1. Proof of Proposition in Section 3.5.2

Proposition. Given a SPD matrix $X_{d \times d}$, let $U = [u_1, u_2, \ldots, u_d]$ and $D = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_d)$ denote the eigenvector and eigenvalue matrices in full, respectively. Let $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_d]^T$ be the power of the eigenvalues of $X$, i.e., $X(\alpha) = UD\alpha U^T$, and $\text{vec}(\cdot)$ denote the vectorisation of a matrix. It can be shown that $\text{vec}(\log(X(\alpha))) = \Gamma \alpha$, where $\Gamma = [v_1, v_2, \ldots, v_d]$ and $v_i \equiv \text{vec}((\log \lambda_i) u_i u_i^T)$, $v_i \in \mathbb{R}^{d^2}$. The columns of $\Gamma$ form a set of $d$ orthogonal bases spanning a $d$-dimensional subspace $\mathcal{V}$ in the whole space of $\mathbb{R}^{d^2}$.

Proof. By eigen-decomposition, we have $X = UDU^T = \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T + \cdots + \lambda_d u_d u_d^T$. It is easy to obtain that $\log(X) = U \log(D) U^T = \sum_{i=1}^d (\log \lambda_i) u_i u_i^T$ and $\log(X(\alpha)) = \sum_{i=1}^d (\alpha_i \log \lambda_i) u_i u_i^T$, where $\log(D)$ denotes the diagonal matrix obtained after applying the natural logarithm to the diagonal elements of $D$ (The square bracket $[\cdot]$ is used to differentiate it from the matrix logarithm). Therefore, it holds that

$$\text{vec}(\log(X(\alpha))) = \text{vec}((\alpha_1 \log \lambda_1) u_1 u_1^T) + \cdots + \text{vec}((\alpha_d \log \lambda_d) u_d u_d^T)$$

(1)

Now let's prove that $\{v_1, v_2, \cdots, v_d\}$ forms a set of $d$ orthogonal bases for a $d$-dimensional subspace (denoted by $\mathcal{V}$) in $\mathbb{R}^{d^2}$, that is, $\langle v_i, v_j \rangle = 0$, for all $1 < i < j < d$.

It can be shown that

$$\langle v_i, v_j \rangle = \langle \text{vec}((\log \lambda_i) u_i u_i^T), \text{vec}((\log \lambda_j) u_j u_j^T) \rangle$$

(2)

$$= \langle (\log \lambda_i) u_i u_i^T, (\log \lambda_j) u_j u_j^T \rangle_F$$

$$= \text{trace}((\log \lambda_i) u_i u_i^T (\log \lambda_j) u_j u_j^T)$$

$$= (\log \lambda_i \log \lambda_j) \text{trace}(u_i u_i^T u_j u_j^T)$$

$$= 0 \quad (\because \ u_i^T u_j = 0 \text{ as two eigenvectors})$$

In addition, it is trivial to show that $\|v_i\|^2 = (\log \lambda_i)^2$. Therefore, $\{v_1, v_1, \cdots, v_d\}$ forms a set of $d$ orthogonal bases for a subspace $\mathcal{V}$ in $\mathbb{R}^{d^2}$. \qed
2. Fifteen most difficult texture pairs in Brodatz data set used in Section 4.1

Figure 1. Fifteen most difficult texture pairs (with class labels) used in the binary classification experiment from Brodatz data set in Section 4.1.