Convex Global 3D Registration with Lagrangian Duality

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Abstract

The registration of 3D models by a Euclidean transformation is a fundamental task at the core of many application in computer vision. This problem is non-convex due to the presence of rotational constraints, making traditional local optimization methods prone to getting stuck in local minima. This paper addresses finding the globally optimal transformation in various 3D registration problems by a unified formulation that integrates common geometric registration modalities (namely point-to-point*,* point-to-line *and* point-to-plane*). This formulation renders the optimization problem independent of both the number and nature of the correspondences.*

The main novelty of our proposal is the introduction of a strengthened *Lagrangian dual relaxation for this problem, which surpasses previous similar approaches [32] in effectiveness. In fact, even though with* no theoretical guarantees*, exhaustive empirical evaluation in both synthetic and real experiments* always *resulted on a tight relaxation that allowed to recover a guaranteed globally optimal solution by exploiting duality theory.*

Thus, our approach allows for effectively solving the 3D registration with global optimality guarantees while running at a fraction of the time for the state-of-the-art alternative [34], based on a more computationally intensive Branch and Bound method.

1. Introduction

The problem of registering 3D geometric data is a classical problem in numerous fields, including computer vision, robotics, photogrammetry or medical imaging [23, 12, 37]. It seeks the transformation that brings closest together different surfaces according to some *meaningful distance function*. Consider the ubiquitous scenario in which a system (usually a sensor) returns 3D points $\{\boldsymbol{x}_i\}_{i=1}^m$ of an object and a model of the same is available consisting of 3D primitives $\{P_i\}$ (typically points, lines and/or planes) [3, 14, 47, 28]. Assuming the correspondences between the sets are given, $\boldsymbol{x}_i \leftrightarrow P_i$, the general problem reduces to finding the *optimal* roto-translation $\boldsymbol{T} = (\boldsymbol{R}, \boldsymbol{t}) \in \mathrm{SE}(3)$ as

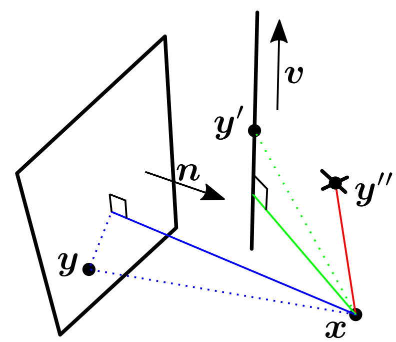


Figure 1. This paper presents a unified formulation for the 3D registration with *point-to-plane*, *point-to-line* and *point-to-point* correspondences, and then provides a certifiable globally optimal solution using Lagrangian duality.

$$\boldsymbol{T}^\star = \underset{\boldsymbol{T}\in\mathrm{SE}(3)}{\arg\min} \sum_{i=1}^{m} \mathrm{d}_{P_i}(\boldsymbol{T} \oplus \boldsymbol{x}_i)^2. \quad (1)$$

Here $\mathrm{SE}(3)$ stands for the usual Special Euclidean group in 3D, $\boldsymbol{T} \oplus \boldsymbol{x}_i$ denotes the Euclidean transformation of the point $\boldsymbol{x}_i$ and $\mathrm{d}_{P_i}(\cdot)$ is the distance to the primitive P_i [34]. Even with known correspondences, the registration problem (1) is tough to solve in a global fashion due to the non-convexity of the constraints in the rotation $\boldsymbol{R} \in \mathrm{SO}(3)$. A closed-form solution exists only if all the correspondences are point-to-point, as given by Horn *et al*. [24]. For the other cases, most pipelines resort to local approximations, with the inherent risk of getting stuck in local minima [33, 34].

In this paper we address the *global optimization* of the general 3D registration problem (1). After providing a thorough overview of potential alternatives for this task in Section 2, we present in Section 3 a unified formulation integrating *point-to-point*, *point-to-line* and *point-to-plane* correspondences into a single quadratic objective that is a function of the rotation $\boldsymbol{R}$ only. Our main contribution is the development of a novel convex relaxation for this formulation under the usual framework of the Lagrangian dual problem, which results on a small constant size *semidefinite program* (SDP). This novel relaxation, fully characterized in Section 4, turned tight for *all* the evaluations performed on synthetic and real data, under an extensive range of different problem conditions (Section 5). This allowed us to solve the original non-convex problem in a global fashion using its connection to the convex relaxation through duality theory [4].

In summary, the proposed algorithm Alg. 1 provides an iterative optimization framework for 3D registration problems that, unlike current local iterative approaches, is *able to* provide a certifiable global solution and, in fact, our *empirical* observation is that it always does so.

We remark that this global performance comes without any theoretical guarantees so far. Whereas we acknowledge the importance of the missing formal proof that justifies this behavior, we hope the empirical performance of the proposed relaxation is motivation enough to encourage further exploitation of this approach.

2. Related work

As a result of its widespread interest, geometric registration has been object of extensive research for decades now. The number of works aiming at guaranteing global optimality in registration problems is much smaller, though, and focuses mainly on the case of *point* cloud registration with unknown correspondences [27, 35, 50, 52].

Multimodal registration

Registration across diverse 3D geometric primitives arises in a variety of problems and applications. In the Iterative Closest Point (ICP) framework, since the original proposal using point-to-point correspondences by Besl and McKay [3], different types of correspondences have been introduced for improved performance: *e.g.* point-to-plane [14] or plane-to-plane [45, 46]. Multimodal correspondences play also an important role in the extrinsic calibration of sensors of different nature, such as camera and lidar [51, 49, 16, 18]. Numerous state-of-the-art SLAM (Simultaneous Localization and Mapping) solutions work with models that include plane and line primitives too [2, 47, 17, 43, 28, 19].

In the context of multimodal registration a significant effort has been devoted for decades to the resolution of minimal problems [26], *e.g.* for line-to-plane correspondences [14, 49, 6, 5] and point-to-plane correspondences [21, 39, 38]. These solutions find application mainly in random sample and consensus (RANSAC) frameworks, but they do not face the more general least squares problem (1). Instead, this optimization problem is traditionally addressed through local linear approximations on the rotation, assuming an initialization close to the solution is given [14], which can easily lead to suboptimal local solutions [33].

Global optimization

Next we review the main approaches available in the literature for global optimization [22] and connect them to the registration problem (1) at hand.

Analytical solutions A classical approach involves the computation of all the stationary points (among which there is the global minimum). This approach is used by Censi [13] for the global resolution of the *2D* registration problem, reducing the problem to solving a 4-th order polynomial equation. However, this approach does not generalize well to the 3D case due to the higher complexity of the rotation space, which produces an explosion in the complexity of the resulting polynomial system [29, 30].

Branch and Bound A commonly used tool for NP-hard optimization problems is Branch and Bound (BnB). This is used by Olsson *et al*. [33, 34] for solving the same problem addressed in this paper, yielding a provably global solution. However, the resolution time is notably high (in the order of seconds) due to the exploratory nature and exponential worst-case performance of BnB.

Convex relaxations Convex relaxation techniques consider approximate, simpler versions of the problem whose global optimum is much easier to reach. If the approximation is good, the solution of the relaxed problem may then provide valuable information about the original problem. Thus, the main task is to find an appropriate relaxation. For problems such as ours (1), which are affected by the non-convexity of rotation constraints, a possible relaxation is the search into the *convex* hull of $\mathrm{SO}(3)$. This has proved to give good approximations in rotation synchronization [44], SLAM [41] and 2D/3D registration [25].

Another generic (and successful) tool for providing relaxations of difficult constrained problems is the Lagrangian dual relaxation [4]. This provides particularly good approximations for many problems that can be reformulated as a Quadratically Constrained Quadratic Program (QCQP), where the relaxed problem becomes a Semidefinite Program (SDP) [4, 15]. Some problems involving rotations can be characterized as QCQPs, such as Pose Graph Optimization, for which recent literature applying the Lagrangian dual relaxation has shown impressive results finding globally optimal solutions based solely on convex relaxations [11, 10, 7, 40, 8].

The Lagrangian dual relaxation has been applied before to the QCQP formulation of 3D registration [32]. In this case the approximation can be very good and even provides the global solution in a certain range of problems, but it deteriorates when the noise level increases or the number of correspondences approaches the minimal cases, that is, when the problem becomes inherently more difficult [33].

Still in the context of the Lagrangian dual relaxation, the relaxation can be *strengthened* (improved) by introducing additional valid constraints [31, Chap. 13]. This has found applicability in the optimization literature [36, 42], and in QCQP problems involving orthonormal constraints, improving the obtained relaxation considerably [1].

To the best of our knowledge, this "trick" has not been considered before for relaxations involving rotations. In this paper, we extend the QCQP formulation of [32] with a whole set of valid quadratic rotational constraints to achieve an improved relaxation that (empirically) returned a globally optimal solution *in all problem instances*, regardless of the noise level or the number of correspondences.

3. Formulation of multimodal registration

In this section we revisit the formulation of the 3D registration problem (1), rewriting it in a suitable form for later applying the Lagrangian dual relaxation.

3.1. Generalized distance function

First, we provide a unified formulation of the distances for the *point-to-point*, *point-to-line* and *point-to-plane* correspondences on the basis of the results presented in [34].

In the registration problem (1), the square distance from a 3D point $\boldsymbol{x}$ to a 3D primitive P minimized in the registration problem (1) is typically that to the closest point in P, defined by

$$\mathrm{d}_P^2(\boldsymbol{x}) = \min_{\boldsymbol{y}' \in P} \|\boldsymbol{x} - \boldsymbol{y}'\|_2^2. \tag{2}$$

For all the primitives considered here (see Fig. 1) the closest distance problem (2) has a simple closed-form solution that fits a generalized distance function of the form

$$\mathrm{d}_P(\boldsymbol{x})^2 = \|\boldsymbol{x} - \boldsymbol{y}\|_{\boldsymbol{C}}^2 = (\boldsymbol{x} - \boldsymbol{y})^\top \boldsymbol{C} (\boldsymbol{x} - \boldsymbol{y}), \tag{3}$$

namely

$$\begin{aligned}
&\min_{\boldsymbol{y}' \in P} \|\boldsymbol{x} - \boldsymbol{y}'\|_2^2 = && (4)\\
&\quad = \|\boldsymbol{x} - \boldsymbol{y}\|_2^2 = \|\boldsymbol{x} - \boldsymbol{y}\|_{\boldsymbol{I}_3}^2, && \text{(point)}\\
&\quad = \|(\boldsymbol{I} - \boldsymbol{v}\boldsymbol{v}^\top)(\boldsymbol{x} - \boldsymbol{y})\|_2^2 = \|\boldsymbol{x} - \boldsymbol{y}\|_{(\boldsymbol{I} - \boldsymbol{v}\boldsymbol{v}^\top)}^2, && \text{(line)}\\
&\quad = (\boldsymbol{n}^\top (\boldsymbol{x} - \boldsymbol{y}))^2 = \|\boldsymbol{x} - \boldsymbol{y}\|_{\boldsymbol{n}\boldsymbol{n}^\top}^2. && \text{(plane)}
\end{aligned}$$

Here $\boldsymbol{y} \in \mathbb{R}^3$ is any point lying in the primitive[1], $\boldsymbol{v}$ is the unit direction vector for a line, $\boldsymbol{n}_i$ is the unit normal vector for a plane, and $\boldsymbol{C} \in \mathbb{S}^3$ is a symmetric matrix whose expression depends on the primitive as reflected by (4). These results stem from applying elementary algebra to each primitive [34], but we provide the full proof in the **supplementary material** for completeness.

[1] For the *point-to-point* case $\boldsymbol{y}_i$ *is* the primitive.

3.2. Quadratic formulation and marginalization

The distances minimized in the registration problem (1) depend on the transformation $\boldsymbol{T}$,

$$\mathrm{d}_{P_i}^2(\boldsymbol{T} \oplus \boldsymbol{x}_i) = (\boldsymbol{T} \oplus \boldsymbol{x}_i - \boldsymbol{y}_i)^\top \boldsymbol{C}_i (\boldsymbol{T} \oplus \boldsymbol{x}_i - \boldsymbol{y}_i). \tag{5}$$

If a matrix representation is chosen for the rotation $\boldsymbol{R}$, the expression of the transformed point $\boldsymbol{T} \oplus \boldsymbol{x}_i$ is linear in the elements of $\boldsymbol{R}$ and $\boldsymbol{t}$:

$$\boldsymbol{T} \oplus \boldsymbol{x}_i = \boldsymbol{R}\boldsymbol{x}_i + \boldsymbol{t} = \underbrace{(\tilde{\boldsymbol{x}}^\top \otimes \boldsymbol{I}_3)}_{\boldsymbol{X}_i} \mathrm{vec}(\boldsymbol{T}), \tag{6}$$

where $\tilde{\boldsymbol{x}} = \left[\boldsymbol{x}^\top, 1\right]^\top$ refers to the homogeneized version of $\boldsymbol{x}$, $\otimes$ is the Kronecker product and $\mathrm{vec}(\boldsymbol{T})$ is the vectorization (applied column-wise) of the transformation matrix,

$$\mathrm{vec}(\boldsymbol{T}) = \begin{bmatrix} \mathrm{vec}(\boldsymbol{R}) \\ \boldsymbol{t} \end{bmatrix}. \tag{7}$$

Proof. **Supplementary material** □

With this linear parameterization of the transformed point, it is easy to see that the generalized distance to minimize is a *quadratic function* of $\boldsymbol{\tau} = \mathrm{vec}(\boldsymbol{T})$, writable as

$$\mathrm{d}_{P_i}^2(\boldsymbol{T} \oplus \boldsymbol{x}_i) = \tilde{\boldsymbol{\tau}}^\top \underbrace{\boldsymbol{N}_i^\top \boldsymbol{C}_i \boldsymbol{N}_i}_{\tilde{\boldsymbol{M}}_i} \tilde{\boldsymbol{\tau}}, \tag{8}$$

with $\boldsymbol{N}_i = \left[\tilde{\boldsymbol{x}}_i^\top \otimes \boldsymbol{I}_3 | -\boldsymbol{y}_i\right]$ and $\tilde{\boldsymbol{\tau}} = \begin{bmatrix} \mathrm{vec}(\boldsymbol{T}) \\ 1 \end{bmatrix}^\top$. Because of the quadratic nature of the cost (8) it is possible to accumulate the observations, *compressing all the data* into a single 13×13 matrix term $\tilde{\boldsymbol{M}}$:

$$f(\boldsymbol{T}) = \sum_{i=1}^{m} \mathrm{d}_{P_i}^2(\boldsymbol{T} \oplus \boldsymbol{x}_i) = \tilde{\boldsymbol{\tau}}^\top \underbrace{\left(\sum_{i=1}^{m} \tilde{\boldsymbol{M}}_i\right)}_{\tilde{\boldsymbol{M}}} \tilde{\boldsymbol{\tau}}. \tag{9}$$

Thanks to this compression step, the size of the following reformulated problem is independent of m.

When minimizing the quadratic objective $f(\boldsymbol{T})$ in (9), the problem can be further reduced if we apply marginalization on the unconstrained part of the unknown $\boldsymbol{T}$, that is, in the translation $\boldsymbol{t}$. It is well known from previous work that $\boldsymbol{t}$ can be derived in terms of $\boldsymbol{R}$ [24, 33]. In the quadratic formulation this is straightforward:

Lemma 1. *The optimal translation for a fixed $\boldsymbol{R}$ is*

$$\boldsymbol{t}^\star(\boldsymbol{R}) = -\tilde{\boldsymbol{M}}_{\boldsymbol{t},\boldsymbol{t}}^{-1} \tilde{\boldsymbol{M}}_{\boldsymbol{t},!\boldsymbol{t}}\, \tilde{\boldsymbol{r}}, \quad \tilde{\boldsymbol{r}} = \begin{bmatrix} \mathrm{vec}(\boldsymbol{R}) \\ 1 \end{bmatrix}. \tag{10}$$

Input: List of correspondences $\{\boldsymbol{x}_i \leftrightarrow P_i\}_{i=1}^m$
Output: Global optimum $\boldsymbol{T}^\star$

```
/* build compressed quadratic form */
initialize M̃ ← zeros(13);
for (x_i, P_i) ∈ {x_i ↔ P_i}_{i=1}^m do
    M̃_i ← buildTerm(x_i, P_i) ;   // depends on P_i
    M̃ ← M̃ + M̃_i;
end
/* minimize quadratic objective */
marginalize Q̃ = M̃/M̃_{t,t} ;   // Schur complement
solve optimal rotation R* using Alg. 2;
get optimal translation t* = t*(R*) from (10);
return T* = (R*, t*)
```

Algorithm 1: Globally optimal 3D registration

Here the subindex $\boldsymbol{t}$ *stands for the set of indexes corresponding to translation variables, whereas* $!\boldsymbol{t}$ *is its complement. The* marginalized *optimization problem is then*

$$f^\star = \min_{\boldsymbol{R}\in SO(3)} \underbrace{\tilde{\boldsymbol{r}}^\top \tilde{\boldsymbol{Q}} \tilde{\boldsymbol{r}}}_{q(\tilde{\boldsymbol{r}})}, \quad \tilde{\boldsymbol{r}} = \begin{bmatrix} \mathrm{vec}(\boldsymbol{R}) \\ 1 \end{bmatrix}, \tag{11}$$

where the marginalized *quadratic form* $\tilde{\boldsymbol{Q}} = \tilde{\boldsymbol{M}}/\tilde{\boldsymbol{M}}_{\boldsymbol{t},\boldsymbol{t}}$ *is the* Schur complement *of the block* $\tilde{\boldsymbol{M}}_{\boldsymbol{t},\boldsymbol{t}}$ *in the matrix* $\tilde{\boldsymbol{M}}$.

Proof. **Supplementary material** □

The registration problem is addressed then following the pipeline depicted in Algorithm 1, where the main complexity remains in solving the marginalized problem (11). This is still a non-convex optimization problem in $\boldsymbol{R}$, for which a convex relaxation is provided in the next session. This relaxation *empirically* proves to be tight in all the evaluated cases, allowing us to recover a globally optimal solution for the marginalized problem (11).

4. Tight dual relaxation

The key ingredient to solve the non-convex problem (11) lies upon an adequate application of Lagrangian duality. We will apply fundamental results from duality theory [4, 9], so some basic properties and notions regarding the Lagrangian dual problem are provided in the **supplementary material** for completeness. Then we present a specific formulation ($\tilde{\mathcal{P}}$) of the constrained problem (11) in Section 4.1 and its corresponding dual problem ($\mathcal{D}$) in Section 4.2. Finally we show how to recover the globally optimal solution in Section 4.3, provided that strong duality holds. The experiments of Section 5 show that, *empirically*, this relaxation is always tight (strong duality holds), even in extreme conditions.

Once the whole dual framework has been developed, its implementation is straightforward and the resolution of the marginalized problem (11) is done following the relatively simple pipeline depicted in Algorithm 2.

4.1. Primal problem

We will address now the task of formulating the optimization problem (11) in such a manner that the approach described above produces successful results.

The constraint $\boldsymbol{R}_i \in \mathrm{SO}(3)$ in this problem states that the 3×3 rotation matrix $\boldsymbol{R}$ fulfills the *orthonormality* and *determinant* constraints, that is,

$$\mathrm{SO}(3) \equiv \{\boldsymbol{R} \in \mathbb{R}^{3\times 3} : \boldsymbol{R}^\top \boldsymbol{R} = \boldsymbol{I}_3, \det(\boldsymbol{R}) = +1\}. \tag{12}$$

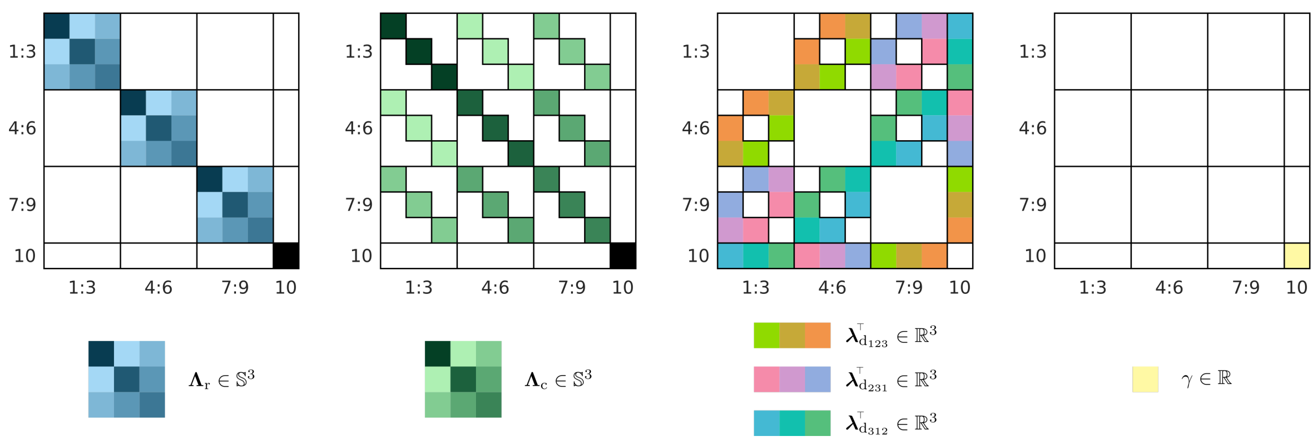
In order to apply the Lagrangian dual relaxation, it is particularly appealing to formulate the primal problem as a Quadratically Constrained Quadratic Program (QCPQ). In the usual characterization of SO(3), the orthonormality constraints are all quadratic *but* the determinant constraint is cubic. Because of this, it has been customary in other problems involving rotations to relax the constraints by dropping the determinant constraint $\det(\boldsymbol{R}) = +1$ and keeping only the orthonormality constraints $\boldsymbol{R}^\top \boldsymbol{R} = \boldsymbol{I}_3$, which amounts to performing the optimization in O(3) rather than in SO(3). This approach has provided tight relaxations for other problems [11, 10, 7, 40]. For the registration problem however it works well only in a certain range of problems [32].

Duality strengthening Let us now make a quick annotation about an important fact concerning the Lagrangian duality that will be key for the success of our proposal: By construction, every time a new scalar constraint $c_{k+1}(\cdot)$ is introduced into the Lagrangian a new dual variable λ_{k+1} appears and the domain of the dual problem increases its dimension in one. As a result, the bound $d^\star_{k+1}$ provided by the new dual problem is at least as good as that of the previous one, $d^\star_k \leq d^\star_{k+1} \leq f^\star$. As a consequece the dual problem *is not intrinsic* [36, 4]: it depends on the particular formulation of the primal problem. In particular, it depends on the specific characterization of the optimization domain or *feasible region*: Adding appropiate redundant *valid constraints* has actually shown to be remarkably effective for improving the quality of the dual relaxations in other problems [31, Chap. 13].

Following this idea, our approach is to characterize the feasible set SO(3) with the largest possible amount of *quadratic* constraints. The chosen constraints need to be linearly independent to introduce any potential improvement [36]. It is important that we keep the complexity of the constraints quadratic in order to maintain the Lagrangian dual problem simple. For the set of orthogonal matrices O(3) it has been shown [1] that a complete set of quadratic constraints is given by the combination of both column-based and row-based orthogonality constraints: $\{\boldsymbol{R}^\top \boldsymbol{R} = \boldsymbol{I}_3, \boldsymbol{R}\boldsymbol{R}^\top = \boldsymbol{I}_3\}$. However, Tron *et al*. show in [48] that the

Table 1. Table of constraints, Lagrange multipliers and penalizations for Problem ($\tilde{\mathcal{P}}$)

Constraint type	Constraint equation	Dual variable	Penalization term
Orthonormal rows	$y^2\boldsymbol{I}_3 - \boldsymbol{R}\boldsymbol{R}^\top = \boldsymbol{0}$	$\boldsymbol{\Lambda}_\mathrm{r} = \begin{bmatrix}\lambda_1 & \lambda_6 & \lambda_5\\ \lambda_6 & \lambda_2 & \lambda_4\\ \lambda_5 & \lambda_4 & \lambda_3\end{bmatrix} \in \mathbb{S}^3$	$\tilde{\boldsymbol{r}}^\top \tilde{\boldsymbol{P}}_\mathrm{r}(\boldsymbol{\Lambda}_\mathrm{r})\tilde{\boldsymbol{r}}$
Orthonormal columns	$y^2\boldsymbol{I}_3 - \boldsymbol{R}^\top\boldsymbol{R} = \boldsymbol{0}$	$\boldsymbol{\Lambda}_\mathrm{c} = \begin{bmatrix}\lambda_7 & \lambda_{12} & \lambda_{11}\\ \lambda_{12} & \lambda_8 & \lambda_{10}\\ \lambda_{11} & \lambda_{10} & \lambda_9\end{bmatrix} \in \mathbb{S}^3$	$\tilde{\boldsymbol{r}}^\top \tilde{\boldsymbol{P}}_\mathrm{c}(\boldsymbol{\Lambda}_\mathrm{c})\tilde{\boldsymbol{r}}$
Handedness	$\boldsymbol{R}^{(1)}\times\boldsymbol{R}^{(2)} - y\boldsymbol{R}^{(3)} = \boldsymbol{0}$ $\boldsymbol{R}^{(2)}\times\boldsymbol{R}^{(3)} - y\boldsymbol{R}^{(1)} = \boldsymbol{0}$ $\boldsymbol{R}^{(3)}\times\boldsymbol{R}^{(1)} - y\boldsymbol{R}^{(2)} = \boldsymbol{0}$	$\boldsymbol{\lambda}_{\mathrm{d}_{123}} = \begin{bmatrix}\lambda_{13} & \lambda_{14} & \lambda_{15}\end{bmatrix}^\top \in \mathbb{R}^3$ $\boldsymbol{\lambda}_{\mathrm{d}_{231}} = \begin{bmatrix}\lambda_{16} & \lambda_{17} & \lambda_{18}\end{bmatrix}^\top \in \mathbb{R}^3$ $\boldsymbol{\lambda}_{\mathrm{d}_{312}} = \begin{bmatrix}\lambda_{19} & \lambda_{20} & \lambda_{21}\end{bmatrix}^\top \in \mathbb{R}^3$	$\tilde{\boldsymbol{r}}^\top \tilde{\boldsymbol{P}}_{\mathrm{d}_{ijk}}(\boldsymbol{\lambda}_{\mathrm{d}_{ijk}})\tilde{\boldsymbol{r}}$
Homogeneization	$1 - y^2 = 0$	$\gamma \equiv \lambda_{22} \in \mathbb{R}$	$\gamma + \tilde{\boldsymbol{r}}^\top \tilde{\boldsymbol{P}}_\mathrm{h}(\gamma)\tilde{\boldsymbol{r}}$



(a) Orthonormal rows: $\tilde{\boldsymbol{P}}_\mathrm{r}(\boldsymbol{\Lambda}_\mathrm{r})$ (b) Orthonormal columns: $\tilde{\boldsymbol{P}}_\mathrm{c}(\boldsymbol{\Lambda}_\mathrm{c})$ (c) Handedness: $\tilde{\boldsymbol{P}}_{\mathrm{d}_{ijk}}(\boldsymbol{\lambda}_{\mathrm{d}_{ijk}})$ (d) Homogeneization: $\tilde{\boldsymbol{P}}_\mathrm{h}(\gamma)$

Figure 2. Pattern of the penalization matrices (top row) and dual variables (bottom row) for the different sets of constraints. A coloured cell indicates its value depends (linearly) only on the corresponding dual variable λ_i. Black cells in (a) and (b) stand for values involving a linear combination of several dual variables.

rotation space $\mathrm{SO}(3)$ has additional quadratic constraints due to the *handedness* property that forces the positive unit determinant: Since $\boldsymbol{R} \in \mathrm{O}(3) \Rightarrow \det(\boldsymbol{R}) = \pm 1$, provided that $\boldsymbol{R} \in \mathrm{O}(3)$ the positive sign is guaranteed if the matrix columns fulfill the well-known right-hand rule, $\boldsymbol{R}^{(1)} \times \boldsymbol{R}^{(2)} = \boldsymbol{R}^{(3)}$, where $\boldsymbol{R}^{(k)}$ is the k-th column of $\boldsymbol{R}$. Taking any of the three possible cyclic permutations of the column indexes for the right-hand rule provides exactly three independent quadratic constraints. Altogether we end up with $2\cdot 6 + 3\cdot 3 = 21$ scalar rotational constraints: 6 for each *symmetric* matrix constraint from orthonormality, and 3 for each vector constraint from the handedness constraint.

Problem homogeneization The optimization objective in (11) as well as the gathered constraints are all quadratic functions but in general not homogeneous. It is very convenient for simplifying the derivation of the dual problem to *homogeneize* the problem by introducing an auxiliary variable y with the constraint $y^2 = 1$ [11, 48]. We define the *equivalent*, *homogeneous*, *strengthened* primal problem:

$$\begin{aligned} \min_{\boldsymbol{R}}\ & \tilde{\boldsymbol{r}}^\top \tilde{\boldsymbol{Q}} \tilde{\boldsymbol{r}}, \quad \tilde{\boldsymbol{r}} = \begin{bmatrix}\mathrm{vec}(\boldsymbol{R})\\ y\end{bmatrix} \\ \text{s.t. }& \boldsymbol{R}^\top\boldsymbol{R} = y^2\boldsymbol{I}_3, \\ & \boldsymbol{R}\boldsymbol{R}^\top = y^2\boldsymbol{I}_3, \\ & \boldsymbol{R}^{(i)}\times\boldsymbol{R}^{(j)} = y\boldsymbol{R}^{(k)},\ i,j,k = \mathrm{cyclic}(1,2,3) \\ & y^2 = 1. \end{aligned} \qquad (\tilde{\mathcal{P}})$$

4.2. Dual problem

Once the primal problem has been clearly defined, the derivation of the dual problem is a mechanical work, basically reduced to the derivation of the penalization term corresponding to each constraint.

The primal problem ($\tilde{\mathcal{P}}$) is a QCQP, so the Lagrangian is

$$\mathrm{L}(\tilde{\boldsymbol{r}}, \tilde{\boldsymbol{\lambda}}) = \gamma + \tilde{\boldsymbol{r}}^\top \underbrace{(\tilde{\boldsymbol{Q}} + \tilde{\boldsymbol{P}}(\tilde{\boldsymbol{\lambda}}))}_{\tilde{\boldsymbol{Z}}} \tilde{\boldsymbol{r}}, \tag{13}$$

where the "homogeneous" dual vector $\tilde{\boldsymbol{\lambda}} = \left[\boldsymbol{\lambda}^\top, \gamma\right]^\top \in \mathbb{R}^{22}$ gathers the dual variables $\boldsymbol{\lambda}$ corresponding to all the rotation constraints (altogether 21) and the dual variable γ from the homogeneization constraint $y^2 = 1$, shown in Tab. 4. The *penalized matrix* $\tilde{\boldsymbol{Z}}$ is the sum of two terms: $\tilde{\boldsymbol{Q}}$ that contains all the data from the original problem, and $\tilde{\boldsymbol{P}}(\tilde{\boldsymbol{\lambda}})$ that accumulates all the penalization terms corresponding to the different kinds of constraints:

$$\tilde{\boldsymbol{P}}(\tilde{\boldsymbol{\lambda}}) = \tilde{\boldsymbol{P}}_{\mathrm{r}}(\boldsymbol{\Lambda}_{\mathrm{r}}) + \tilde{\boldsymbol{P}}_{\mathrm{c}}(\boldsymbol{\Lambda}_{\mathrm{c}}) + \tilde{\boldsymbol{P}}_{d}(\{\boldsymbol{\lambda}_{\mathrm{d}_{ijk}}\}) + \tilde{\boldsymbol{P}}_{\mathrm{h}}(\gamma).$$

This matrix is (by definition) a linear function of the dual variables, and the pattern of the different matrix components can be seen in Fig. 2. A detailed overview of the construction and formulae for $\tilde{\boldsymbol{P}}(\tilde{\boldsymbol{\lambda}})$ is available in the accompanying **supplementary material**.

With this particularly simple expression for the Lagrangian function, the Lagrangian relaxation is an unconstrained problem which can be solved in closed-form as

$$d(\tilde{\boldsymbol{\lambda}}) = \min_{\tilde{\boldsymbol{r}}} \mathrm{L}(\tilde{\boldsymbol{r}}, \tilde{\boldsymbol{\lambda}}) = \min_{\tilde{\boldsymbol{r}}} \gamma + \tilde{\boldsymbol{r}}^\top \tilde{\boldsymbol{Z}} \tilde{\boldsymbol{r}} \tag{14}$$

$$= \begin{cases} \gamma & \text{if } \tilde{\boldsymbol{Z}} \succcurlyeq \mathbf{0}, \\ -\infty & \text{otherwise.} \end{cases} \tag{15}$$

The Lagrangian relaxation is unbounded below unless the penalized matrix $\tilde{\boldsymbol{Z}}$ is *positive semidefinite* (PSD). As a result, the *maximization* of the dual objective $d(\tilde{\boldsymbol{\lambda}})$ can be safely restricted to those vectors $\tilde{\boldsymbol{\lambda}}$ preserving the positive semidefiniteness of $\tilde{\boldsymbol{Z}}$. Thus, the *dual problem* corresponding to the *homogeneous* primal problem ($\tilde{\mathcal{P}}$) is a *Semidefinite Program* (SDP):

$$d^\star = \max_{\tilde{\boldsymbol{\lambda}}} \gamma, \quad \text{s.t. } \tilde{\boldsymbol{Z}}(\tilde{\boldsymbol{\lambda}}) = \tilde{\boldsymbol{Q}} + \tilde{\boldsymbol{P}}(\tilde{\boldsymbol{\lambda}}) \succcurlyeq \mathbf{0}. \tag{$\mathcal{D}$}$$

This problem is *convex* and off-the-shelf specialized solvers exist for it [20].

4.3. Primal-via-dual resolution

In this section we begin by assuming that the duality gap for our primal-dual pair is zero (we will see in the experiments that this assumption always holds in practice). By duality theory [4], $\tilde{\boldsymbol{r}}^\star$ must be a minimizer of the Lagrangian (13) evaluated at $\tilde{\boldsymbol{\lambda}}^\star$,

$$\boldsymbol{x}^\star = \arg\min_{\boldsymbol{x}} \mathrm{L}(\boldsymbol{x}, \boldsymbol{\lambda}^\star) \Rightarrow (\tilde{\boldsymbol{r}}^\star)^\top \tilde{\boldsymbol{Z}}^\star \tilde{\boldsymbol{r}}^\star = 0. \tag{16}$$

Since $\boldsymbol{Z}^\star \succcurlyeq \mathbf{0}$, this means that the primal optimum $\tilde{\boldsymbol{r}}^\star$ must lie in the nullspace of $\tilde{\boldsymbol{Z}}^\star$:

$$\boldsymbol{x}^\star = \arg\min_{\boldsymbol{x}} \mathrm{L}(\boldsymbol{x}, \boldsymbol{\lambda}^\star) \Rightarrow \tilde{\boldsymbol{r}}^\star \in \mathrm{null}(\tilde{\boldsymbol{Z}}^\star). \tag{17}$$

Input: Marginalized quadratic form $q(\tilde{\boldsymbol{r}})$ (11)
Output: Global optimum $\boldsymbol{R}^\star$ (if no error occurs)
build symbolic penalized matrix $\tilde{\boldsymbol{Z}}(\tilde{\boldsymbol{\lambda}}) \leftarrow \tilde{\boldsymbol{Q}} + \tilde{\boldsymbol{P}}(\tilde{\boldsymbol{\lambda}})$;
solve the SDP problem ($\mathcal{D}$) to get $\tilde{\boldsymbol{\lambda}}^\star$ and $\tilde{\boldsymbol{Z}}^\star$; `// CVX`
get nullspace $\boldsymbol{V} = \mathrm{null}(\tilde{\boldsymbol{Z}}^\star)$;
ASSERT($\mathrm{rank}(\boldsymbol{V}) == 1$);
dehomogeneize $\tilde{\boldsymbol{r}}^\star \leftarrow \boldsymbol{V}/\boldsymbol{V}(end)$;
ASSERT($q(\tilde{\boldsymbol{r}}^\star) - d^\star \approx 0$) ; `// strong duality`
$\boldsymbol{R}^\star \hookleftarrow \tilde{\boldsymbol{r}}^\star$; `// reshape`
return $\boldsymbol{R}$

Algorithm 2: Convex global optimizer

If the nullspace has rank 1, the solution $\tilde{\boldsymbol{r}}^\star$ is recovered up to a scale factor. Then, since the solution must also fulfill the original constraints in the primal problem ($\tilde{\mathcal{P}}$), we fully determine the solution by setting $y = 1$, which in practice reduces to *dehomogeneizing* the solution $\tilde{\boldsymbol{r}}^\star$.

Then with the obtained primal solution $\boldsymbol{r}^\star$ we can check that our initial assumption holds and the duality gap is effectively zero, $d^\star = f(\boldsymbol{R}^\star)$. As we will see in the experiments, both conditions $\mathrm{rank}(\mathrm{null}(\tilde{\boldsymbol{Z}}^\star)) = 1$ and $d^\star = f^\star$ were fulfilled for absolutely all the experiments considered, even under the most extreme situations (in terms of noise and number of correspondences).

5. Experiments

In this section we show that in practice the *strong duality assumption* holds in our primal-dual formulation for any considered problem. As a result, using Algorithm 1 it is always possible to recover the primal optimal solution, and we do this at a fraction of the time necessary for the more complex exploratory techniques. The SDP ($\mathcal{D}$) is solved using CVX [20].

We assess the performance of our method, `Ours`, in both synthetic and real data and compare it to that of two different state-of-the-art approaches for solving the 3D registration problem globally: a provably optimal exploratory approach based on Branch and Bound, `BnB` [34], and a dual-based approach which provides a relaxation, referred to as `Olsson` [32].

But first of all, we define the main parameters that will characterize our general registration problem (1) as well as the metrics used in the assessment of performance.

***Effective* number of correspondences** The number of measurements has a notable impact in the complexity of the registration problem. Namely, Olsson and Eriksson [32] justify and illustrate that the problem becomes easier to solve as the number of measurements increase.

Following this intuition, the most difficult registration problems should be those close to minimal cases. In order to measure how close a problem is to being minimal we consider the unifying framework presented by Ramalingam and Taguchi [38]: Point-to-point and point-to-line correspondences can be transformed into equivalent sets of 3 and 2 point-to-plane correspondences, respectively. We define then the *effective number of correspondences*, $\hat{m}$, as the equivalent number of independent point-to-plane correspondences

$$\hat{m} = 3m_{\text{point}} + 2m_L + 1m_\Pi. \tag{18}$$

The minimum value of $\hat{m}$ for which a general 3D registration problem *may* have a unique global minimum is 7 [38].

Geometry Even if the effective number of correspondences is higher than 6, degeneracies and symmetries with multiple global minima may still occur depending on the geometric distribution of the correspondences. These cases are identified as well in [38]. We took care during the evaluation on both synthetic and real data to discard these degenerate configurations where the true global minimum cannot be found from the data only.

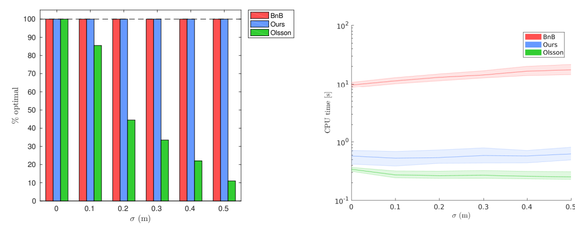
Measurement noise This models the quality of the measurements (we consider no oultliers). If there is no noise, a simple linear relaxation would provide the global solution. Then, as the noise level σ increases the problem becomes harder to solve [32].

Metrics Several metrics can be used to measure the effectiveness of the relaxation approaches, `Ours` and `Olsson`. Due to space issues we choose to show here the most significant metric, which is in our view the optimality ratio, that is, the percentage of cases in which a globally optimal solution was attained. A solution is considered *globally optimal* if the suboptimality gap $\Delta = f - f^\star$ is zero up to numerical precision. The global minimum $f^\star$ can be found from a provably global algorithm such as `BnB` or, as we will see next, also from our *tight* relaxation.

In order to measure the computational performance of the different methods we also plot the resolution times. In particular, we use *shaded error bars* to display the median values plus the 1st and 3rd quartiles reflecting the distribution of the values.

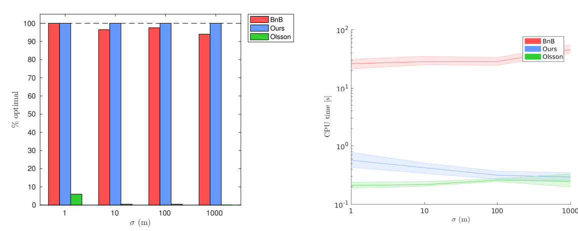
The statistics shown in the figures were generated from a population of 100 registration problems in each case.

Other interesting metrics, as well as other parameters ranges beyond those displayed in this document are available in the **supplementary material**. These additional results support the same conclusions about the methods reached in this document and previous works [32, 34].



(a) (b)

Figure 3. Results in the synthetic scenario for increasing level of noise σ, with $\hat{m} = 10$. (a) The effectivenes of the reference relaxation `Olsson` [32] drops steadily as the level of noise increases. `Ours` remained optimal in all cases. (b) Both convex relaxations `Ours` and `Olsson` take roughly the same time, whereas `BnB` gets *two orders of magnitude* slower.



(a) (b)

Figure 4. Results in the synthetic scenario for extremely severe σ noise conditions and reduced amount of effective number measurements $\hat{m} = 7$. These conditions are of reduced practical interest, but are used to show the general robustness of the tight condition for our relaxation compared to the reference relaxation `Olsson` [32]. In these regimes `BnB` suffers of numerical issues.

5.1. Synthetic data

For obtaining synthetic problems we generated a set of random model primitives that added up to $\hat{m}$ effective correspondences. Similarly to [32], each primitive was determined by randomly taking a point inside a sphere of radius 10 m (plus a random unit direction for the case of lines and planes). Then a "measured" point was randomly picked from the set defined by the primitive and we corrupted it with a Gaussian noise of standard deviation σ.

We show the behaviour w.r.t. the noise level in Fig. 3 in a challenging case with $\hat{m} = 10$. The results for a varying value of $\hat{m}$ were similar to those shown for the real data in Fig. 6(a). These and other evaluations for different ranges of parameters are shown in the **supplementary material**. To sum up, our method attained the globally optimal solution in *all* the considered cases, without exception, even in the most severe cases where the number of features $\hat{m}$ remained almost minimal and the measurement noise σ was raised way beyond any expectable value in real scenarios (see Fig. 4). The reference relaxation `Olsson` in contrast was rarely tight in the challenging scenarios and returned a suboptimal solution.

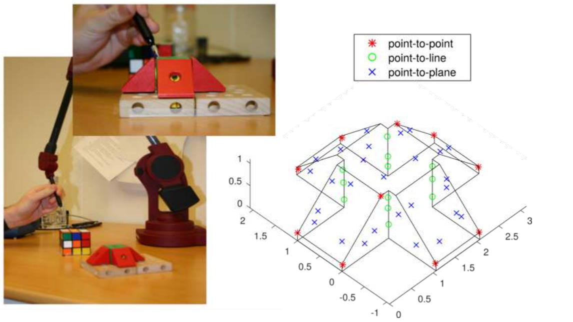


Figure 5. The experimental setup for obtaining real data, taken from [32], and optimal registration of the measurements to the model.

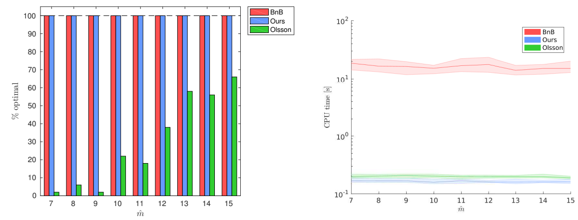
5.2. Real data

Thanks to the corresponding authors, it was possible to exploit the same real data employed in the references [32, 34]. Their experimental setup consisted of using a *MicroScribe-3DLX* 3D scanner to measure the 3D coordinates of some points on the real object, as shown in Fig. 5. For the *Space Station* model 49 points were measured on different primitives of the object, namely 27 on planes, 12 in lines and 10 on corner points. The registration of the complete set to the computer model is shown in 5. In [32] it was shown that for this complete problem `Olsson` works fine, attaining the global optimum as `BnB` but in much less time.

We use the same data to generate a more extensive set of challenging real problems. In this case, we can produce significantly more difficult problems by sampling a smaller set of measurements from the data: We choose different combinations of *point-to-point*, *point-to-line* and *point-to-plane* correspondences that result in a particular *effective number of measurements* $\hat{m}$. The precision σ of the measurements in this case is fixed by the sensor, with errors of about 0.5 millimeters according to the authors of the dataset.

The obtained results are consistent with those observed from the evaluation on synthetic data. The optimality ratio is displayed in Fig. 6(a). The behaviour for `Olsson` w.r.t. the parameter $\hat{m}$ was consistent with that predicted in the original work [32]: It hardly attained global optimality in near-minimal cases, and the performance improved steadily with the increase on $\hat{m}$. Meanwhile, our approach succeeded again in all the cases, always returning the globally optimal solution. Again, both convex relaxations `Ours` and `Olsson` took roughly the same time, whereas `BnB` is *two orders of magnitude* slower (see Fig. 6(b)).

In conclusion, our approach attained the same optimality ratio guaranteed so far only for the provably optimal BnB method, whereas `Ours` was *two* orders of magnitudes faster than `BnB`.



(a) (b)

Figure 6. Results in the real scenario for a low effective number of measuremens $\hat{m}$. (a) The effectivenes of the reference relaxation `Olsson` [32] improves with the increase in $\hat{m}$, yet it hardly attains optimality in most cases whereas ours remained optimal in all cases. (b) Both convex relaxations `Ours` and `Olsson` take roughly the same time, whereas `BnB` is *two orders of magnitude* slower.

6. Conclusions

We have presented a unified formulation for the 3D registration problem involving point-to-point, point-to-line and point-to-plane correspondences that compresses the objective into a single quadratic function of the rotation. Thanks to its generality and flexibility, this formulation should have the potential to introduce further types of correspondences beyond those explored in this work.

The remaining optimization problem has then been characterized as a Quadratically Constrained Quadratic Program. Exploiting a full set of quadratic rotational constraints we obtain a Lagrangian dual relaxation from which a globally optimal solution could be recovered in 100% of the tested cases, although it remains open the theoretical question of why strong duality holds for this relaxation in virtually all cases.

Finally, even though the current approach is already 2 orders of magnitude faster than the competing BnB approach, we are just taking an off-the-shelf generic SDP solver so this performance could be improved further by using specialized solvers that exploit the low-rank structure of the underlying SDP problem.

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