Optimal and strong duality will hold for \((\mathbf{R}_i, \mathbf{R}_j, \mathbf{R}_k)\).

**Proof.** A sufficient condition for strong duality to hold is given by (23) and (24). For a cycle graph, we get \(\alpha\) as in Theorem 4.2. Let \(\mathbf{D}(\mathbf{R})\) denote the matrix block in (46) is positive semidefinite if and only if the two vectors \(v_\pm = (v_{\pm,1}, v_{\pm,2}, \ldots, v_{\pm,n})^T\), where \(v_{\pm,i} = \cos(\frac{\pi i}{n})\) for \(i = 1, \ldots, n\). Now it is straightforward to check that \(v_\pm\) are eigenvectors to (49) with eigenvalues \(4\sin(\frac{\pi}{n})\). The sign of the smallest of these two eigenvalues determines the positive definiteness of the matrix in (49). In other words, we have shown that if \(|\alpha| \leq \frac{\pi}{n}\) then \(\mathbf{D}(\mathbf{R}) \succeq 0\).

**Proof of Lemma 5.1.** Let \(\mathbf{B}\) be a positive semidefinite matrix. Then, the solution to (46) is given by:

\[
S^* = -\mathbf{B}W \left[ \left( \mathbf{W}^T \mathbf{B} \right)^2 \right]^{\frac{1}{2}}.
\]

**Proof.** From the Schur complement, we have that the two vectors \(v_\pm\) are eigenvectors to (49) with eigenvalues \(4\sin(\frac{\pi}{n})\). The sign of the smallest of these two eigenvalues determines the positive definiteness of the matrix in (49). In other words, we have shown that if \(|\alpha| \leq \frac{\pi}{n}\) then \(\mathbf{D}(\mathbf{R}) \succeq 0\).

**Proof of Theorem 4.2**

**Theorem 4.2.** Let \(\mathbf{R}_i\), \(i = 1, \ldots, n\) denote a stationary point to the primal problem \((P)\) for a cycle graph with \(n\) vertices. Let \(\alpha_{ij}\) denote the angular residuals, i.e., \(\alpha_{ij} = \angle(\mathbf{R}_i, \mathbf{R}_j, \mathbf{R}_k)\). Then, \(\mathbf{R}_i\), \(i = 1, \ldots, n\) will be globally optimal and strong duality will hold for \((P)\) if

\[
|\alpha_{ij}| \leq \frac{\pi}{n}, \quad \forall (i, j) \in E.
\]

**Proof.** A sufficient condition for strong duality to hold is that \(\Lambda^* - \mathbf{R} \succeq 0\) (Lemma 3.2), which is equivalent to \(\mathbf{D}(\Lambda^* - \mathbf{R}) \succeq 0\) with the same notation and argument as in (23) and (24). For a cycle graph, we get

\[
\mathbf{D}(\Lambda^* - \mathbf{R}) \mathbf{D}(\Lambda^* - \mathbf{R})^T = \begin{bmatrix}
\mathbf{E}_{12} + \mathbf{E}_{1n} - \mathbf{E}_{12} - \mathbf{E}_{1n} \\
-\mathbf{E}_{12}^T & \mathbf{E}_{12}^T + \mathbf{E}_{23} - \mathbf{E}_{23} \\
-\mathbf{E}_{23}^T & \ddots & \ddots \\
-\mathbf{E}_{1n}^T & \ddots & \ddots & -\mathbf{E}_{1n}^T \\
-\mathbf{E} & \mathbf{E} & \cdots & \mathbf{E} \\
-\mathbf{E} & \mathbf{E} & \cdots & \mathbf{E} \\
\mathbf{E} & \mathbf{E} & \cdots & \mathbf{E} \\
\mathbf{E} & \mathbf{E} & \cdots & \mathbf{E}
\end{bmatrix}. 
\]

(48)

As this matrix is symmetric, it implies for the first diagonal block that \(\mathbf{E}_{12} = \mathbf{E}_{12}^T = \mathbf{E}_{1n} = \mathbf{E}_{1n}^T\). As all \(\alpha_{ij} \in \text{SO}(3)\), it follows that \(\mathbf{E}_{12} = \mathbf{E}_{1n} = \mathbf{E}\) for some rotation \(\mathbf{E} \in \text{SO}(3)\).

Similarly, for the second diagonal block \(\mathbf{E}_{23} = \mathbf{E}_{23}^T = \mathbf{E}\) and by induction, the matrix \(\mathbf{D}(\Lambda^* - \mathbf{R}) \mathbf{D}(\Lambda^* - \mathbf{R})^T\) has the following tridiagonal (Laplacian-like) structure

\[
\begin{bmatrix}
\mathbf{E} & \mathbf{E} & \cdots & \mathbf{E} \\
-\mathbf{E} & \mathbf{E} & \cdots & \mathbf{E} \\
\mathbf{E} & \mathbf{E} & \cdots & \mathbf{E} \\
\mathbf{E} & \mathbf{E} & \cdots & \mathbf{E} \\
\end{bmatrix}.
\]

(49)

Note that this means that the total error is equally distributed in an optimal solution among all the residuals, in particular, \(\alpha_{ij} = \alpha\) for all \((i, j) \in E\), where \(\alpha\) is the residual rotation angle of \(\mathbf{E}\).

Let \(u\) denote the rotation axis of \(\mathbf{E}\) and let \(v\) and \(w\) be an orthogonal base which is orthogonal to \(v\). Then, define

\[
\min_{S \in \mathbb{R}^{3 \times 3}} \langle S, W \rangle \quad \text{s.t.} \quad I - S^T B^T S \succeq 0,
\]

(53a)

\[
(I - BB^T)S = 0.
\]

(53b)

The KKT conditions for (53), with Lagrangian multipliers \(\Gamma\) and \(\Upsilon\), become

\[
W + 2B^T S \Gamma + (I - BB^T) \Upsilon = 0,
\]

(54)

\[
I - S^T B^T S \succeq 0,
\]

(55)

\[
(I - BB^T)S = 0.
\]

(56)

\[
\Gamma \succeq 0,
\]

(57)

\[
(I - S^T B^T S) \Gamma = 0.
\]

(58)

Rewrite (54) and (58) as

\[
B^T S \Gamma = -\frac{1}{2} W - \frac{1}{2} (I - BB^T) \Upsilon,
\]

(59)

\[
\Gamma^T \Gamma = \Gamma^T S^T B^T S \Gamma.
\]

(60)
Since the pseudoinverse fulfills $B^\dagger BB^\dagger = B^\dagger$, combining (59) and (60) we obtain

$$\Gamma^2 = \Gamma^T S^T B^\dagger BB^\dagger S \Gamma = \frac{1}{4} (W + (I - BB^\dagger)\Upsilon)^T B (W + (I - BB^\dagger)\Upsilon) = \frac{1}{4} W^T BW. \quad (61)$$

Here the last equality follows since $B(I - BB^\dagger) = 0$. This gives

$$\Gamma = \frac{1}{2} \left( W^T BW \right)^{\frac{1}{2}}. \quad (62)$$

Inserting (64) in (59)

$$B^\dagger S \left( W^T BW \right)^{\frac{1}{2}} = -W - (I - BB^\dagger)\Upsilon, \quad (63)$$

multiplying with $B$ form the left on both sides and using (56), $BB^\dagger S = S$, we arrive at

$$S \left( W^T BW \right)^{\frac{1}{2}} = -BW, \quad (64)$$

and consequently

$$S = -BW \left[ \left( W^T BW \right)^{\frac{1}{2}} \right]^\dagger. \quad (65)$$

Finally, since

$$\Gamma = \frac{1}{2} \left( W^T BW \right)^{\frac{1}{2}} \succeq 0, \quad (66)$$

$$I - S^T B^\dagger S = I - \left[ \left( W^T BW \right)^{\frac{1}{2}} \right]^\dagger W^T BW \left[ \left( W^T BW \right)^{\frac{1}{2}} \right]^\dagger \succeq 0, \quad (67)$$

the conditions (55) and (57) are satisfied then (50) must be a feasible and optimal solution to (53) and consequently also to (46).