Bounds relating the two formulations

1 Notations and problem formulation

Denote by D_1, D_2 two $n \times n$ symmetric pairwise geodesic distance matrices of two corresponding shapes. Assume that the shapes are nearly isometric, with some unknown correspondence. D_2 can be thought of some perturbation of PD_1P^T, denoted by $D_2 = PD_1P^T + \Delta D$, where P encodes the correspondence. Denote by

$$\begin{aligned} D_1 &= Q_1 \Lambda_1 Q_1^T \\ D_2 &= Q_2 \Lambda_2 Q_2^T \end{aligned} \tag{1}$$

the eigendecomposition of D_1 and D_2. Define the elements of the $n \times n$ matrix W to be $W_{ij} = \sqrt{\Lambda_{ij}}$ for both shapes. Define $X_1 = Q_1 W_1$ and $X_2 = Q_2 W_2$, and note that $D_1 = X_1 X_1^T$ and $D_2 = X_2 X_2^T$. Denote by $\pi(n)$ the group of $n \times n$ permutation matrices, by $\mathcal{U}(n)$ the group of $n \times n$ unitary matrices, and by $\mathcal{O}(n)$ the group of $n \times n$ orthogonal matrices. Here, the norm $\|F\|_{2,\infty}$ stands for $\max\{\|f_i\|_2\}_{i=1}^n$, where $\{f_i\}$ are the rows of F.

Consider the minimization

$$\arg\min_{P \in \pi(n)} \|PD_1P^T - D_2\|_\infty, \tag{2}$$

and the related form,

$$\arg\min_{P \in \pi(n), C \in \mathcal{O}(n)} \|PX_1C - X_2\|_{2,\infty}. \tag{3}$$

We claim that the minimal value of (2), which approximates the Gromov-Hausdorff distance, is small if and only if the value of (3) is small. Hence, for nearly isometric shapes, the alternative form could be used. In the following sections, we present the relative bounds supporting this claim.

2 First direction

Let A_1 and A_2 be two matrices of the same size, and define $B_1 = A_1 A_1^T$ and $B_2 = A_2 A_2^T$. Denote by $a_{1,i}$ row i in A_1 and by $a_{2,i}$ row i in A_2, such that $B_{1,ij} = a_{1,i} a_{1,j}^T$ and $B_{2,ij} = a_{2,i} a_{2,j}^T$. Suppose that A_1 and A_2 satisfy

$$\|A_1 - A_2\|_{2,\infty} < \epsilon. \tag{4}$$

Hence, for any i,

$$\|a_{1,i} - a_{2,i}\|_2 < \epsilon. \tag{5}$$

Denote $e_i = a_{1,i} - a_{2,i}$, then, we have

$$\begin{aligned} |B_{1,ij} - B_{2,ij}| &= |a_{1,i} a_{1,j}^T - a_{2,i} a_{2,j}^T| \\ &= |(a_{2,i} + e_i)(a_{2,j} + e_j)^T - a_{2,i} a_{2,j}^T| \\ &= |a_{2,i} e_j^T + e_i a_{2,j}^T + e_i e_j^T| \\ &\le |a_{2,i} e_j^T| + |e_i a_{2,j}^T| + |e_i e_j^T| \\ &\le \|a_{2,i}\|_2 \|e_j\|_2 + \|e_i\|_2 \|a_{2,j}\|_2 + \|e_i\|_2^2 \\ &< \epsilon \|a_{2,i}\|_2 + \epsilon \|a_{2,j}\|_2 + \epsilon^2, \end{aligned} \tag{6}$$

where we used the triangle inequality and Cauchy–Schwarz inequality. Since it applies for any i and j, we obtained that

$$|B_1 - B_2|_\infty \quad < \quad \epsilon \|A_2\|_{2,\infty} + \epsilon \|A_2\|_{2,\infty} + \epsilon^2$$

$$= \quad 2\epsilon\|A_2\|_{2,\infty} + \epsilon^2. \tag{7}$$

Returning back to our problem, assume we have found a solution such that

$$\|PX_1C - X_2\|_{2,\infty} < \epsilon. \tag{8}$$

By plugging $A_1 = PX_1C$ and $A_2 = X_2$, we obtain

$$\begin{aligned} A_1A_1^T &= PX_1CC^TX_1^TP^T = PX_1X_1^TP^T = PD_1P^T \\ A_2A_2^T &= X_2X_2^T = D_2, \end{aligned} \tag{9}$$

where we used the fact that C is orthogonal. Finally,

$$|PD_1P^T - D_2|_\infty < 2\epsilon\|X_2\|_{2,\infty} + \epsilon^2. \tag{10}$$

Furthermore, notice that

$$\begin{aligned} \|X_2\|_{2,\infty} &= \|Q_2W_2\|_{2,\infty} \leq \|Q_2\|_{2,\infty}\|W_2\|_{2,\infty} \\ &= \|W_2\|_{2,\infty} = \sqrt{\lambda_{\max}(D_2)} \end{aligned} \tag{11}$$

Thus, if we manage to find a good solution to (3), it guaranties some bound on the minimum of (2).

3 Second direction

Let H and $\tilde{H}$ be some $n \times n$ real symmetric matrices and assume that $\tilde{H}$ is a small perturbation of H, $\tilde{H} = H + \Delta H$. Denote by $H = U\Lambda U^T$ and $\tilde{H} = \tilde{U}\tilde{\Lambda}\tilde{U}^T$ the eigendecompositions of H and $\tilde{H}$. Suppose we denote the eigendecompositions by

$$H = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \begin{bmatrix} U_1^T \\ U_2^T \end{bmatrix} \tag{12}$$

and

$$\tilde{H} = \begin{bmatrix} \tilde{U}_1 & \tilde{U}_2 \end{bmatrix} \begin{bmatrix} \tilde{\Lambda}_1 & 0 \\ 0 & \tilde{\Lambda}_2 \end{bmatrix} \begin{bmatrix} \tilde{U}_1^T \\ \tilde{U}_2^T \end{bmatrix}, \tag{13}$$

such that

$$\begin{aligned} \Lambda_1 &= \operatorname{diag}(\lambda_1, \lambda_2, ..., \lambda_k) \\ \Lambda_2 &= \operatorname{diag}(\lambda_{k+1}, \lambda_{k+2}, ..., \lambda_n) \\ \tilde{\Lambda}_1 &= \operatorname{diag}(\tilde{\lambda}_1, \tilde{\lambda}_2, ..., \tilde{\lambda}_k) \\ \tilde{\Lambda}_2 &= \operatorname{diag}(\tilde{\lambda}_{k+1}, \tilde{\lambda}_{k+2}, ..., \tilde{\lambda}_n), \end{aligned} \tag{14}$$

where we construct the split as follows. Assume we choose an eigenvalue $\lambda \in \lambda(\Lambda)$ and split Λ into Λ_1, Λ_2, such that $\lambda = \lambda_1 = \lambda_2 = ... = \lambda_k$ and $\lambda \notin \lambda(\Lambda_2)$. So, if we define

$$\delta = \min_{\lambda_2 \in \lambda(\Lambda_2)} |\lambda_2 - \lambda|, \tag{15}$$

then $\delta > 0$. Then, we split $\tilde{\Lambda}$ into $\tilde{\Lambda}_1, \tilde{\Lambda}_2$, such that $\tilde{\Lambda}_1$ contains the eigenvalues that are nearest to λ, or, in other words, we would like to maximize δ_1, where

$$\delta_1 = \min_{\tilde{\lambda} \in \lambda(\tilde{\Lambda}_2)} |\tilde{\lambda} - \lambda|. \tag{16}$$

According to the *Bauer-Fike* theorem [1], for any $\lambda_i \in \lambda(\Lambda)$ there exists $\tilde{\lambda}_j \in \lambda(\tilde{\Lambda})$ such that

$$|\lambda_i - \tilde{\lambda}_j| \leq \|U\|_2\|U^{-1}\|_2\|\Delta H\|_2 = \|\Delta H\|_2, \tag{17}$$

and vice-versa - for any $\tilde{\lambda}_i \in \lambda(\tilde{\Lambda})$ there exists $\lambda_j \in \lambda(\Lambda)$ such that

$$|\lambda_j - \tilde{\lambda}_i| \leq \|\Delta H\|_2. \tag{18}$$

Note that when $\|\Delta H\|_2 \to 0$, we have $\tilde{\Lambda}_1 \to \Lambda_1$ and $\delta_1 \to \delta$.
Therefore, assuming $\|\Delta H\|_2$ is small enough, δ_1 can be bounded by

$$\delta_1 > \delta - \|\Delta H\|_2 > 0. \tag{19}$$

Next, let $\Theta(U_1, \tilde{U}_1) = \arccos\left((U_1^T \tilde{U}_1 \tilde{U}_1^T U_1)^{1/2}\right)$ be the canonical angle between the column spaces of U_1 and $\tilde{U}_1$, where the *arccos* acts as a matrix operator [4]. Then, according to [2], which based its results on [3],

$$\|E\|_F = \|\sin(\Theta(U_1, \tilde{U}_1))\|_F \leq \frac{\|\Delta H U_1\|_F}{\delta_1}, \tag{20}$$

where we defined $E = \sin(\Theta(U_1, \tilde{U}_1))$.
Since U_1 holds k orthonormal columns, $\|U_1\|_F^2 = k$, and we have

$$\|\Delta H U_1\|_F \leq \|\Delta H\|_F \|U_1\|_F = \sqrt{k}\|\Delta H\|_F. \tag{21}$$

Define $\epsilon = \frac{\sqrt{k}\|\Delta H\|_F}{\delta_1}$, so that we finally get

$$\|E\|_F \leq \epsilon. \tag{22}$$

Matrix operators such as *sin*, *cos* and *square-root* on some matrix A can be defined as applying them element-wise on the eigenvalues of A. Define $B = U_1^T \tilde{U}_1 \tilde{U}_1^T U_1$. B is symmetric, and hence, it is easy to see that $\Theta = \arccos(B^{1/2})$ and $E = \sin(\theta)$ must be symmetric matrices as well.

Denote by Λ_E, Λ_Θ and Λ_B the diagonal eigenvalues matrices of E, Θ and B and by $\lambda_{E,i}$, $\lambda_{\Theta,i}$ and $\lambda_{B,i}$ the i-th eigenvalue of the corresponding terms. We have,

$$\|\Lambda_E\|_F = \|E\|_F \leq \epsilon. \tag{23}$$

Suppose that $\|\Delta H\|_F \to 0$. Since $\delta_1 \to \delta > 0$ is bounded, we get $\epsilon \to 0$, and $\|\Lambda_E\|_F \to 0$, and consequently, for any i, $\lambda_{E,i} \to 0$.
By definition, $\lambda_{\Theta,i} = \arcsin(\lambda_{E,i})$. Since *arcsin* is continuous around 0, $\lambda_{\Theta,i} \to 0$. Similarly, *cos* and *square* are continuous so $\lambda_{B,i} \to 1$ and consequently $\Lambda_B \to I(n)$. B is symmetric and hence it can be written as an eigendecomposition

$$B = C_B \Lambda_B C_B^T, \tag{24}$$

where C_B is orthogonal. We have

$$B = U_1^T \tilde{U}_1 \tilde{U}_1^T U_1 = C_B \Lambda_B C_B^T. \tag{25}$$

Moving the orthogonal matrix C_B to the left, and denoting $V = \tilde{U}_1^T U_1 C_B$, we obtain

$$V^T V = C_B^T U_1^T \tilde{U}_1 \tilde{U}_1^T U_1 C_B = \Lambda_B. \tag{26}$$

This implies that the columns of V are orthogonal and their squared norms are equal to the eigenvalues of B. Therefore, there exists some orthogonal matrix C_V such that $V = C_V \sqrt{\Lambda_B}$. We have

$$V = \tilde{U}_1^T U_1 C_B = C_V \sqrt{\Lambda_B}, \tag{27}$$

and since U_1 and $\tilde{U}_1$ are orthogonal,

$$\tilde{U}_1^T = C_V \sqrt{\Lambda_B} C_B^T U_1^T. \tag{28}$$

or

$$U_1 C_U = \tilde{U}_1, \tag{29}$$

where $C_U = C_B \sqrt{\Lambda_B} C_V^T$.
Next, recall that by our construction, all values in the diagonal of Λ_1 are equal, or in other words, $\Lambda_1 = \lambda I$. Similarly, $\sqrt{\Lambda_1} = \sqrt{\lambda} I$, so it can be swapped with any matrix. We obtain

$$U_1 \sqrt{\Lambda_1} C_U = U_1 C_U \sqrt{\Lambda_1} = \tilde{U}_1 \sqrt{\Lambda_1}. \tag{30}$$

Define $C_\lambda = C_B C_V^T$, and note that if $\epsilon \to 0$ then $\Lambda_B \to I$ and hence $C_U = C_B\sqrt{\Lambda_B}C_V^T \to C_\lambda$. Therefore, there exists some orthogonal matrix C_λ such that for $\epsilon \to 0$,

$$\tilde{U}_1\sqrt{\tilde{\Lambda}_1} \to \tilde{U}_1\sqrt{\Lambda_1} = U_1\sqrt{\Lambda_1}C_U \to U_1\sqrt{\Lambda_1}C_\lambda. \tag{31}$$

To recap, after choosing some $\lambda \in \lambda(\Lambda)$ and splitting the eigendecomposition of H and $\tilde{H}$ accordingly, we obtained that there exists some orthogonal matrix C_λ such that

$$\tilde{U}_1\sqrt{\tilde{\Lambda}_1} \to U_1\sqrt{\Lambda_1}C_\lambda. \tag{32}$$

for $\|\Delta H\|_F \to 0$. It is possible now to choose a different $\lambda \in \lambda(\Lambda)$ and repeat this process. The obtained matrices C_λ can be then collected and inserted as blocks into the diagonal of a large matrix C, such that

$$\tilde{U}\sqrt{\tilde{\Lambda}} \to U\sqrt{\Lambda}C. \tag{33}$$

for $\|\Delta H\|_F \to 0$.
Back to our problem, define $H = PD_1P^T$ and $\tilde{H} = D_2 = PD_1P^T + \Delta D$. Then by construction, there exists some orthogonal matrix C such that

$$Q_2\sqrt{\Lambda_2} \to Q_1\sqrt{\Lambda_1}C, \tag{34}$$

or

$$\|PX_1C - X_2\|_{2,\infty} \to 0, \tag{35}$$

for $\|\Delta D\| \to 0$.

References

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