) \\
&\Leftrightarrow \hat{L}^\circ = \hat{K}^\circ \cap \hat{\boldsymbol{c}}^\perp.
\end{aligned} \tag{3}$$

where we used the basic fact that $Im(\hat{\mathcal{M}}^*) = \hat{\boldsymbol{c}}^\perp$. $\square$

Proposition 4. *A family $L_1, \ldots, L_n$ of convex sets in $\mathbb{P}^2$ is consistent for a set of projections $\mathcal{M}_1, \ldots, \mathcal{M}_n$ if and only if $L_1^\circ, \ldots, L_n^\circ$ are sectionally consistent for the embeddings $\mathcal{M}_1^*, \ldots, \mathcal{M}_n^*$. Moreover, if consistency holds, and H is the* visual hull *associated with $L_1, \ldots, L_n$, then $H = K^\circ$, where K is the* convex hull *of $\mathcal{M}_1^*(L_1^\circ), \ldots, \mathcal{M}_n^*(L_n^\circ)$.*

Proof. The only thing that remains to be shown is that $(\mathcal{M}_i^*(L_i^\circ))^\circ$ is in fact the visual cone C_i. This prove this, we use the following relations for cones in $\mathbb{R}^4$:

$$\begin{aligned}
(\hat{\mathcal{M}}_i^*(\hat{L}_i^\circ))^\circ &= \{x \in \mathbb{R}^4 \mid \psi(x) \leq 0 \ \forall \psi \in \hat{\mathcal{M}}_i^*(\hat{L}_i^\circ)\} \\
&= \{x \in \mathbb{R}^4 \mid \varphi(\hat{\mathcal{M}}(x)) \leq 0 \ \forall \varphi \in \hat{L}_i^\circ\} \\
&= \{x \in \mathbb{R}^4 \mid \hat{\mathcal{M}}(x) \in \hat{L}_i\}.
\end{aligned} \tag{4}$$

Note that while $\{x \in \mathbb{R}^4 \mid \hat{\mathcal{M}}(x) \in \hat{L}_i\}$ is a convex cone in $\mathbb{R}^4$, its projectivization coincides with the two sided projective cone C_i, which is *not* convex for Definition 1 (indeed, the set $\mathcal{M}_i^*(L_i^\circ)$ has empty interior). $\square$

References

[1] S. Boyd and L. Vandenberghe. *Convex optimization*. Cambridge university press, 2004. 1

[2] R. Schneider. *Convex bodies: the Brunn–Minkowski theory*. Number 151. Cambridge University Press, 2013. 1