The Geometry of First-Returning Photons for Non-Line-of-Sight Imaging (Supplemental Material)

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1. Introduction

In this supplemental material, we cover the following topics:

- In Section 2, we visualize scenarios when the uniqueness of shortest path is violated.
- In Section 3, we prove that the supporting hyperplane at $\mathbf{p}^*$ is tangential to the ellipsoidal constraint.
- In Section 4, since the optimization for finding the mirrored light source is non-convex, we use algebraic manipulation on the objective function and use the algebraic minimizer to initialize the optimization process.
- Supplementary video that includes explanation of proposed method and 3D visualization of results.
  https://youtu.be/DbU2m0LEvbU

2. Uniqueness of the Shortest Path

We assume that the first-returning photon is generated from a single NLOS scene point or equivalently, the shortest three-bounce light path given a pair of sensing and illumination spots is unique. This assumption is always valid for convex NLOS scenes. However, there are some NLOS scenes that contain multiple scene points contributing to the first-returning photon observation. An example of this is shown in Fig. 1(a). However, when we move the illumination and sensing points — even slightly — the shortest light path becomes unique (see Fig. 1(b)).

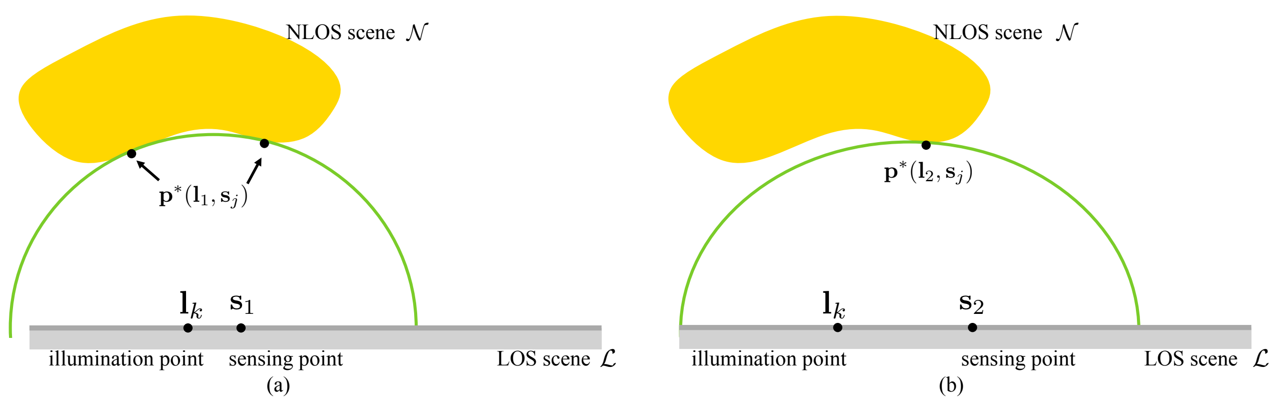

Figure 1. **The uniqueness of the shortest path.** (a) We show an example where there are multiple scene points contributing to the shortest path. That is, the NLOS scene intersects with the ellipsoidal constraint $E(\mathbf{l}_k, \mathbf{s}_1)$ at multiple points. Even though this means that the shortest path between $\mathbf{l}_k$ and $\mathbf{s}_1$ is not unique, when we move the illumination or the sensing point, this is no longer the case. (b) When we move the sensing to $\mathbf{s}_2$, now the NLOS scene only intersect with the ellipsoidal constraint at one location.

3. Algebraic Proof of Observation 2

Observation 2 states, suppose that NLOS scene is locally smooth at $\mathbf{p}^*$, the unique supporting hyperplane at $\mathbf{p}^*$ is tangential to the ellipsoid $E(\mathbf{l}_k, \mathbf{s}_j)$. Recall that from the ellipsoid property, the surface normal of supporting hyperplane of ellipsoid is the angular bisector. We now prove that prove the normal at $\mathbf{p}^*$ is also the angle bisector and hence, the supporting planes to the NLOS surface and the ellipsoid at $\mathbf{p}^*$ are one and the same.

Consider the setup in Figure 2. Light emitted from a source at $\mathbf{l}_k$ bounces off an object and reaches the point $\mathbf{s}_j$. The shortest among all such paths passes through the point $\mathbf{p}^*$ on the object. Given this, we prove that the surface normal at $\mathbf{p}^*$ — assuming that the object is locally smooth — satisfies

$$\mathbf{n} \propto \frac{\mathbf{l}_k - \mathbf{p}^*}{\|\mathbf{l}_k - \mathbf{p}^*\|} + \frac{\mathbf{s}_j - \mathbf{p}^*}{\|\mathbf{s}_j - \mathbf{p}^*\|},$$

or equivalently, the surface normal at $\mathbf{p}^*$ is the angular bisector of the vectors to $\mathbf{l}_k$ and $\mathbf{s}_j$, from $\mathbf{p}^*$.

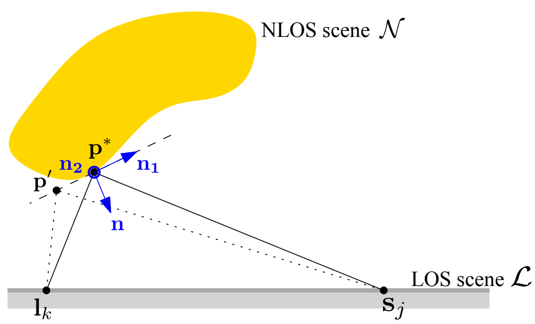

Figure 2. **Normal estimation of discovered NLOS scene point.** The normal of the scene patch corresponding to the first photon is the angle bisector. For all other points $\mathbf{p}'$ in the neighboring area, the time of flight will be larger compared to $\mathbf{p}$.

Since the NLOS object is locally smooth, therefore, points in an infinitesimal local neighborhood of the discovered $\mathbf{p}^*$ can be represented as

$$\mathbf{p}' = \mathbf{p}^* + \alpha\mathbf{n}_1 + \beta\mathbf{n}_2, \tag{1}$$

where $\mathbf{n}_1$ and $\mathbf{n}_2$ are the basis spanning the local planar approximation. That is, $\langle \mathbf{n}, \mathbf{n}_1 \rangle = 0$, $\langle \mathbf{n}, \mathbf{n}_2 \rangle = 0$, and $\langle \mathbf{n}_1, \mathbf{n}_2 \rangle = 0$, where $\mathbf{n}$ is the surface normal. We can now compute the distance from $\mathbf{l}_k$ to $\mathbf{s}_j$ via points on the local planar patch at $\mathbf{p}'$.

$$\begin{aligned} d(\mathbf{p}') &= \|\mathbf{p}' - \mathbf{l}_k\| + \|\mathbf{p}' - \mathbf{s}_j\| \\ &= \|\mathbf{p}^* + \alpha\mathbf{n}_1 + \beta\mathbf{n}_2 - \mathbf{l}_k\| + \|\mathbf{p}^* + \alpha\mathbf{n}_1 + \beta\mathbf{n}_2 - \mathbf{s}_j\| \end{aligned}$$

Since the shortest path in this local neighborhood happens for $\mathbf{p}' = \mathbf{p}^*$, the derivative of $d(\cdot)$ with respect to α and β is zero at $\mathbf{p}^* = \mathbf{p}'$, or equivalently, when $\alpha = \beta = 0$.

$$\begin{aligned} \frac{\partial d(\mathbf{p}')}{\partial \alpha} &= \frac{\mathbf{n}_1^T(\mathbf{p}^* + \alpha\mathbf{n}_1 + \beta\mathbf{n}_2 - \mathbf{l}_k)}{\|\mathbf{p}^* + \alpha\mathbf{n}_1 + \beta\mathbf{n}_2 - \mathbf{l}_k\|} + \frac{\mathbf{n}_1^T(\mathbf{p}^* + \alpha\mathbf{n}_1 + \beta\mathbf{n}_2 - \mathbf{s}_j)}{\|\mathbf{p}^* + \alpha\mathbf{n}_1 + \beta\mathbf{n}_2 - \mathbf{s}_j\|} \\ \frac{\partial d(\mathbf{p}')}{\partial \beta} &= \frac{\mathbf{n}_2^T(\mathbf{p}^* + \alpha\mathbf{n}_1 + \beta\mathbf{n}_2 - \mathbf{l}_k)}{\|\mathbf{p}^* + \alpha\mathbf{n}_1 + \beta\mathbf{n}_2 - \mathbf{l}_k\|} + \frac{\mathbf{n}_2^T(\mathbf{p}^* + \alpha\mathbf{n}_1 + \beta\mathbf{n}_2 - \mathbf{s}_j)}{\|\mathbf{p}^* + \alpha\mathbf{n}_1 + \beta\mathbf{n}_2 - \mathbf{s}_j\|} \end{aligned}$$

The minimum is reached when $\mathbf{p}' = \mathbf{p}^*$. That is, the derivative reaches 0 when $\alpha = 0$ and $\beta = 0$.

$$\begin{aligned} \frac{\partial d(\mathbf{p}')}{\partial \alpha}|_{\alpha=0,\beta=0} &= \mathbf{n}_1^T \left(\frac{\mathbf{p}^* - \mathbf{l}_k}{\|\mathbf{p}^* - \mathbf{l}_k\|} + \frac{\mathbf{p}^* - \mathbf{s}_j}{\|\mathbf{p}^* - \mathbf{s}_j\|} \right) = 0 \\ \frac{\partial d(\mathbf{p}')}{\partial \alpha}|_{\alpha=0,\beta=0} &= \mathbf{n}_2^T \left(\frac{\mathbf{p}^* - \mathbf{l}_k}{\|\mathbf{p}^* - \mathbf{l}_k\|} + \frac{\mathbf{p}^* - \mathbf{s}_j}{\|\mathbf{p}^* - \mathbf{s}_j\|} \right) = 0 \end{aligned}$$

The relationships above suggest that the vector $\left(\frac{\mathbf{p}^*-\mathbf{l}_k}{\|\mathbf{p}^*-\mathbf{l}_k\|}+\frac{\mathbf{p}^*-\mathbf{s}_j}{\|\mathbf{p}^*-\mathbf{s}_j\|}\right)$ is perpendicular to both $\mathbf{n}_1$ and $\mathbf{n}_2$ and hence, it must be aligned along the surface normal since the surface normal is perpendicular to both $\mathbf{n}_1$ and $\mathbf{n}_2$. Therefore,

$$\mathbf{n} \propto \frac{\mathbf{p}^*-\mathbf{l}_k}{\|\mathbf{p}^*-\mathbf{l}_k\|}+\frac{\mathbf{p}^*-\mathbf{s}_j}{\|\mathbf{p}^*-\mathbf{s}_j\|} \tag{2}$$

Here we show that the normal at $\mathbf{p}^*$ is equal to the angle bisector, which is also the normal of the supporting hyperplane of the ellipsoid at $\mathbf{p}^*$. Therefore, the supporting hyperplane is tangential to the ellipsoid.

4. Initialization for Algorithm 2

If we assume that the observations from sensing positions in a local neighborhood are caused by a locally planar NLOS scene, we can use the observed ToFs to jointly recover the plane. We can recover the locally planar scene patch by finding the mirrored position of the light source by solving

$$\min_{\mathbf{x}} \sum_{\mathbf{s}_j \in \Omega} (d(\mathbf{l}_k, \mathbf{s}_j) - \|\mathbf{x}-\mathbf{s}_j\|)^2. \tag{3}$$

The optimization problem in (3) is non-convex, thus the result depends on the initialization. We find algebraic minimizer of the objective function is a good initialization method.

Without loss of generosity, we assume the sensor are all on the same plane $z=0$. That is, $\mathbf{s}_j = (s_j^x, s_j^y, 0)$. We seek to find the location for the mirrored light source $\mathbf{l}'_k = (l^x, l^y, l^z)$.

By squaring the path length, we get

$$d^2(\mathbf{l}_k, \mathbf{s}_j) = (l^x - s_j^x)^2 + (l^y - s_j^y)^2 + (l^z - 0)^2 \tag{4}$$

For simplicity, we will denote $d(\mathbf{l}_k, \mathbf{s}_j)$ as d_j.

We use another sensor location $\mathbf{s}_i$ that belong to the same local neighborhood to jointly estimate (l^x, l^y, l^z).

$$d_i^2 = (l^x - s_i^x)^2 + (l^y - s_i^y)^2 + (l^z - 0)^2 \tag{5}$$

(5) - (4), we get

$$\begin{aligned} d_i^2 - d_j^2 &= \left[(l^x - s_i^x)^2 - (l^x - s_j^x)^2\right] + \left[(l^y - s_i^y)^2 - +(l^y - s_j^y)^2\right] \\ &= (2l^x - s_i^x - s_j^2)(s_j^x - s_i^x) + (2l^y - s_i^y - s_j^y)(s_j^y - s_i^y) \\ &= 2(s_j^x - s_i^x)l^x + 2(s_j^y - s_i^y)l^y - \left[(s_j^x)^2 - (s_i^x)^2\right] - \left[(s_j^y)^2 - (s_i^y)^2\right] \end{aligned}$$

We can see that the relationship is linear in l^x and l^y. Therefore, we can construct a linear system. Here, for ease for interpretation, we use $\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_m$ to represent sensors in the local neighborhood.

$$2\begin{bmatrix} (s_1^x - s_2^x) & (s_1^y - s_2^y) \\ (s_1^x - s_3^x) & (s_1^y - s_3^y) \\ \vdots & \vdots \\ (s_1^x - s_m^x) & (s_1^y - s_m^y) \end{bmatrix} \begin{bmatrix} l^x \\ l^y \end{bmatrix} = \begin{bmatrix} d_2^2 - d_1^2 + \left[(s_1^x)^2 - (s_2^x)^2\right] + \left[(s_1^y)^2 - (s_2^y)^2\right] \\ d_3^2 - d_1^2 + \left[(s_1^x)^2 - (s_3^x)^2\right] + \left[(s_1^y)^2 - (s_3^y)^2\right] \\ \vdots \\ d_m^2 - d_1^2 + \left[(s_1^x)^2 - (s_m^x)^2\right] + \left[(s_1^y)^2 - (s_m^y)^2\right] \end{bmatrix}$$

(l^x, l^y) can be solved using pseudo inverse. Finally, the mirrored light source location can be solved

$$l^z = \sqrt{d_j^2 - (l^x - s_j^x)^2 - (l^y - s_j^y)^2}$$