# A clever elimination strategy for efficient minimal solvers
# Appendix

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This appendix includes (1) additional details on the Elimination Theorem (in Sec. 1), (2) derivation of the constraints on the projection for planar scenes by cameras with unknown focal length (in Sec. 2), (3) details of focal length extraction (in Sec. 3), (4) detailed presentation of the generators of $E+f$ problem (in Sec. 4), (5) results on the solvers' sparsity (in Sec. 5), and (6) noise experiment for the E+f+k problem (in Sec. 6).

## 1. Details on the elimination theorem

Here we provide additional details for

**Theorem 1.1 (Elimination theorem [1])** *Let* $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$ *be an ideal and let* $G$ *be a Gröbner basis of* $I$ *with respect to the lexicographic monomial order where* $x_1 > x_2 > \cdots > x_n$*. Then, for every* $0 \leq l \leq n$*, the set* $G_l = G \cap \mathbb{C}[x_{l+1}, \ldots, x_n]$ *is a Gröbner basis of the* $l$*-th elimination ideal* $I_l = I \cap \mathbb{C}[x_1, \ldots, x_l]$ *.*

See [1] for a full account of the theory.

The *ring* $\mathbb{C}[x_1, \ldots, x_n]$ stands for all polynomials in $n$ unknowns $x_1, \ldots, x_n$ with complex coefficients. In computer vision applications, however, coefficients of polynomial systems are always real (in fact, rational) numbers and our systems consist of a finite number $s$ of polynomial equations $f_i(x_1, \ldots, x_n) = 0$, $i = 1, \ldots, s$.

The *ideal* $I = \{\sum_{i=1}^{s} h_i f_i \,|\, h_i, \ldots\ h_s \in \mathbb{C}[x_1, \ldots, x_n]\}$ generated by $s$ polynomials (generators) $f_i$ is the set of all polynomial linear combinations of the polynomials $f_i$. Here the multipliers $h_i$ are polynomials. All elements in the ideal $I$ evaluate to zero (are satisfied) at the solutions to the equations $f_i(x_1, \ldots, x_n) = 0$.

The *Gröbner basis* $G = \{g_1, \ldots, g_m\}$ of an ideal $I$ is a particularly convenient set of the generators of $I$, which can be used to find solutions to the original system $f_i$ in an easy way. For instance, for linear (polynomial) equations, a Gröbner basis of the ideal generated by the linear polynomials is obtained by Gaussian elimination. After Gaussian elimination, equations appear in a triangular form allowing one to solve for one unknown after another. This pattern carries on in a similar way to (some) Gröbner bases of general polynomial systems and thus it makes Gröbner bases a convenient tool for solving general polynomial systems.

Algorithmic construction of Gröbner bases relies on an ordering of monomials to specify in which order to deal with monomials of a polynomial. *Lexicographic monomial order* (LEX) is a particularly convenient order, which can be used to produce Gröbner bases that are in the triangular form. LEX orders monomials as words in a dictionary. An important parameter of a LEX order (i.e. ordering of words) is the order of the unknowns (i.e. ordering of letters). For instance, monomial $xy^2z = xyyz > xyzz = xyz^2$ when $x > y > z$ (i.e. $xyyz$ is before $xyzz$ in a standard dictionary). However, when $x < y < z$, then $xy^2z = xyyz < xyzz = xyz^2$. We see that there are $n!$ possible LEX orders when dealing with $n$ unknowns.

The set $G_l = G \cap \mathbb{C}[x_{l+1}, \ldots, x_n]$ contains all the polynomials in Gröbner basis $G$ that contain only unknowns $x_{l+1}, \ldots, x_n$. For instance, if $G$ is a Gröbner basis in the triangular form, then $G_l = \{g_m(x_n), g_{m-1}(x_{n-1}, x_n), \ldots, g_{m-l}(x_1, \ldots, x_{l+1})\}$ contains polynomials in one, two, …, $l$ unknowns.

The polynomials $G_l$ generate the *elimination ideal* $I_l = I \cap \mathbb{C}[x_{l+1}, \ldots, n]$, containing all polynomials from $I$ that use the unknowns $x_{l+1}, \ldots, x_n$ only. Hence, for each of $n!$ orderings, we get $n$ elimination ideals $I_l$.

## 2. 3D planar homograpy with unknown focal length

We assume that a planar object (say, simply a plane) is observed by an unknown camera with the projection matrix [2]

$$\mathtt{P} = \mathtt{K}[\mathtt{R}\,|\,\mathbf{t}], \tag{1}$$

where $\mathtt{K} = diag(f, f, 1)$ is the calibration matrix with the unknown focal length $f$, $\mathtt{R} = [r_{ij}]_{ij=1}^{3} \in SO(3)$ is the unknown rotation, and $\mathbf{t} = [t_1, t_2, t_3]^\top \in \mathbb{R}^3$ the unknown translation.

Without loss of generality, we assume that the plane is defined by $z = 0$, *i.e.* all 3D points with homogeneous coordinates $\mathbf{X}_i = [x_i, y_i, z_i, 1]^\top$ have the $3^{rd}$ coordinate $z_i = 0$. Then, the image points $\mathbf{u}_i = [u_i, v_i, 1]^\top$ and the corresponding 3D points $\mathbf{X}_i = [x_i, y_i, 0, 1]^\top$ are related by

$$\alpha_i\,\mathbf{u}_i = \mathtt{H}\,\hat{\mathbf{X}}_i, \tag{2}$$

where $\alpha_i$ are unknown scalars, $\hat{\mathbf{X}}_i = [x_i, y_i, 1]$, and $\mathtt{H} = [h_{ij}]_{ij=1}^{3} \in \mathbb{R}^{3\times 3}$ is a homography matrix that has the form

$$\mathtt{H} = \begin{bmatrix} \mathbf{p}_1 \; \mathbf{p}_2 \; \mathbf{p}_4 \end{bmatrix} = \begin{bmatrix} f\,r_{11} & f\,r_{12} & t_1 \\ f\,r_{21} & f\,r_{22} & t_2 \\ r_{31} & r_{32} & t_3 \end{bmatrix} \tag{3}$$

where $\mathbf{p}_j$ is the $j^{th}$ column of the projection matrix $\mathtt{P}$ (1).

Next, from the projection equation (2), we eliminate the scalar values $\alpha_i$. This can be done by multiplying (2) by the skew symmetric matrix $[\mathbf{u}]_\times$ [2] to get

$$\begin{bmatrix} 0 & -1 & v_i \\ 1 & 0 & -u_i \\ -v_i & u_i & 0 \end{bmatrix} \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix} = 0 \tag{4}$$

The matrix equation (4) contains three polynomial equations, two of which are linearly independent. This means that we need at least 3.5 2D $\leftrightarrow$ 3D point correspondences to estimate the unknown homography $\mathtt{H}$, because $\mathtt{H}$ has 7 degrees of freedom: three parameters for the rotation, three parameters for the translation and also the focal length.

For the 3.5 point correspondences, matrix equation (4) results in seven linearly independent linear homogeneous equations in nine elements of the homography matrix $\mathtt{H}$.

Moreover, we have here two additional polynomial constraints on elements of $\mathtt{H}$. For the first two columns of the rotation matrix $\mathtt{R}$, there holds

$$r_{11}r_{12} + r_{21}r_{22} + r_{31}r_{32} = 0 \tag{5}$$

$$r_{11}^2 + r_{21}^2 + r_{31}^2 - r_{12}^2 - r_{22}^2 - r_{32}^2 = 0 \tag{6}$$

This means that the elements of the first two columns of the homography matrix $\mathtt{H} = [h_{ij}]_{ij=1}^{3}$ (3) satisfy

$$w^2\,h_{11}\,h_{12} + w^2\,h_{21}h_{22} + h_{31}h_{32} = 0 \tag{7}$$

$$w^2\,h_{11}^2 + w^2\,h_{21}^2 + h_{31}^2 - w^2\,h_{12}^2 - w^2\,h_{22}^2 - h_{32}^2 = 0 \tag{8}$$

where $w = 1/f$.

Hence, estimating 3D planar homography with unknown focal length results in seven linear homogeneous equations and two non-linear homogeneous equations in $X = \{h_{11}, h_{12}, h_{13}, h_{21}, h_{22}, h_{23}, h_{31}, h_{32}, h_{33}, w\}$. This system of nine homogeneous equations has the same form as that presented in Section 2.2 of the main paper. Therefore this system can be efficiently solved using the new elimination strategy presented in Section 2.1.3. This strategy results in solving one fourth-degree equation in one unknown (see Section 2.2 in the main paper).

## 3. Extraction of the focal length

In this section we present formulas for extracting the focal length from a given fundamental matrix $\mathtt{F}$ for two cases

1. $\mathtt{E} = \mathtt{F}\,\mathtt{K}$
2. $\mathtt{E} = \mathtt{K}\,\mathtt{F}\,\mathtt{K}$

where $\mathtt{K} = diag(f, f, 1)$ is a diagonal calibration matrix. Unlike most of the existing formulas and methods for extracting the focal length from the fundamental matrix $\mathtt{F}$, the presented formulas contain directly elements of the fundamental matrix. They don't require an SVD decomposition of the fundamental matrix or computation of the epipoles.

### 3.1. E+f problem

Here we will assume that the principal points [2] are at the origin (which can be always achieved by shifting the known principal points) and use the recent result [3, Lemma 5.1] which we restate in our notation:

**Lemma 3.1** *Let* $\mathtt{F}$ *be a fundamental matrix of the form that satisfies* $\mathtt{E} = \mathtt{F}\mathtt{K}$. *Then there are exactly two pairs of essential matrix and focal length* $(\mathtt{X} = \mathtt{E}, f)$ *and* $(\mathtt{X} = \mathrm{diag}(-1, -1, 1)\mathtt{E}, -f)$. *The positive* $f$ *is recovered from* $\mathtt{F} = [f_{ij}]_{1\le i,j\le 3}$ *by the following formula*

$$f^2 = \frac{f_{23}f_{31}^2 + f_{23}f_{32}^2 - 2f_{21}f_{31}f_{33} - 2f_{22}f_{32}f_{33} - f_{23}f_{33}^2}{2f_{11}f_{13}f_{21} + 2f_{12}f_{13}f_{22} - f_{23}(f_{11}^2 - f_{12}^2 + f_{13}^2 + f_{21}^2 + f_{22}^2 + f_{23}^2}$$

### 3.2. f+E+f problem

To derive formulas for the extraction of $f$ from $\mathtt{F}$ computed from images with the same unknown focal length, we follow methods developed in [3]. In this case, the result is the following formula for $f^2$, namely: $-f_{13}^2f_{32}f_{33} - f_{23}^2f_{32}f_{33} + f_{12}f_{13}f_{33}^2 + f_{22}f_{23}f_{33}^2$ quantity divided by $f_{11}f_{13}f_{31}f_{32} + f_{21}f_{23}f_{31}f_{32} + f_{12}f_{13}f_{32}^2 + f_{22}f_{23}f_{32}^2 - f_{11}f_{12}f_{31}f_{33} - f_{21}f_{22}f_{31}f_{33} - f_{12}^2f_{32}f_{33} - f_{22}^2f_{32}f_{33})$, which can be obtained by the following `Macaulay2` code

```
R = QQ[f,f11,f12,f13,f21,f22,f23,f31,f32,f33]
F = matrix{{f11,f12,f13},{f21,f22,f23},
           {f31,f32,f33}};
K = matrix{{f, 0, 0}, {0, f, 0}, {0, 0, 1}};
E = K*F*K;
```

```
G = ideal(det(E))+minors(1,2*E*transpose(E)*E
            -trace(E*transpose(E))*E);
Gs = saturate(G,ideal(f));
gse = flatten entries mingens gb Gs;
cofs = g->coefficients(g,Variables=>{f});
cofsg = apply(gse,cofs);
cofsg_2
```

# 4. The elimination ideal for the E+f problem

We consider the E+f problem from Section 3.2, *i.e.* the problem of estimating epipolar geometry of one calibrated and one up to focal length calibrated camera. Here, in this case

$$\mathtt{E} = \mathtt{F}\,\mathtt{K}, \tag{9}$$

where $\mathtt{K} = diag(f, f, 1)$ is a diagonal calibration matrix for the first camera, containing the unknown focal length $f$. Here, $\mathtt{F}$ is the $3\times3$ fundamental matrix and $\mathtt{E}$ is the $3\times3$ essential matrix [2]

For the E+f problem, we have the ideal $I \subset \mathbb{C}[f_{11}, f_{12}, f_{13}, f_{21}, f_{22}, f_{23}, f_{31}, f_{32}, f_{33}, f]$ generated by ten equations, one cubic from the rank constraint

$$\det(\mathtt{F}) = 0, \tag{10}$$

and nine polynomials from the trace constraint

$$2\,\mathtt{F}\,\mathtt{Q}\,\mathtt{F}^\top\mathtt{F} - trace(\mathtt{F}\,\mathtt{Q}\,\mathtt{F}^\top)\mathtt{F} = 0, \tag{11}$$

where $\mathtt{Q} = \mathtt{K}\,\mathtt{K}$.

For this problem, the new elimination strategy from Section 2.1.3 leads to computing the generators of the elimination ideal $I_f = I \cap \mathbb{C}[f_{11}, f_{12}, f_{13}, f_{21}, f_{22}, f_{23}, f_{31}, f_{32}, f_{33}]$, i.e. the generators that do not contain $f$. To compute these generators we can use the following `Macaulay2` [4] code:

```
R = QQ[f,f11,f12,f13,f21,f22,f23,f31,f32,f33];
F = matrix {{f11,f12,f13},{f21,f22,f23},
{f31,f32,f33}};
K = matrix {{f,0,0},{0,f,0},{0,0,1}};
E = F*K;
I = minors(1,2*E*transpose(E)*E
    -trace(E*transpose(E))*E)+ideal(det(E));
G = eliminate({f},saturate(I,ideal(f)))
dim G, degree G, mingens G
```

For the E+f problem, the variety $G$ has dimension 6 and degree 9 in $\mathbb{P}^8$ and is defined by one cubic and three quartics. It can be verified that these four polynomials correspond to the four maximal minors of the $3\times4$ matrix:

$$\begin{pmatrix} f_{11} & f_{12} & f_{13} & f_{21}f_{31} + f_{22}f_{32} + f_{23}f_{33} \\ f_{21} & f_{22} & f_{23} & -f_{11}f_{31} - f_{12}f_{32} - f_{13}f_{33} \\ f_{31} & f_{32} & f_{33} & 0 \end{pmatrix}. \tag{12}$$

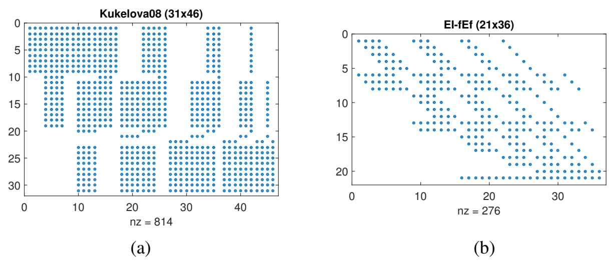



Figure 1. **Sparsity patterns for the solvers to the f+E+f problem:** (a) state-of-the-art $31\times46$ Kukelova08 [5] solver (b) the new $21\times36$ EI-fEf solver.

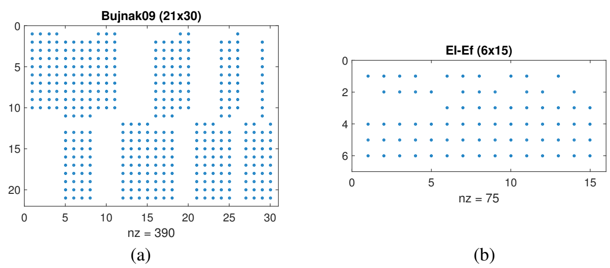



Figure 2. **Sparsity patterns for the solvers to the E+f problem:** (a) state-of-the-art $21\times30$ Bujnak09 [6] solver (b) the new $6\times9$ EI-Ef solver.

# 5. Sparsity patterns of solvers

Here, we show a comparison of the sparsity patterns of our new elimination-based solvers (EI-fEf, EI-Ef, EI-Efk) and of the SOTA solvers [5, 6, 7].

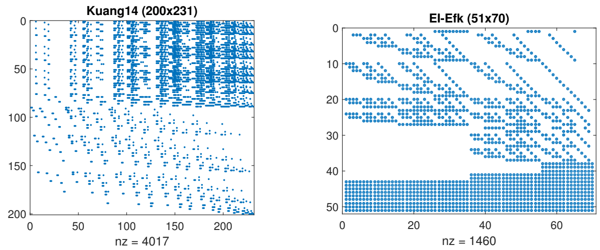
Figure 1 shows the sparsity patterns of the (a) state-of-the-art (SOTA) $31\times46$ Kukelova08 [5] solver for the f+E+f problem and (b) the new $21\times36$ EI-fEf solver for this problem. In this case the new EI-fEf solver is not only smaller but also sparser. The ratio of the number of non-zero elements of the $31\times46$ template matrix of the SOTA solver Kukelova08 [5] ($nz_S$) and the number of non-zero elements of the $21\times36$ matrix of the EI-fEf solver ($nz_{EI}$) is 3.

Figure 2 shows the sparsity patterns of the (a) SOTA $21\times30$ Bujnak09 [6] solver and (b) the new $6\times15$ EI-Ef solver for the E+f problem. Here the ratio of the number of non-zero elements of the template matrix of the SOTA solver [6] and the number of non-zero elements of the template matrix of our new EI-Ef solver is 5.2.

Finally, Figure 3 shows the sparsity patterns of the (a) SOTA $200\times231$ Kuang14 [7] solver and (b) the new $51\times70$ EI-Efk solver for the E+f+k problem. Here, the ratio $nz_S/nz_{EI}$ is approximately 2.8.

# 6. Noise experiment for the E+f+k problem

Next, Figure 4 shows the results of experiments with noise simulation for the E+f+k problem. We show the estimated radial distortion parameters for the ground truth ra-




(a) (b)

Figure 3. **Sparsity patterns for the solvers to the E+f+k problem:** (a) state-of-the-art 200×231 Kuang14 [7] solver (b) the new 51×70 EI-Efk solver.

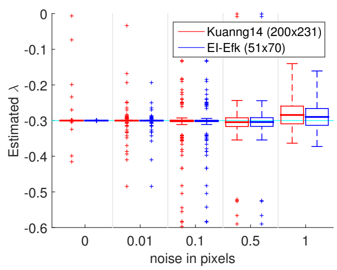



Figure 4. Comparison of the new EI-Efk solver (blue) with the SOTA Kuang14 [7] solver (red). Boxplots of estimated $\lambda$'s for different noise levels and $\lambda_{gt} = -0.3$

dial distortion $\lambda_{gt} = -0.3$ and 200 runs for each noise level. We compared our new E+f+k solver with the SOTA Kuang solver [7]. Figure 4 shows resuts by MATLAB `boxplot`. In the presence of noise, our new EI-Efk solver (blue) gives similar or even better estimates than the SOTA solver Kuang14 [7] for which we observed more failures (crosses).

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