A Clever Elimination Strategy for Efficient Minimal Solvers

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Motivation

- New strategy for solving minimal problems
- Do **more computation** in an **off-line** stage and **less** in an **on-line** stage
- Many minimal problems in computer vision lead to coupled sets of linear and polynomial equations where image measurements enter the linear equations only
- New strategy
  1. **Eliminate all unknowns which do not appear in the linear equations** - do this only once in the pre-processing step (offline)
  2. Extend solutions to the other unknowns
- Can be generalized to fully non-linear systems by monomial lifting
- New constraints on the fundamental matrix of partially calibrated cameras

Basic algebraic geometry

Consider a system of m polynomial equations in n unknowns $X = \{x_1, \ldots, x_n\}$

$$F = \{f_1(x_1, \ldots, x_n) = 0, \ldots, f_m(x_1, \ldots, x_n) = 0\} \tag{1}$$

Assume that the set $F = \{f_1, \ldots, f_m\}$ generates a zero dimensional **ideal**

$$I = \left\{ \sum_{i=1}^{m} g_i f_i, | g_1, \ldots, g_m \in \mathbb{C}[X] \right\} \subset \mathbb{C}[X], \tag{2}$$

Gröbner basis G in Lex order - a special basis of the ideal $I = \langle G \rangle$ which contains polynomial in one variable

Elimination theory - classical algorithmic approach to eliminating some variables between polynomials of several variables

The j^{th} **elimination ideal** of I is $I_j = I \cap \mathbb{C}[x_{j+1}, \ldots, x_n]$ - we eliminate the first j variables to obtain I_j

Theorem: Let G be a Gröbner basis for I in the lex order. Then $G \cap \mathbb{C}[x_{j+1}, \ldots, x_n]$ is a Gröbner basis for the j^th elimination ideal I_j.

Example: Let $I = \langle x^3 + xy + 1, x^2y^2, y^4 \rangle$. Its Gröbner basis in the lex order is $y^2, x^3 + xy + 1$. Therefore $I_x = \langle y^2 \rangle$

Standard elimination strategy

Partitioned F into two subsets:

$$F_L = \{f_i \in F \mid deg(f_i) = 1\}, \tag{3}$$
$$F_N = \{f_i \in F \mid deg(f_i) > 1\}. \tag{4}$$

F_L - linear polynomials from F
F_N - polynomials of higher degrees

SOTA Gröbner basis solvers - variables are mostly eliminated in the online phase
Online:

1. Rewrite the linear equations F_L in the unknowns X_L as $\mathtt{M}X_L = 0$, where $\mathtt{M}$ is a coefficient matrix.
2. Compute a null space basis $\mathtt{N}$ of $\mathtt{M}$, re-parametrize the unknowns $X_L = \mathtt{N}Y$.
3. Substitute $X_L = \mathtt{N}Y$ into the non-linear equations F_N.
4. Solve the system $F_N(Y \cup (X \setminus X_L)) = 0$ (e.g. by the automatic generator [5]).

Clever elimination strategy

Divide the n unknowns X into two subsets:

$$X_L = \{x_i \in X \mid x_i \text{ appears in some } f \in F_L\} \tag{5}$$
$$X_N = X \setminus X_L. \tag{6}$$

X_L - unknowns that appear in linear equations
X_N - unknowns that appear in equations of higher degree only
Notation:
$|F_L| = m_L, |F_N| = m_N, |X_L| = n_L, |X_N| = n_N, m = m_L + m_N, n = n_L + n_N$.

Idea: Eliminate variables X_N from F_N in the pre-processing (offline) step

Offline:

1. Let $I = \langle F_N \rangle$ and consider the elimination ideal $I_{X_N} = I \cap \mathbb{C}[X_L]$.
2. Compute the generators G of I_{X_N} containing only unknowns in X_L.

Online:

3. Rewrite the linear equations F_L in the unknowns X_L as $\mathtt{M}X_L = 0$.
4. Compute null space basis $\mathtt{N}$ of $\mathtt{M}$ and re-parametrize the unknowns $X_L = \mathtt{N}Y$. For rank $\mathtt{M} = m_L$, Y contains $k = n_L - m_L$ new unknowns.
5. Substitute $X_L = \mathtt{N}Y$ into the generators G of elimination ideal I_{X_N}.
6. Solve the new system of polynomial equations $G(Y) = 0$ in k unknowns (e.g. by the automatic generator [5]).
7. Back-substitute to recover $X_L = \mathtt{N}Y$.
8. Extend partial solutions for X_L to solutions for X.

f+E+f relative pose problem

Problem of estimating relative pose and the common unknown focal length of two cameras from six image point correspondences:
Epipolar constraints

$$\mathbf{x}_i^\top \mathtt{F} \mathbf{x}_i' = 0 \tag{7}$$

for six image point correspondences $\mathbf{x}_i, \mathbf{x}_i'$, $i = 1, \ldots, 6$,.
Rewrite in a matrix form

$$\mathtt{M}\,\mathtt{f} = \mathbf{0}, \tag{8}$$

where $\mathtt{M}$ is a 6×9 coefficient matrix and $\mathtt{f}$ is a vector of 9 elements of $\mathtt{F}$.
For six image correspondences $\mathtt{F}$ can be parametrized by two unknowns as

$$\mathtt{F} = x\,\mathtt{F}_1 + y\,\mathtt{F}_2 + \mathtt{F}_3, \tag{9}$$

where $\mathtt{F}_1, \mathtt{F}_2, \mathtt{F}_3$ are matrices from the three-dimensional null space of $\mathtt{M}$ and x and y are new unknowns.
The rank constraint

$$\det(\mathtt{F}) = 0 \tag{10}$$

The trace constraint

$$2\,\mathtt{F}\,\mathtt{Q}\,\mathtt{F}^\top \mathtt{Q}\,\mathtt{F} - trace(\mathtt{F}\,\mathtt{Q}\,\mathtt{F}^\top \mathtt{Q})\,\mathtt{F} = 0, \tag{11}$$

where $\mathtt{Q} = \mathtt{K}\mathtt{K} = diag(f^2, f^2, 1)$.

All SOTA solvers solve these **ten third- and fifth-order polynomial equations** (10) **and** (11) **in three unknowns** x, y and $w = 1/f^2$

f+E+f relative pose problem - NEW SOLVER

After **eliminating the unknown focal length** f from (10) and (11) we get

one **qubic**

$$\det(\mathtt{F}) \tag{12}$$

and the **quintic**

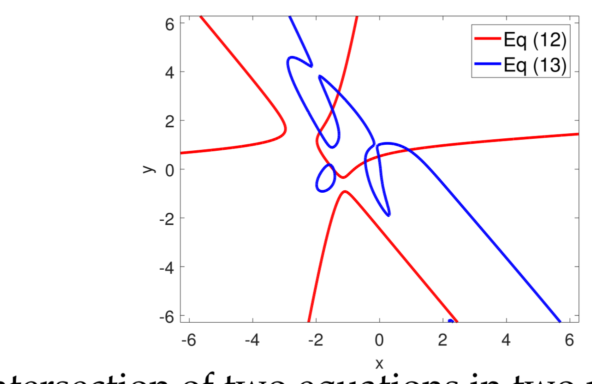
$$\begin{aligned}
&f_{11}f_{13}^3f_{31} + f_{13}^2f_{21}f_{23}f_{31} + f_{11}f_{13}f_{23}^2f_{31} + f_{21}f_{23}^3f_{31} - f_{11}f_{13}f_{31}^3 \\
&-f_{21}f_{23}f_{31}^3 + f_{12}f_{13}^3f_{32} + f_{13}^2f_{22}f_{23}f_{32} + f_{12}f_{13}f_{23}^2f_{32} + f_{22}f_{23}^3f_{32} \\
&-f_{12}f_{13}f_{31}^2f_{32} - f_{12}^2f_{13}^2f_{33} - f_{11}f_{13}f_{31}f_{32}^2 - f_{21}f_{23}f_{31}f_{32}^2 - f_{12}f_{13}f_{32}^3 \\
&-f_{22}f_{23}f_{32}^3 - f_{11}^2f_{13}^2f_{33} - f_{22}f_{23}f_{31}^2f_{32} - 2f_{11}f_{13}f_{21}f_{23}f_{33} - 2f_{12}f_{13}f_{22}f_{23}f_{33} \\
&-f_{21}^2f_{23}^2f_{33} - f_{22}^2f_{23}^2f_{33} + f_{11}^2f_{31}^2f_{33} + f_{21}^2f_{31}^2f_{33} + 2f_{11}f_{12}f_{31}f_{32}f_{33} + \\
&2f_{21}f_{22}f_{31}f_{32}f_{33} + f_{12}^2f_{32}^2f_{33} + f_{22}^2f_{32}^2f_{33}.
\end{aligned} \tag{13}$$

in two unknowns x and y.

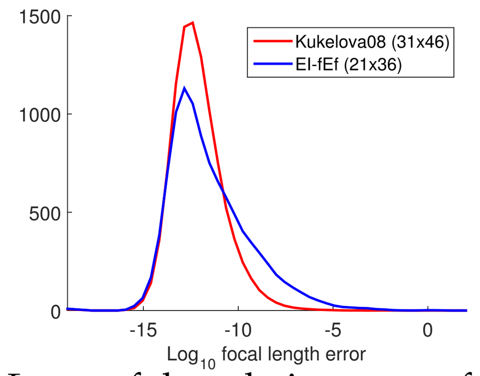
Result: The zero set of (12) and (13) equals the space of all fundamental matrices $\mathtt{F}$ that can be decomposed into $\mathtt{F} = \mathtt{K}^{-1}\mathtt{E}\,\mathtt{K}^{-1}$, for $\mathtt{K} = diag(f, f, 1)$.

The generators (12) and (13) of the elimination ideal $I_f = I \cap \mathbb{C}[f_{11}, f_{12}, f_{13}, f_{21}, f_{22}, f_{23}, f_{31}, f_{32}, f_{33}]$, i.e. the elements that do not contain the focal length f, can be computed offline using the following `Macaulay2` [3] code:

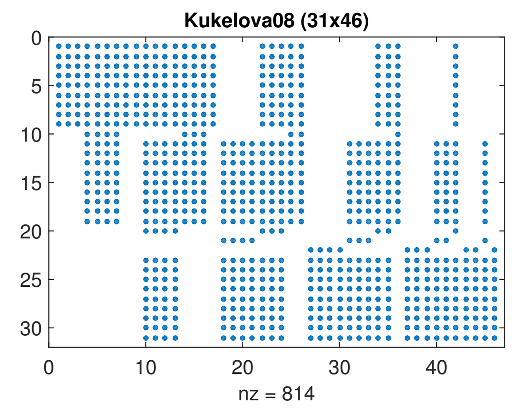
```
R = QQ[f,f11,f12,f13,f21,f22,f23,f31,f32,f33];
F = matrix{{f11,f12,f13},{f21,f22,f23}, {f31,f32,f33}};
K = matrix{{f,0,0},{0,f,0},{0,0,1}};
E = K*F*K;
I = minors(1,2*E*transpose(E)*E
    -trace(E*transpose(E))*E)+ideal(det(E));
G = eliminate(f,saturate(I,ideal(f)))
dim G, degree G, mingens G
```



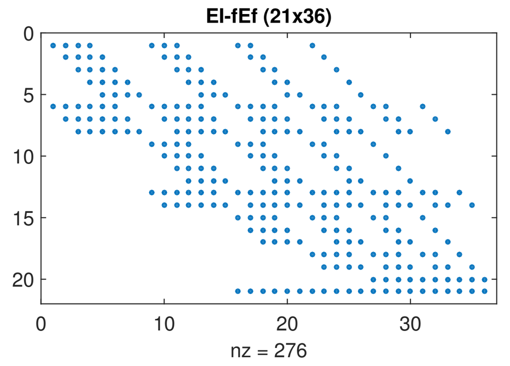
Intersection of two equations in two unknowns



Log_{10} of the relative error of f



(a)



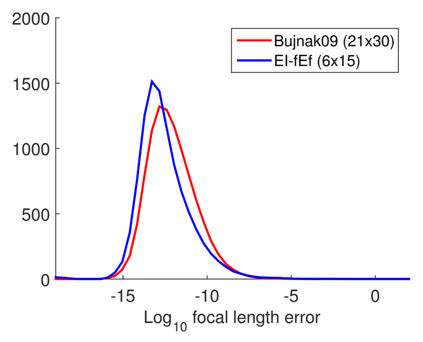
(b)

Sparsity patterns for the solvers to the f+F+f problem: (a) state-of-the-art 31×46 Kukelova08 [5] solver (b) the new 21×36 EI-fEf solver.

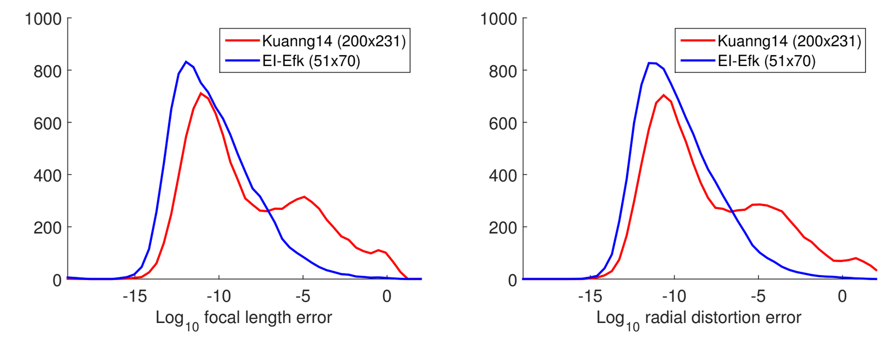
New solvers

E+f relative pose problem

- Relative pose of one calibrated and one up to focal length calibrated camera from 6 point correspondences
- Similar constraints to f+E+f problem - $\mathtt{E} = \mathtt{F}\,\mathtt{K}; \mathtt{K} = diag(f, f, 1)$
- SOTA solver [2] - ten 3^{rd} and 4^{th} order polynomial equations in three unknowns, G-J elimination of a **21×30** matrix
- After eliminating the unknown focal length f from (10) and (11) we get **one cubic and three quartics in two unknowns**
- The new EI-Ef solver performs G-J elimination of a **6×15** matrix



E+f relative problem



E+f+k relative pose problem

E+f+k relative pose problem

- Relative pose of one calibrated camera and one camera with unknown focal length and unknown radial distortion from 7 correspondences
- One-parameter division model for distortion

$$\mathbf{x}_{u_i}(\lambda) = \left[x_{d_i}, y_{d_i}, 1 + \lambda(x_{d_i}^2 + y_{d_i}^2)\right]^\top$$

- Epipolar constraint - $\mathbf{x}_{u_i}^\top \mathtt{F}\,\mathbf{x}_{u_i}'(\lambda) = 0$, - results in **non-linear polynomial equations**
- SOTA solver [4] - careful manual manipulation of the input equations, a large solver $\rightarrow$ G-J elimination of a **200×231** matrix
- The new solver - first **linearize the equations** from the epipolar constraint via monomial lifting
- Then **eliminate the focal length f and the radial distortion parameter λ**
- New generators - **two cubics and nine quartics in four unknowns**
- The new EI-Efk solver performs G-J elimination of a **51×70** matrix

F+β relative pose of unsynchronized cameras

- Simultaneously estimation of camera geometry and time shift β for unsynchronized cameras
- Original system of 7 non-linear equations in 7 unknowns - **633×649**
- After eliminating β - 18 equations in 6 unknowns - **194×210** solver
- For more details see **Poster 2003 [1]** - (Tuesday, July 25, 10am)

Summary

	f+F+f	E+f	E+f+k	F+β
SOTA	31×46 [5]	21×30 [2]	200×231 [4]	633×649 [5]
EI (new)	21×36	6×15	51×70	194×210

References

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