Consensus Maximization with Linear Matrix Inequality Constraints

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Code: http://cvg.ethz.ch/research/conmax/

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Motivation

Consensus maximization has proven to be a useful tool for robust estimation. In this paper, we show the solution space can be reduced by introducing Linear Matrix Inequality (LMI) constraints. This leads to significant speed ups of the optimization time even for large amounts of outliers, while maintaining global optimality.

Contributions

- General **LMI constraints** can be used in a variety of geometric problems. We show derivations for rigid-body, rigid-body + scale, restricted rotations, essential matrix.
- LMI constraints used within Branch-and-Bound (BnB) paradigm to optimally solve the **consensus maximization**.
- LMI constraints **speeds up** the search process.

Notation

$A \succeq 0$ — A is positive semi-definite

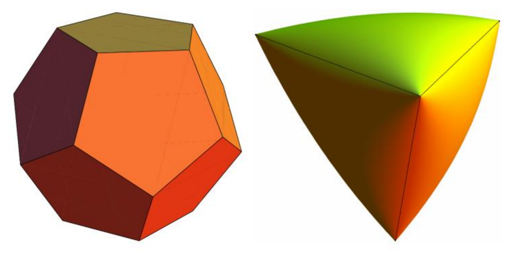
Spectrahedron — A *spectrahedron* is the intersection of positive semi-definite matrices with an affine-linear space.

$$\mathcal{S} = \{y \in \mathbb{R}^n : A(y) \succeq 0\}$$

where

$$A(y) = A_0 + \sum_i^n y_i A_i$$

LMI (Linear Matrix Inequality) — $A(y) \succeq 0$



Problem Formulation

Consensus Maximization

Given a set of measurement pairs $\mathcal{Z} = \{\mathcal{P}_i\}_{i=1}^n$ and a threshold ϵ,

$$\begin{aligned} \underset{x,\zeta\subseteq\mathcal{Z}}{\text{maximize}} \quad & |\zeta|, \\ \text{subject to} \quad & \gamma_i(x) \le \epsilon, \quad \forall \mathcal{P}_i \in \zeta, \\ & A(x) \succeq 0. \end{aligned}$$

Consider a geometric transformation $T(x) : U \to V$ that relates a pair of measurements $\mathcal{P} = \{U, V\}$. Let $\gamma(x)$ be the residual error for a known $\mathcal{P}$ and the estimate x.

$$U \xrightarrow{T(x)} V$$

$$v_i = S(x)\, u_i + t(x)$$

(Similarity Transformation)

$$\gamma_i(x) = [S(x)\,u_i + t(x) - v_i]^\intercal \Sigma^{-1} [S(x)\,u_i + t(x) - v_i]$$
$$= x^\intercal Q_i x + q_i^\intercal x + r_i \textit{ with } Q_i \succeq 0.$$

Mixed Integer Programming

The consensus maximization problem can be restated as Mixed Integer Semi-Definite Programming (MI-SDP), allowing for global optimization:

$$\begin{aligned} \underset{x,z}{\text{minimize}} \quad & \sum_i z_i, \\ \text{subject to} \quad & \gamma_i(x) \le \epsilon + z_i\,\mathcal{M}, \quad \forall i, && \longrightarrow \textbf{Residual bounds} \\ & z_i \in \{0,1\}, && \longrightarrow \textbf{Binary variables} \\ & A(x) \succeq 0. && \longrightarrow \textbf{LMI} \end{aligned}$$

Theory

Definition (Orbitope [1]) An orbitope is the convex hull of an orbit of a compact algebraic group that acts linearly on a real vector space. The orbit has the structure of a real algebraic variety, and the orbitope is a convex semi-algebraic set.

A 3-dimensional rotation matrix R ∈ SO(3) has dimension three. However, its tautological orbitope is a convex body of dimension nine. The following theorem is a key ingredient of this work.

Theorem (SO(3) Orbitope [1]) The tautological orbitope conv(SO(3)) is a spectrahedron whose boundary is a quartic hypersurface. A 3×3 matrix A lies in conv(SO(3)) if and only if,

$$I_{4\times4} + \mathcal{L}(A) \succeq 0$$

Proposition (SSO(3) and SO(3) Orbitope) ∀ S ∈ SSO(3) there exists A ∈ conv(SO(3)) such that S = α A, if and only if ∃ α > 0:

$$\alpha\, I_{4\times4} + \mathcal{L}(S) \succeq 0$$

where,

$$\mathcal{L}(A) = \begin{bmatrix} a_{11}+a_{22}+a_{33} & a_{32}-a_{23} & a_{13}-a_{31} & a_{21}-a_{12} \\ a_{32}-a_{23} & a_{11}-a_{22}-a_{33} & a_{21}+a_{12} & a_{13}+a_{31} \\ a_{13}-a_{31} & a_{21}+a_{12} & a_{22}-a_{11}-a_{33} & a_{32}+a_{23} \\ a_{21}-a_{12} & a_{13}+a_{31} & a_{32}+a_{23} & a_{33}-a_{11}-a_{22} \end{bmatrix}$$

References

[1] R. Sanyal, F. Sottile, and B. Sturmfels. Orbitopes. Mathematika, 57(02):275–314, 2011.

Generalization

Transformation:

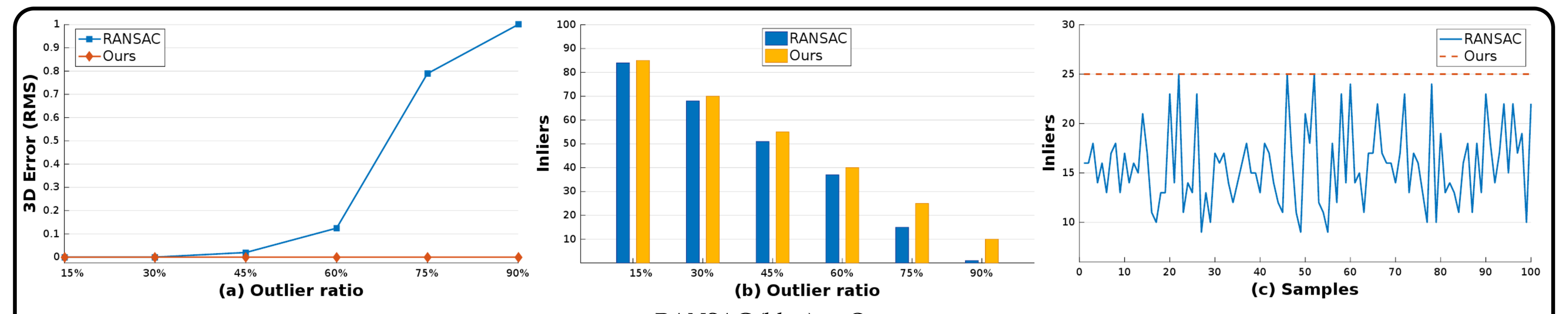
$$\beta_i(x)\, v_i = B_i(x)\, u_i + b(x)$$

New Residual:

$$\gamma_i(x) = \Delta_i(x)^\intercal \Sigma^{-1} \Delta_i(x),$$
$$\Delta_i(x) = B_i(x)\, u_i + b(x) - \beta_i(x)\, v_i$$

Transformations	Constraints	$\beta_i(x)$	$B_i(x)$	b(x)	LMIs
Similarity	$S(x) \in SSO(3)$	1	S(x)	t(x)	$\mathcal{K}_s \succeq 0$
Absolute Pose	$R(x) \in SO(3)$	$r_3(x)^\intercal u_i + t_3(x)$	R(x)	t(x)	$\mathcal{K}_a \succeq 0$
Relative Pose	$E(x) \in \mathcal{E}$	$[(n_i)_1 e_2(x) - (n_i)_2 e_1(x)]^\intercal u_i$	$[n_i]_\times E(x)$	0	$\mathcal{K}_r \succeq 0$

Results - Similarity Transformation (Synthetic Data)

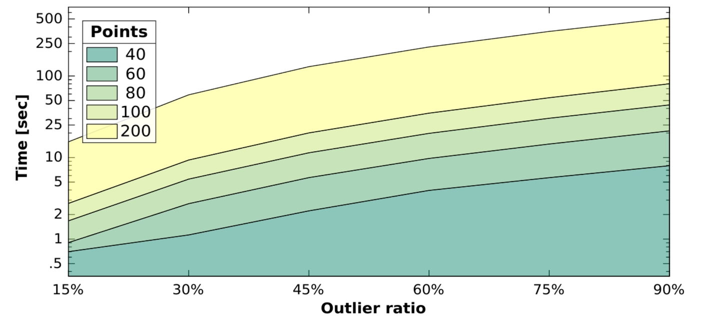


RANSAC (blue) vs Ours

Outliers	N=30		N=40		N=50	
	W	W/O	W	W/O	W	W/O
15 %	0.26	0.61	0.44	2	0.64	7
30 %	0.62	9	1.17	47	2.05	315
45 %	1.24	48	2.68	467	4.38	-
60 %	2.30	498	4.63	-	7.84	-
75 %	3.42	-	7.04	-	11.81	-

W: with constraints (Ours). **W/O**: without constraints. **(–)**: greater than 3600 sec.

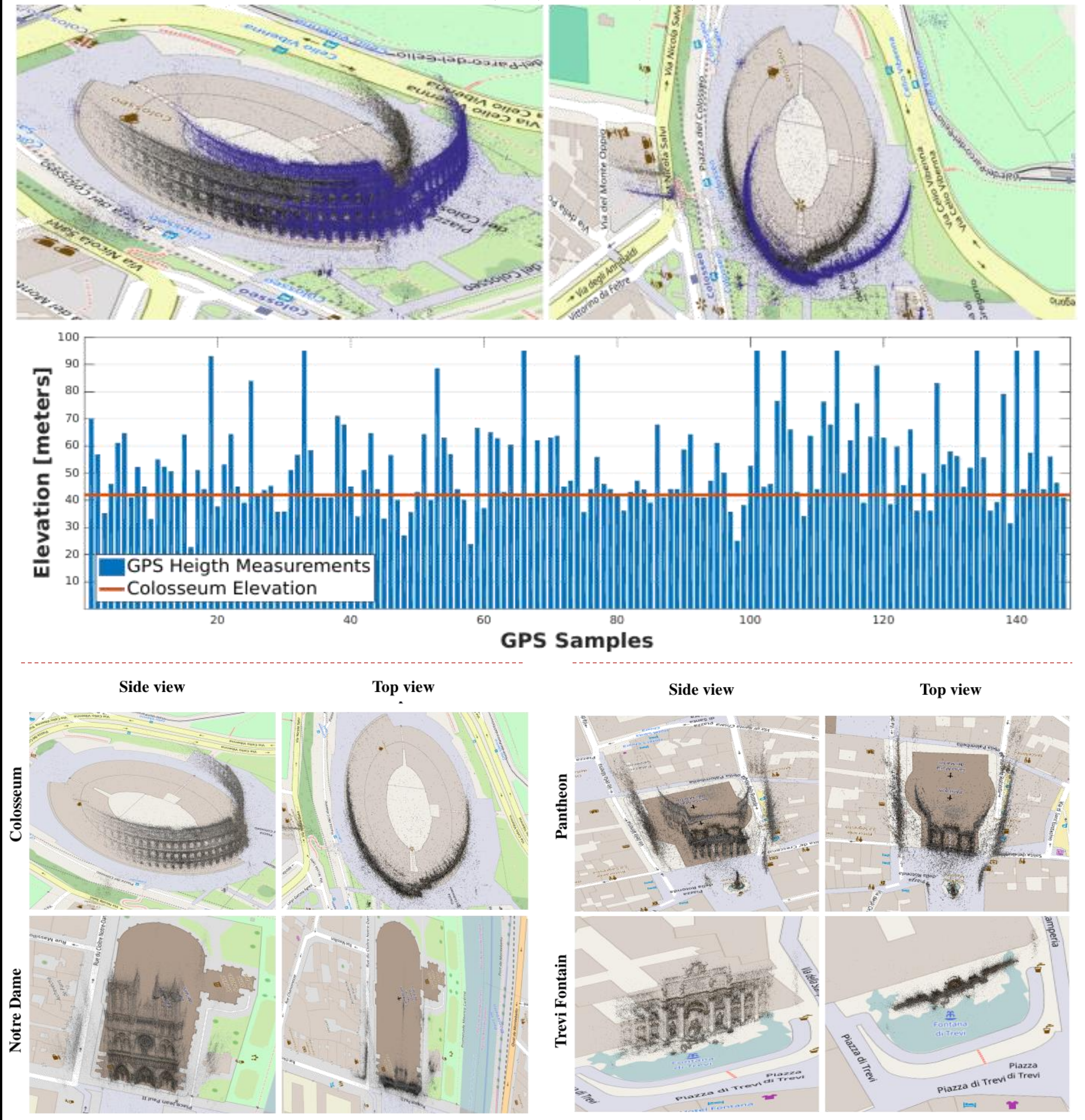
Time comparison obtained with and without the LMI constraint.



Runtime for increasing number of points and outlier ratio.

Results - Similarity Transformation (Real Data)

RANSAC (blue) vs. Ours (black)



Scene	$\Delta\theta$ (Yaw)	$\Delta\theta$ (Pitch)	$\Delta\theta$ (Roll)	ΔT	Height	Scale	$\lvert\zeta^*\rvert / N$	Time [sec]
Colosseum	$< 1^o$	$< 1^o$	$< 1^o$	$< 1m$	$< 1m$	$< 1\%$	117/147	88.26 s
Notre Dame	$< 3^o$	$< 2^o$	$< 1^o$	$< 2m$	$< 1m$	$< 1\%$	103/144	43.17 s
Pantheon	$< 3^o$	$< 5^o$	$< 2^o$	$< 3m$	$< 2m$	$< 7\%$	14/ 47	16.12 s
Trevi Fontain	$< 2^o$	$< 1^o$	$< 3^o$	$< 1m$	$< 1m$	$< 3\%$	104/140	65.68 s

$\Delta\theta$ [degree]: rotation error (Yaw, Pitch and Roll). ΔT [meters]: translation error. ζ^*: maximum consensus set. N: number of available GPS tags.

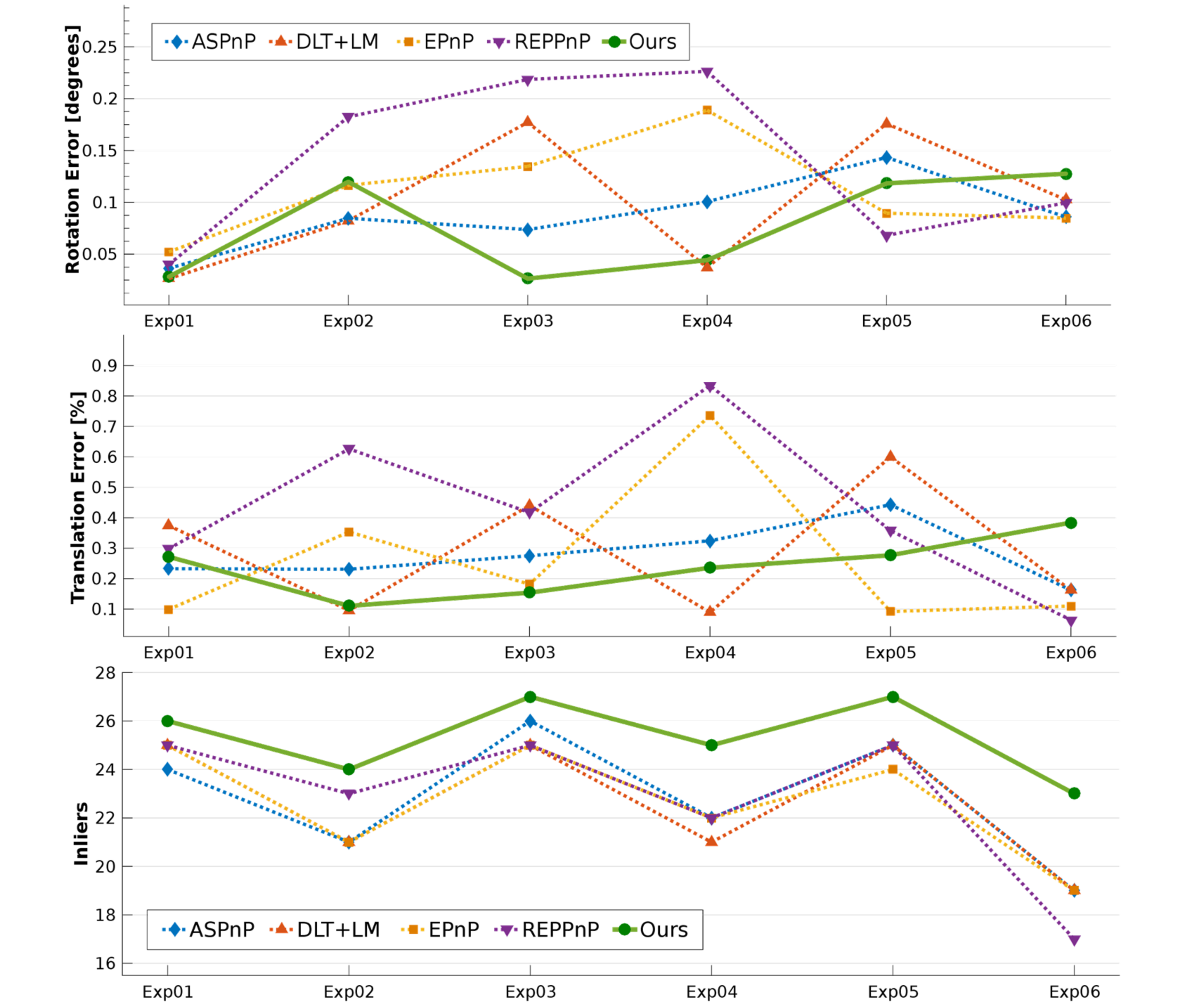
Results – Relative Pose (Real Data)

RANSAC vs. Our method with and without LMI constraints.

Scene	Image	Method	$\lvert\zeta\rvert / N$	ΔR [degree]	ΔT [%]	Time [sec]
Fountain		RANSAC	20 / 39	0.29	4.81	**0.61**
		Ours	**25** / 39	**0.15**	**1.76**	3.35
Herz-Jesu		RANSAC	35 / 70	2.12	3.20	**0.63**
		Ours	**49** / 70	**0.12**	**2.87**	23.84

$\lvert\zeta\rvert$: number of inliers. **Ours**: method with constraints. ΔR [degree]: rotation error. ΔT [%]: translation error.

Results – Absolute Pose (Real Data)



	Error		Inliers	Time [sec]	
	ΔR	ΔT	$\lvert\zeta^*\rvert / N$	$Ours$	$Ours^*$
Exp01	0.03	0.27	26 / 42	13.86	2.96
Exp02	0.12	0.28	24 / 45	25.71	2.78
Exp03	0.12	0.11	27 / 46	50.27	13.80
Exp04	0.13	0.41	25 / 46	61.18	20.44
Exp05	0.04	0.24	27 / 47	174.81	10.40
Exp06	0.18	0.31	23 / 44	120.24	58.06

ΔR [degree]: rotation error. ΔT [%]: translation error. ζ^*: maximum consensus.
$Ours^*$: imposing the additional constraint (16): $R + R^T \succeq I$.

Conclusions

- We present a general **global optimization framework** for consensus maximization with LMI constraints.
- Proposed LMI constraints offer a significant **speedup** in computation time, under a globally optimal framework, by reducing the solution search space.
- Experiments on problems of **similarity transformation**, **absolute pose**, and **relative pose** estimation were successfully conducted.

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