Analytical Modeling of Vanishing Points and Curves in Catadioptric Cameras

(Supplementary Material)

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Contents

In this supplementary material, we show some important derivations and numerical examples supporting the paper *Analytical Modeling of Vanishing Points and Curves in Catadioptric Cameras*.

A. Appendix for the computation of vanishing points for a given directions Secs. 2.1 and 2.2

In this appendix we show the coefficients of the polynomial equations used to compute the coordinates of the vanishing point in the mirror, i.e. $\kappa_9^3[y,z]$ and $\kappa_{10}^4[y,z]$:

$$a_1 = 8 s_3 c_2 \tag{1}$$
$$a_2 = 4 s_3 A^2 - 4 s_3 A \tag{2}$$
$$a_3 = 4 A B s_3 - 8 A s_2 c_2 + 8 A s_3 c_3 \tag{3}$$
$$a_4 = B^2 s_3 - 4 C s_3 - 4 B s_2 c_2 + 4 B s_3 c_3 \tag{4}$$
$$a_5 = -4 s_2 A^2 + 4 s_2 A \tag{5}$$
$$a_6 = 4 B s_2 - 8 A B s_2 - 4 A s_2 c_3 + 4 A s_3 c_2 - 4 A^2 s_2 c_3 - 4 A^2 s_3 c_2 \tag{6}$$
$$a_7 = 8 A C s_2 - 3 B^2 s_2 - 4 C s_2 - 4 B s_2 c_3 + 4 B s_3 c_2 - 4 A B s_2 c_3 - 4 A B s_3 c_2 \tag{7}$$
$$a_8 = 4 B C s_2 + 4 C s_2 c_3 - 4 C s_3 c_2 - B^2 s_2 c_3 - B^2 s_3 c_2, \tag{8}$$

and

$$b_1 = s_1^2 (2A - 2)^2 + (2 s_2 - 2 A s_2)^2 \tag{9}$$
$$b_2 = 2(B + 2c_3)(2A - 2) s_1^2 - 2(2 s_2 - 2 A s_2)(B s_2 + 2 s_2 c_3 - 2 s_3 c_2) \tag{10}$$
$$b_3 = (B s_2 + 2 s_2 c_3 - 2 s_3 c_2)^2 + s_1^2 (B + 2 c_3)^2 \tag{11}$$
$$b_4 = -4 A s_1^2 c_2 (2A - 2) \tag{12}$$
$$b_5 = -s_1^2 (2 B c_2 (2A - 2) + 4 A c_2 (B + 2 c_3)) \tag{13}$$
$$b_6 = -2 B s_1^2 c_2 (B + 2 c_3) \tag{14}$$
$$b_7 = A (2 s_2 - 2 A s_2)^2 \tag{15}$$
$$b_8 = B (2 s_2 - 2 A s_2)^2 - 2 A (2 s_2 - 2 A s_2)(B s_2 + 2 s_2 c_3 - 2 s_3 c_2) \tag{16}$$
$$b_9 = A (B s_2 + 2 s_2 c_3 - 2 s_3 c_2)^2 - C (2 s_2 - 2 A s_2)^2 + 4 A^2 s_1^2 c_2^2 - 2 B (2 s_2 - 2 A s_2)(B s_2 + 2 s_2 c_3 - 2 s_3 c_2) \tag{17}$$
$$b_{10} = B (B s_2 + 2 s_2 c_3 - 2 s_3 c_2)^2 + 2 C (2 s_2 - 2 A s_2)(B s_2 + 2 s_2 c_3 - 2 s_3 c_2) + 4 A B s_1^2 c_2^2 \tag{18}$$
$$b_{11} = B^2 s_1^2 c_2^2 - C (B s_2 + 2 s_2 c_3 - 2 s_3 c_2)^2. \tag{19}$$

The coefficients of the polynomial equations shown above were derived for the general case. When considering specific cases, these coefficients are simplified significantly and, in many cases, they are eliminated. As a consequence, in many cases, the degree of polynomial equations are reduced. Results for general axial, axial with a spherical mirror, axial with an ellipsoid mirror, axial with conical mirror, and axial with cylindrical mirror are shown below.

A.1. Axial Case

In the axial case, the camera movement has to be restricted to motion in the u_z axis, therefore, $c_2 = 0$. The general equations become:

$$\kappa_9^3[y,z] = a_1 y z^2 + a_2 y z + a_3 y + a_4 z^3 + a_5 z^2 + a_6 z + a_7, \tag{20}$$

with:

$$a_1 = 4s_3A^2 - 4s_3A \tag{21}$$
$$a_2 = 4ABs_3 + 8As_3c_3 \tag{22}$$
$$a_3 = s_3B^2 + 4s_3c_3B - 4Cs_3 \tag{23}$$
$$a_4 = -4s_2A^2 + 4s_2A \tag{24}$$
$$a_5 = 4Bs_2 - 8ABs_2 - 4As_2c_3 - 4A^2s_2c_3 \tag{25}$$
$$a_6 = 48ACs_2 - 3B^2s_2 - 4Cs_2 - 4Bs_2c_3 - 4ABs_2c_3 \tag{26}$$
$$a_7 = s_2c_3B^2 + 4Cs_2B + 4Cs_2c_3, \tag{27}$$

and:

$$\kappa_{10}^4\,[y,z] = b_1y^2z^2 + b_2y^2z + b_3y^2 + b_4z^4 + b_5z^3 + b_6z^2 + b_7z + b_8, \tag{28}$$

with:

$$b_1 = s_1^2(2A-2)^2 + (2s_2 - 2As_2)^2 \tag{29}$$
$$b_2 = 2(B+2c_3)(2A-2)s_1^2 - 2(2s_2 - 2As_2)(Bs_2 + 2s_2c_3) \tag{30}$$
$$b_3 = s_1^2(B+2c_3)^2 + (Bs_2 + 2s_2c_3)^2 \tag{31}$$
$$b_4 = A(2s_2 - 2As_2)^2 \tag{32}$$
$$b_5 = B(2s_2 - 2As_2)^2 - 2A(2s_2 - 2As_2)(Bs_2 + 2s_2c_3) \tag{33}$$
$$b_6 = A(Bs_2 + 2s_2c_3)^2 - C(2s_2 - 2As_2)^2 - 2B(2s_2 - 2As_2)(Bs_2 + 2s_2c_3) \tag{34}$$
$$b_7 = B(Bs_2 + 2s_2c_3)^2 + 2C(2s_2 - 2As_2)(Bs_2 + 2s_2c_3) \tag{35}$$
$$b_8 = -C(Bs_2 + 2s_2c_3)^2. \tag{36}$$
$$\tag{37}$$

A.2. Spherical mirror ($A = 1$, $B = 0$)

Constraints for the spherical mirror are:

$$\kappa_9^2\,[y,z] = a_1yz + a_2y + a_3z^2 + a_4z + a_5, \tag{38}$$

with:

$$a_1 = 8s_3c_3 \tag{39}$$
$$a_2 = -4Cs_3 \tag{40}$$
$$a_3 = -8s_2c_3 \tag{41}$$
$$a_4 = 4Cs_2 \tag{42}$$
$$a_5 = 4Cs_2c_3, \tag{43}$$

and:

$$\kappa_{10}^2\,[y,z] = b_1y^2 + b_2z^2 + b_3 \tag{44}$$

with:

$$b_1 = 4s_1^2c_3^2 + 4s_2^2c_3^2 \tag{45}$$
$$b_2 = 4s_2^2c_3^2 \tag{46}$$
$$b_3 = -4Cs_2^2c_3^2. \tag{47}$$

A.3. Ellipsoid mirror ($A = 0$, $C = 0$)

Constraints for the ellipsoid mirror are:

$$\kappa_9^2\,[y,z] = a_1y + a_2z^2 + a_3z + a_4, \tag{48}$$

with:

$$a_1 = s_3 B^2 + 4 s_3 c_3 B \quad (49)$$
$$a_2 = 4 B s_2 \quad (50)$$
$$a_3 = -3 s_2 B^2 - 4 s_2 c_3 B \quad (51)$$
$$a_4 = -B^2 s_2 c_3, \quad (52)$$

and:

$$\kappa_{10}^4 [y, z] = b_1 y^2 z^2 + b_2 y^2 z + b_3 y^2 + b_4 z^3 + b_5 z^2 + b_6 z, \quad (53)$$

with:

$$b_1 = 4 s_1^2 + 4 s_2^2 \quad (54)$$
$$b_2 = (-4B - 8c_3) s_1^2 - 4 s_2 (B s_2 + 2 s_2 c_3) \quad (55)$$
$$b_3 = s_1^2 (B + 2c_3)^2 + (B s_2 + 2 s_2 c_3)^2 \quad (56)$$
$$b_4 = 4 B s_2^2 \quad (57)$$
$$b_5 = -4 B s_2 (B s_2 + 2 s_2 c_3) \quad (58)$$
$$b_6 = B (B s_2 + 2 s_2 c_3)^2. \quad (59)$$

A.4. Conical mirror ($B = 0$, $C = 0$)

Constraints for the conical mirror are:

$$\kappa_9^3 [y, z] = a_1 y z^2 + a_2 y z + a_3 z^3 + a_4 z^2, \quad (60)$$

with:

$$a_1 = 4 s_3 A^2 - 4 s_3 A \quad (61)$$
$$a_2 = 8 A s_3 c_3 \quad (62)$$
$$a_3 = -4 s_2 A^2 + 4 s_2 A \quad (63)$$
$$a_4 = -4 s_2 c_3 A^2 - 4 s_2 c_3 A, \quad (64)$$

and:

$$\kappa_{10}^4 [y, z] = b_1 y^2 z^2 + b_2 y^2 z + b_3 y^2 + b_4 z^4 + b_5 z^3 + b_6 z^2, \quad (65)$$

with:

$$b_1 = s_1^2 (2A - 2)^2 + (2 s_2 - 2 A s_2)^2 \quad (66)$$
$$b_2 = 4 c_3 (2A - 2) s_1^2 - 4 s_2 c_3 (2 s_2 - 2 A s_2) \quad (67)$$
$$b_3 = 4 s_1^2 c_3^2 + 4 s_2^2 c_3^2 \quad (68)$$
$$b_4 = A (2 s_2 - 2 A s_2)^2 \quad (69)$$
$$b_5 = -4 A s_2 c_3 (2 s_2 - 2 A s_2) \quad (70)$$
$$b_6 = 4 A s_2^2 c_3^2. \quad (71)$$

A.5. Cylindrical mirror ($A = 0$, $B = 0$)

Constraints for the cylindrical mirror are:

$$\kappa_9^1 [y, z] = a_1 y + a_2 z + a_3, \quad (72)$$

with:

$$a_1 = -4 C s_3 \quad (73) \qquad a_2 = -4 C s_2 \quad (74) \qquad a_3 = 4 C s_2 c_3, \quad (75)$$

and:

$$\kappa_{10}^4 [y, z] = b_1 y^2 z^2 + b_2 y^2 z + b_3 y^2 + b_4 z^2 + b_5 z + b_6, \quad (76)$$

with:

$$b_1 = 4s_1^2 + 4s_2^2 \quad (77)$$

$$b_2 = -8c_3s_1^2 - 8c_3s_2^2 \quad (78)$$

$$b_3 = 4s_1^2c_3^2 + 4s_2^2c_3^2 \quad (79)$$

$$b_4 = -4Cs_2^2 \quad (80)$$

$$b_5 = 8Cs_2^2c_3 \quad (81)$$

$$b_6 = -4Cs_2^2c_3^2. \quad (82)$$

A.6. Compute Vanishing Points

Following the derivations described in Sec. 2.2 (Computing Vanishing Points from a Given Direction), to compute vanishing points, we end up with a 10th degree polynomial (in the general case), as defined below:

$$\kappa_{16}^{10}[z] = c_1z^{10} + c_2z^9 + c_3z^8 + c_4z^7 + c_5z^6 + c_6z^5 + c_7z^4 + c_8z^3 + c_9z^2 + c_{10}z^1 + c_{11}, \quad (83)$$

where:

$$c_1 = (b_7a_2^2b_1 + a_5^2b_1^2)/a_1^2 \quad (84)$$

$$c_2 = (b_8a_2^2b_1 + b_2b_7a_2^2 - b_4a_2a_5b_1 + 2a_3b_7a_2b_1 + 2b_2a_5^2b_1 + 2a_6a_5b_1^2 - 2a_1b_7a_5b_1)/a_1^2 \quad (85)$$

$$\begin{aligned} c_3 = (&a_1^2b_7^2 - b_4a_1a_2b_7 - 2b_8a_1a_5b_1 - 2a_1a_5b_2b_7 - 2a_1a_6b_1b_7 + b_9a_2^2b_1 + b_8a_2^2b_2 + b_3a_2^2b_7 + \\ &+ 2b_8a_2a_3b_1 + 2a_2a_3b_2b_7 - b_5a_2a_5b_1 - b_4a_2a_5b_2 - b_4a_2a_6b_1 + 2a_4a_2b_1b_7 + a_3^2b_1b_7 - b_4a_3a_5b_1 + \\ &+ 2b_3a_5^2b_1 + a_5^2b_2^2 + 4a_5a_6b_1b_2 + 2a_7a_5b_1^2 + a_6^2b_1^2)/a_1^2 \end{aligned} \quad (86)$$

$$\begin{aligned} c_4 = (&2b_7b_8a_1^2 - b_8a_1a_2b_4 - b_5b_7a_1a_2 - b_7a_1a_3b_4 - 2b_9a_1a_5b_1 - 2b_8a_1a_5b_2 + a_1a_5b_4^2 - 2b_3b_7a_1a_5 - \\ &- 2b_8a_1a_6b_1 - 2b_7a_1a_6b_2 - 2a_7b_7a_1b_1 + b_{10}a_2^2b_1 + b_9a_2^2b_2 + b_3b_8a_2^2 + 2b_9a_2a_3b_1 + 2b_8a_2a_3b_2 + \\ &+ 2b_3b_7a_2a_3 - b_6a_2a_5b_1 - b_5a_2a_5b_2 - b_3a_2a_5b_4 - b_5a_2a_6b_1 - a_2a_6b_2b_4 - a_7a_2b_1b_4 + 2a_4b_8a_2b_1 + \\ &+ 2a_4b_7a_2b_2 + b_8a_3^2b_1 + b_7a_3^2b_2 - b_5a_3a_5b_1 - a_3a_5b_2b_4 - a_3a_6b_1b_4 + 2a_4b_7a_3b_1 + 2b_3a_5^2b_2 + \\ &+ 4b_3a_5a_6b_1 + 2a_5a_6b_2^2 + 2a_8a_5b_1^2 + 4a_7a_5b_1b_2 - a_4a_5b_1b_4 + 2a_6^2b_1b_2 + 2a_7a_6b_1^2)/a_1^2 \end{aligned} \quad (87)$$

$$\begin{aligned} c_5 = -(&-a_1^2b_8^2 - 2b_7b_9a_1^2 + b_9a_1a_2b_4 + b_5a_1a_2b_8 + b_6b_7a_1a_2 + a_1a_3b_4b_8 + b_5b_7a_1a_3 + b_7a_1a_4b_4 + \\ &+ 2b_{10}a_1a_5b_1 + 2b_9a_1a_5b_2 + 2a_1a_5b_3b_8 - 2b_5a_1a_5b_4 + 2b_9a_1a_6b_1 + 2a_1a_6b_2b_8 + 2b_7a_1a_6b_3 - a_1a_6b_4^2 + \\ &+ 2a_1a_7b_1b_8 + 2b_7a_1a_7b_2 + 2a_8b_7a_1b_1 - b_{11}a_2^2b_1 - b_{10}a_2^2b_2 - b_9a_2^2b_3 - 2b_{10}a_2a_3b_1 - 2b_9a_2a_3b_2 - \\ &- 2a_2a_3b_3b_8 - 2b_9a_2a_4b_1 - 2a_2a_4b_2b_8 - 2b_7a_2a_4b_3 + b_6a_2a_5b_2 + b_5a_2a_5b_3 + b_6a_2a_6b_1 + b_5a_2a_6b_2 + \\ &+ a_2a_6b_3b_4 + b_5a_2a_7b_1 + a_2a_7b_2b_4 + a_8a_2b_1b_4 - b_9a_3^2b_1 - a_3^2b_2b_8 - b_7a_3^2b_3 - 2a_3a_4b_1b_8 - \\ &- 2b_7a_3a_4b_2 + b_6a_3a_5b_1 + b_5a_3a_5b_2 + a_3a_5b_3b_4 + b_5a_3a_6b_1 + a_3a_6b_2b_4 + a_3a_7b_1b_4 - b_7a_4^2b_1 + \\ &+ b_5a_4a_5b_1 + a_4a_5b_2b_4 + a_4a_6b_1b_4 - a_5^2b_3^2 - 4a_5a_6b_2b_3 - 4a_5a_7b_1b_3 - 2a_5a_7b_2^2 - 4a_8a_5b_1b_2 - \\ &- 2a_6^2b_1b_3 - a_6^2b_2^2 - 4a_6a_7b_1b_2 - 2a_8a_6b_1^2 - a_7^2b_1^2)/a_1^2 \end{aligned} \quad (88)$$

$$\begin{aligned} c_6 = -(&a_2a_5b_3b_6 - a_1a_7b_4^2 - 2a_5a_6b_3^2 - 2a_5a_8b_2^2 - 2a_6a_7b_2^2 - 2a_7a_8b_1^2 - 2a_7^2b_1b_2 - 2a_6^2b_2b_3 - a_4^2b_1b_8 - \\ &- a_4^2b_2b_7 - a_3^2b_1b_{10} - a_3^2b_2b_9 - a_3^2b_3b_8 - a_2^2b_2b_{11} - a_2^2b_3b_{10} - 2a_1^2b_7b_{10} - 2a_1^2b_8b_9 - 2a_1a_5b_4b_6 - \\ &- 2a_1a_6b_4b_5 - a_1a_5b_5^2 + a_2a_6b_2b_6 + a_2a_6b_3b_5 + a_2a_7b_1b_6 + a_2a_7b_2b_5 + a_2a_7b_3b_4 + a_2a_8b_1b_5 + \\ &+ a_2a_8b_2b_4 + a_3a_5b_2b_6 + a_3a_5b_3b_5 + a_3a_6b_1b_6 + a_3a_6b_2b_5 + a_3a_6b_3b_4 + a_3a_7b_1b_5 + a_3a_7b_2b_4 + \\ &+ a_3a_8b_1b_4 + a_4a_5b_1b_6 + a_4a_5b_2b_5 + a_4a_5b_3b_4 + a_4a_6b_1b_5 + a_4a_6b_2b_4 + a_4a_7b_1b_4 + a_1a_2b_4b_{10} + \\ &+ a_1a_2b_5b_9 + a_1a_2b_6b_8 + a_1a_3b_4b_9 + a_1a_3b_5b_8 + a_1a_3b_6b_7 + a_1a_4b_4b_8 + a_1a_4b_5b_7 - 2a_2a_3b_1b_{11} - \\ &- 2a_2a_3b_2b_{10} - 2a_2a_3b_3b_9 - 2a_2a_4b_1b_{10} - 2a_2a_4b_2b_9 - 2a_2a_4b_3b_8 - 2a_3a_4b_1b_9 - 2a_3a_4b_2b_8 - \\ &- 2a_3a_4b_3b_7 - 4a_5a_7b_2b_3 - 4a_5a_8b_1b_3 - 4a_6a_7b_1b_3 - 4a_6a_8b_1b_2 + 2a_1a_5b_1b_{11} + 2a_1a_5b_2b_{10} + \\ &+ 2a_1a_5b_3b_9 + 2a_1a_6b_1b_{10} + 2a_1a_6b_2b_9 + 2a_1a_6b_3b_8 + 2a_1a_7b_1b_9 + 2a_1a_7b_2b_8 + 2a_1a_7b_3b_7 + \\ &+ 2a_1a_8b_1b_8 + 2a_1a_8b_2b_7)/a_1^2 \end{aligned} \quad (89)$$

$$
\begin{aligned}
c_7 =& -(a_2a_6b_3b_6 - a_7^2b_2^2 - a_8^2b_1^2 - a_1^2b_9^2 - a_1a_6b_5^2 - a_1a_8b_4^2 - 2a_5a_7b_3^2 - 2a_6a_8b_2^2 - 2a_7^2b_1b_3 - a_4^2b_1b_9 - \\
&- a_4^2b_2b_8 - a_4^2b_3b_7 - a_3^2b_1b_{11} - a_3^2b_2b_{10} - a_3^2b_3b_9 - a_2^2b_3b_{11} - 2a_1^2b_7b_{11} - 2a_1^2b_8b_{10} - 2a_1a_5b_5b_6 - \\
&- 2a_1a_6b_4b_6 - 2a_1a_7b_4b_5 - a_6^2b_3^2 + a_2a_7b_2b_6 + a_2a_7b_3b_5 + a_2a_8b_1b_6 + a_2a_8b_2b_5 + a_2a_8b_3b_4 + \\
&+ a_3a_5b_3b_6 + a_3a_6b_2b_6 + a_3a_6b_3b_5 + a_3a_7b_1b_6 + a_3a_7b_2b_5 + a_3a_7b_3b_4 + a_3a_8b_1b_5 + a_3a_8b_2b_4 + \\
&+ a_4a_5b_2b_6 + a_4a_5b_3b_5 + a_4a_6b_1b_6 + a_4a_6b_2b_5 + a_4a_6b_3b_4 + a_4a_7b_1b_5 + a_4a_7b_2b_4 + a_4a_8b_1b_4 + \\
&+ a_1a_2b_4b_{11} + a_1a_2b_5b_{10} + a_1a_2b_6b_9 + a_1a_3b_4b_{10} + a_1a_3b_5b_9 + a_1a_3b_6b_8 + a_1a_4b_4b_9 + a_1a_4b_5b_8 + \\
&+ a_1a_4b_6b_7 - 2a_2a_3b_2b_{11} - 2a_2a_3b_3b_{10} - 2a_2a_4b_1b_{11} - 2a_2a_4b_2b_{10} - 2a_2a_4b_3b_9 - 2a_3a_4b_1b_{10} - \\
&- 2a_3a_4b_2b_9 - 2a_3a_4b_3b_8 - 4a_5a_8b_2b_3 - 4a_6a_7b_2b_3 - 4a_6a_8b_1b_3 - 4a_7a_8b_1b_2 + 2a_1a_5b_2b_{11} + \\
&+ 2a_1a_5b_3b_{10} + 2a_1a_6b_1b_{11} + 2a_1a_6b_2b_{10} + 2a_1a_6b_3b_9 + 2a_1a_7b_1b_{10} + 2a_1a_7b_2b_9 + 2a_1a_7b_3b_8 + \\
&+ 2a_1a_8b_1b_9 + 2a_1a_8b_2b_8 + 2a_1a_8b_3b_7)/a_1^2
\end{aligned} \tag{90}
$$

$$
\begin{aligned}
c_8 =& -(a_2a_7b_3b_6 - a_1a_7b_5^2 - 2a_5a_8b_3^2 - 2a_6a_7b_3^2 - 2a_7a_8b_2^2 - 2a_8^2b_1b_2 - 2a_7^2b_2b_3 - a_4^2b_1b_{10} - \\
&- a_4^2b_2b_9 - a_4^2b_3b_8 - a_3^2b_2b_{11} - a_3^2b_3b_{10} - 2a_1^2b_8b_{11} - 2a_1^2b_9b_{10} - 2a_1a_6b_5b_6 - 2a_1a_7b_4b_6 - \\
&- 2a_1a_8b_4b_5 - a_1a_5b_6^2 + a_2a_8b_2b_6 + a_2a_8b_3b_5 + a_3a_6b_3b_6 + a_3a_7b_2b_6 + a_3a_7b_3b_5 + a_3a_8b_1b_6 + \\
&+ a_3a_8b_2b_5 + a_3a_8b_3b_4 + a_4a_5b_3b_6 + a_4a_6b_2b_6 + a_4a_6b_3b_5 + a_4a_7b_1b_6 + a_4a_7b_2b_5 + a_4a_7b_3b_4 + \\
&+ a_4a_8b_1b_5 + a_4a_8b_2b_4 + a_1a_2b_5b_{11} + a_1a_2b_6b_{10} + a_1a_3b_4b_{11} + a_1a_3b_5b_{10} + a_1a_3b_6b_9 + \\
&+ a_1a_4b_4b_{10} + a_1a_4b_5b_9 + a_1a_4b_6b_8 - 2a_2a_3b_3b_{11} - 2a_2a_4b_2b_{11} - 2a_2a_4b_3b_{10} - 2a_3a_4b_1b_{11} - \\
&- 2a_3a_4b_2b_{10} - 2a_3a_4b_3b_9 - 4a_6a_8b_2b_3 - 4a_7a_8b_1b_3 + 2a_1a_5b_3b_{11} + 2a_1a_6b_2b_{11} + 2a_1a_6b_3b_{10} + \\
&+ 2a_1a_7b_1b_{11} + 2a_1a_7b_2b_{10} + 2a_1a_7b_3b_9 + 2a_1a_8b_1b_{10} + 2a_1a_8b_2b_9 + 2a_1a_8b_3b_8)/a_1^2
\end{aligned} \tag{91}
$$

$$
\begin{aligned}
c_9 =& -(-a_1^2b_{10}^2 - 2b_9b_{11}a_1^2 + b_{11}a_1a_3b_5 + a_1a_3b_6b_{10} + a_1a_4b_5b_{10} + b_9a_1a_4b_6 + b_4b_{11}a_1a_4 + 2b_{11}a_1a_7b_2 + \\
&+ 2a_1a_7b_3b_{10} - 2a_1a_7b_5b_6 + 2a_1a_8b_2b_{10} + 2b_9a_1a_8b_3 - a_1a_8b_5^2 - 2b_4a_1a_8b_6 + 2b_1b_{11}a_1a_8 + \\
&+ 2a_6b_{11}a_1b_3 - a_6a_1b_6^2 + a_2b_{11}a_1b_6 - b_{11}a_3^2b_3 - 2b_{11}a_3a_4b_2 - 2a_3a_4b_3b_{10} + a_3a_7b_3b_6 + a_3a_8b_2b_6 + \\
&+ a_3a_8b_3b_5 - a_4^2b_2b_{10} - b_9a_4^2b_3 - b_1b_{11}a_4^2 + a_4a_7b_2b_6 + a_4a_7b_3b_5 + a_4a_8b_2b_5 + b_4a_4a_8b_3 + \\
&+ b_1a_4a_8b_6 + a_6a_4b_3b_6 - 2a_2b_{11}a_4b_3 - a_7^2b_3^2 - 4a_7a_8b_2b_3 - a_8^2b_2^2 - 2b_1a_8^2b_3 - 2a_6a_8b_3^2 + \\
&+ a_2a_8b_3b_6)/a_1^2
\end{aligned} \tag{92}
$$

$$
\begin{aligned}
c_{10} =& -(-2b_{10}b_{11}a_1^2 + b_{10}a_1a_4b_6 + b_5b_{11}a_1a_4 + 2b_{10}a_1a_8b_3 - 2b_5a_1a_8b_6 + 2b_2b_{11}a_1a_8 + \\
&+ 2a_7b_{11}a_1b_3 - a_7a_1b_6^2 + a_3b_{11}a_1b_6 - b_{10}a_4^2b_3 - b_2b_{11}a_4^2 + b_5a_4a_8b_3 + b_2a_4a_8b_6 + a_7a_4b_3b_6 - \\
&- 2a_3b_{11}a_4b_3 - 2b_2a_8^2b_3 - 2a_7a_8b_3^2 + a_3a_8b_3b_6)/a_1^2
\end{aligned} \tag{93}
$$

$$
c_{11} = (a_1^2b_{11}^2 - a_1a_4b_6b_{11} - 2a_1a_8b_3b_{11} + a_1a_8b_6^2 + a_4^2b_3b_{11} - a_4a_8b_3b_6 + a_8^2b_3^2)/a_1^2 \tag{94}
$$

This is a 10th degree polynomial in z that can be solved by computing its real roots. Afterwards, we can back-substitute this value of z into the equations as specified in Sec. 2.2, and get the respective coordinates of the vanishing point on the mirror.

B. Appendix for the computation of direction for a given vanishing points

This section includes some useful information for the derivation of the proposed solution to compute directions, from a given vanishing points.

B.1. Coefficients of the polynomial equations Sec. 2.3

In this section we present the coefficients of the polynomial equations $\kappa_{17}^1[s_2, s_3]$ and $\kappa_{18}^2[s_1, s_2, s_3]$:

$$\kappa_{17}^{1}\left[s_1, s_2\right] = a_1 s_1 + a_2 s_2, \tag{95}$$

where

$$\begin{aligned} a_1 = {} & 4Cc_3 - 4Cz_1 - B^2 c_3 + 4Az_1^3 + 4Bz_1^2 - 3B^2 z_1 - 4A^2 z_1^3 + 4BC \\ & - 4A^2 c_3 z_1^2 + 8ACz_1 - 4Bc_2 y_1 - 4Bc_3 z_1 - 8ABz_1^2 - 4Ac_3 z_1^2 - 8Ac_2 y_1 z_1 - 4ABc_3 z_1; \end{aligned} \tag{96}$$

$$\begin{aligned} a_2 = {} & 4A^2 y_1 z_1^2 - 4c_2 A^2 z_1^2 + 4ABy_1 z_1 - 4c_2 ABz_1 - 4Ay_1 z_1^2 + \\ & + 8c_3 Ay_1 z_1 + 4c_2 Az_1^2 + B^2 y_1 - c_2 B^2 + +4c_3 By_1 + 4c_2 Bz_1 + 8c_2 y_1^2 - 4Cy_1 - 4Cc_2, \end{aligned} \tag{97}$$

and:

$$\kappa_{18}^{2}\left[s_1, s_2, s_3\right] = b_1 s_1^2 + b_2 s_2^2 + b_3 s_2 s_3 + b_4 s_3^2 \tag{98}$$

with:

$$b_1 = (Bc_2 - By_1 - 2c_3 y_1 + 2y_1 z_1 + 2Ac_2 z_1 - 2Ay_1 z_1)^2 \tag{99}$$

$$b_2 = (y_1^2 + z_1^2 - 1)(B + 2c_3 - 2z_1 + 2Az_1)^2 \tag{100}$$

$$b_3 = -4c_2(y_1^2 + z_1^2 - 1)(B + 2c_3 - 2z_1 + 2Az_1) \tag{101}$$

$$b_4 = 4c_2^2(y_1^2 + z_1^2 - 1) \tag{102}$$

B.2. Numerical Example

Although it was not mentioned in the paper, we ran some numerical examples, to validate the proposed techniques. For that purpose, let us consider the example of a mirror defined by $A = -0.15$, $B = -0.30$ and $C = -0.03$ (hyperbolic), and two chessboard in the planes $z = 2$ and $y = -2$ as shown in Fig. B.1, in which $c_2 = -0.5$ and $c_3 = -0.8$. By observing the infinity line, we can pin-point a vanishing point at $[u, v] = [748.1, 650.0]$ (in red). Using the techniques described in previous section, we will determine the direction that generated this vanishing point.

By backward projecting the point into the mirror, we discover its coordinates as being $\mathbf{v} = [x, y, z] \in \Omega = [-0.0670, -0.0463, 0.1155]$. We then substitute it in equations (96) to (102) to obtain:

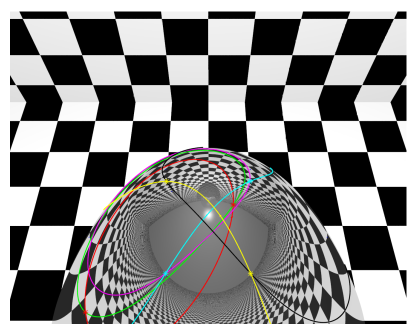

Figure B.1: Image of a hyperbolic catadioptric camera, with several parallel lines, and their respective vanishing points.

$$a_1 = 0.1003 \tag{103}$$

$$a_2 = 0 \tag{104}$$

$$b_1 = 0.0045 \tag{105}$$

$$b_2 = -0.0211 \tag{106}$$

$$b_3 = 0.0195 \tag{107}$$

$$b_4 = -0.0045. \tag{108}$$

By substituting in (25) of the main article, we obtain:

$$s_2 = 0\,, \tag{109}$$

and:

$$s_3 = -\,0.7071 \quad (110)$$
$$s_1 = \pm\,0.7071\;. \quad (111)$$

Hence, we discover that the 3D straight line in the world that generated the vanishing point $\mathbf{v}$ had the direction $\mathbf{s} = [-0.7071, 0, -0.7071]$ by considering the underlying camera system (we can ignore the solution $\mathbf{s}' = [-0.7071, 0, 0.7071]$).

The other points shown in Fig. B.1, have the correspondent directions:

$$\begin{aligned}\text{Red points: } \mathbf{s} &= [-0.7071, 0, -0.7071]\\ \text{Yellow points: } \mathbf{s} &= [-0.7071, 0.7071, 0]\\ \text{Cyan points: } \mathbf{s} &= [0.7071, 0.7071, 0]\;.\end{aligned} \quad (112)$$

C. Appendix for computations related to Vanishing Curves

In this section we present some details regarding the results obtained for the parameterization of the vanishing curves.

C.1. Computing a Curve at the Infinity

In this subsection, we present the full derivation of $\gamma(\mathbf{r})$ as presented in Sec. 3 (Curves at the Infinity). Consider the equation defined in Sec. 2.1 (planar constraint):

$$\kappa_1^1[y, z]x + \kappa_3^2[y, z] = 0 \quad (113)$$

We start by using $\mathbf{s}$ as defined in Eq. 28 of the main article in (113), and obtain a new equation:

$$\kappa_{22}^2[y, z, \alpha]x + \kappa_{23}^3[y, z, \alpha] = 0 \quad (114)$$

In addition, by replacing $\mathbf{s}$ as shown in Eq. 28 of the main article in $\kappa_9[y, z]$, we obtain:

$$\kappa_{24}^3\,[y, z, \alpha] = 0. \quad (115)$$

Since (115) is linear on α, one can use this constraint to define α as:

$$\alpha = \frac{\kappa_{25}^3[y,z]}{\kappa_{26}^3[y,z]}. \quad (116)$$

To conclude the definition of the curve at the infinity, we replace α, as derived in the previous equation, in (114), resulting in:

$$\Gamma(\mathbf{r}) := \kappa_{20}^2[y, z]x + \kappa_{21}^3[y, z] = 0 \quad (117)$$

where the coefficients of $\kappa_{20}^2[y, z]$ and $\kappa_{21}^2[y, z]$ are defined in Appendix. C.2. As already presented in the main paper (Sec. 3), the curves in the infinity are projected onto the mirror as follows:

$$\gamma(\mathbf{r}) := \left\{\mathbf{r} = [x, y, z] \in \mathbb{R}^3 : \Gamma(\mathbf{r}) \wedge \Omega(\mathbf{r}) = 0\right\}. \quad (118)$$

To be able to define a curve in the mirror, we solve $\Gamma(\mathbf{r})$ for x and replace the result in $\Omega(\mathbf{r})$, defining:

$$\kappa_{27}^6[y, z] = 0, \quad (119)$$

obtaining a different (though equally valid) definition of the curve as:

$$\gamma(\mathbf{r}) = \left\{[x, y, z] \in \mathbb{R}^3 : \kappa_{27}^6[y, z] = 0 \wedge x^2 = -y^2 - Az^2 - Bz + C\right\}. \quad (120)$$

Table 1: Degrees of the polynomial equation that can be used to compute lines at the infinity, for specific catadioptric camera systems. We use D denotes the degree of the polynomial equation, and N the number of coefficients. The * in the table implies that one needs to consider a possibility of $z = 0$.

Mirror Type	D	N
General	6	22
General Axial ($c_2 = 0$)	6	18
Spherical Axial ($A = 1$, $B = 0$)	4	12
Ellipsoid Axial ($A = 0$, $C = 0$)	4	9
Conical Axial ($B = 0$, $C = 0$)	4*	9
Cylindrical Axial ($A = 0$, $B = 0$)	2	6

The reason why we could want this alternative definition would be to simplify the process of calculating the actual points that belong to the curve. The coefficient of the general expression for $\kappa^6_{27}[y, z]$ are presented in Appendix C.8.

Similarly to what happens in the estimation of the vanishing points, the complexity of the polynomial equation $\kappa^6_{27}[y, z]$ can be significantly reduced when considering specific camera configurations. In Tab. 1, we present a table with the variation of the degree of the polynomial equation $\kappa^6_{27}[y, z]$ for some specific cases.

C.2. Coefficients Curves at the Infinity

In this subsection we show the coefficients for the parameterization of $\Gamma(\mathbf{r})$ (namely polynomial equations $\kappa^2_{20}[y, z]$ and $\kappa^3_{21}[y, z]$), that can define the curve at the infinity. We define:

$$\kappa^2_{20}[y, z] = a_1 y + a_2 z^2 + a_3 z + a_4 \tag{121}$$

where:

$$\begin{aligned} a_1 =& 8c_2(s_{1,2}s_{2,3} - s_{2,2}s_{1,3}) \\ a_2 =& -(s_{1,2}s_{2,3} - s_{2,2}s_{1,3})(-4A^2 + 4A) \\ a_3 =& (8Ac_3 + 4AB)(s_{1,2}s_{2,3} - s_{2,2}s_{1,3}) \\ a_4 =& (s_{1,2}s_{2,3} - s_{2,2}s_{1,3})(B^2 + 4c_3B - 4C), \end{aligned} \tag{122}$$

and:

$$\kappa^3_{21}[y, z] = b_1 y^2 + b_2 y z^2 + b_3 y z + b_4 y + b_5 z^3 + b_6 z^2 + b_7 z + b_8 \tag{123}$$

where:

$$b_1 = 8s_{2,1}s_{1,3}c_2 - 8s_{1,1}s_{2,3}c_2 \tag{124}$$

$$b_2 = 4As_{1,1}s_{2,3} - 4As_{2,1}s_{1,3} - 4A^2s_{1,1}s_{2,3} + 4A^2s_{2,1}s_{1,3} \tag{125}$$

$$b_3 = 4ABs_{2,1}s_{1,3} - 4ABs_{1,1}s_{2,3} + 8As_{1,1}s_{2,2}c_2 - 8As_{2,1}s_{1,2}c_2 - 8As_{1,1}s_{2,3}c_3 + 8As_{2,1}s_{1,3}c_3 \tag{126}$$

$$\begin{aligned} b_4 =& 4Cs_{1,1}s_{2,3} - 4Cs_{2,1}s_{1,3} - B^2s_{1,1}s_{2,3} + B^2s_{2,1}s_{1,3} + 4Bs_{1,1}s_{2,2}c_2 - 4Bs_{2,1}s_{1,2}c_2 - \\ & - 4Bs_{1,1}s_{2,3}c_3 + 4Bs_{2,1}s_{1,3}c_3 \end{aligned} \tag{127}$$

$$b_5 = 4As_{2,1}s_{1,2} - 4As_{1,1}s_{2,2} + 4A^2s_{1,1}s_{2,2} - 4A^2s_{2,1}s_{1,2} \tag{128}$$

$$
\begin{aligned}
b_6 =& 4Bs_{2,1}s_{1,2} - 4Bs_{1,1}s_{2,2} + 4A^2s_{1,1}s_{2,3}c_2 - 4A^2s_{2,1}s_{1,3}c_2 + 4A^2s_{1,1}s_{2,2}c_3 - 4A^2s_{2,1}s_{1,2}c_3 + \\
&+ 8ABs_{1,1}s_{2,2} - 8ABs_{2,1}s_{1,2} - 4As_{1,1}s_{2,3}c_2 + 4As_{2,1}s_{1,3}c_2 + 4As_{1,1}s_{2,2}c_3 - 4As_{2,1}s_{1,2}c_3 \quad (129)\\
b_7 =& 4Cs_{1,1}s_{2,2} - 4Cs_{2,1}s_{1,2} + 3B^2s_{1,1}s_{2,2} - 3B^2s_{2,1}s_{1,2} - 8ACs_{1,1}s_{2,2} + 8ACs_{2,1}s_{1,2} - \\
&- 4Bs_{1,1}s_{2,3}c_2 + 4Bs_{2,1}s_{1,3}c_2 + 4Bs_{1,1}s_{2,2}c_3 - 4Bs_{2,1}s_{1,2}c_3 + 4ABs_{1,1}s_{2,3}c_2 - \\
&- 4ABs_{2,1}s_{1,3}c_2 + 4ABs_{1,1}s_{2,2}c_3 - 4ABs_{2,1}s_{1,2}c_3 \quad (130)\\
b_8 =& B^2s_{1,1}s_{2,3}c_2 - B^2s_{2,1}s_{1,3}c_2 + B^2s_{1,1}s_{2,2}c_3 - B^2s_{2,1}s_{1,2}c_3 - 4BCs_{1,1}s_{2,2} + 4BCs_{2,1}s_{1,2} + \\
&+ 4Cs_{1,1}s_{2,3}c_2 - 4Cs_{2,1}s_{1,3}c_2 - 4Cs_{1,1}s_{2,2}c_3 + 4Cs_{2,1}s_{1,2}c_3. \quad (131)
\end{aligned}
$$

Specific cases for specific types of camera position and type of mirror are shown in Appendix. C.3 to C.7.

C.3. Coefficients of curves at the infinity: Axial case

In the axial case, the camera movement has to be restricted to motion in the u_z axis, therefore, $c_2 = 0$:

$$\kappa_{20}^2\left[y,z\right] = (s_{1,2}s_{2,3} - s_{2,2}s_{1,3})(a_1z^2 + a_2z + a_3), \quad (132)$$

with:

$$a_1 = 4A^2 - 4A \quad (133) \qquad a_2 = 8Ac_3 + 4AB \quad (134) \qquad a_3 = B^2 + 4c_3B - 4C, \quad (135)$$

and:

$$\kappa_{21}^3\left[y,z\right] = b_1yz^2 + b_2yz + b_3y + b_4z^3 + b_5z^2 + b_6z + b_7, \quad (136)$$

with:

$$
\begin{aligned}
b_1 =& 4As_{1,1}s_{2,3} - 4As_{2,1}s_{1,3} - 4A^2s_{1,1}s_{2,3} + 4A^2s_{2,1}s_{1,3} \quad (137)\\
b_2 =& 4ABs_{2,1}s_{1,3} - 4ABs_{1,1}s_{2,3} - 8As_{1,1}s_{2,3}c_3 + 8As_{2,1}s_{1,3}c_3 \quad (138)\\
b_3 =& 4Cs_{1,1}s_{2,3} - 4Cs_{2,1}s_{1,3} - B^2s_{1,1}s_{2,3} + B^2s_{2,1}s_{1,3} - 4Bs_{1,1}s_{2,3}c_3 + 4Bs_{2,1}s_{1,3}c_3 \quad (139)\\
b_4 =& 4As_{2,1}s_{1,2} - 4As_{1,1}s_{2,2} + 4A^2s_{1,1}s_{2,2} - 4A^2s_{2,1}s_{1,2} \quad (140)\\
b_5 =& 4Bs_{2,1}s_{1,2} - 4Bs_{1,1}s_{2,2} + 4A^2s_{1,1}s_{2,2}c_3 - 4A^2s_{2,1}s_{1,2}c_3 + 8ABs_{1,1}s_{2,2} - 8ABs_{2,1}s_{1,2} + \\
&+ 4As_{1,1}s_{2,2}c_3 - 4As_{2,1}s_{1,2}c_3 \quad (141)\\
b_6 =& 4Cs_{1,1}s_{2,2} - 4Cs_{2,1}s_{1,2} + 3B^2s_{1,1}s_{2,2} - 3B^2s_{2,1}s_{1,2} - 8ACs_{1,1}s_{2,2} + 8ACs_{2,1}s_{1,2} + \\
&+ 4Bs_{1,1}s_{2,2}c_3 - 4Bs_{2,1}s_{1,2}c_3 + 4ABs_{1,1}s_{2,2}c_3 - 4ABs_{2,1}s_{1,2}c_3 \quad (142)\\
b_7 =& B^2s_{1,1}s_{2,2}c_3 - B^2s_{2,1}s_{1,2}c_3 - 4BCs_{1,1}s_{2,2} + 4BCs_{2,1}s_{1,2} - 4Cs_{1,1}s_{2,2}c_3 + 4Cs_{2,1}s_{1,2}c_3. \quad (143)
\end{aligned}
$$

C.4. Coefficients of curves at the infinity: Spherical mirror ($A = 1$, $B = 0$)

Constraints for the spherical mirror are:

$$\kappa_{20}^1\left[y,z\right] = (s_{1,2}s_{2,3} - s_{2,2}s_{1,3})(a_1z + a_2), \quad (144)$$

with:

$$a_1 = 8c_3 \quad (145) \qquad a_2 = -4C, \quad (146)$$

and:

$$\kappa_{21}^2\left[y,z\right] = b_1yz + b_2y + b_3z^2 + b_4z + b_5, \quad (147)$$

with:

$$b_1 = 8s_{2,1}s_{1,3}c_3 - 8s_{1,1}s_{2,3}c_3 \quad (148)$$
$$b_2 = 4Cs_{1,1}s_{2,3} - 4Cs_{2,1}s_{1,3} \quad (149)$$
$$b_3 = 8s_{1,1}s_{2,2}c_3 - 8s_{2,1}s_{1,2}c_3 \quad (150)$$
$$b_4 = 4Cs_{2,1}s_{1,2} - 4Cs_{1,1}s_{2,2} \quad (151)$$
$$b_5 = 4Cs_{2,1}s_{1,2}c_3 - 4Cs_{1,1}s_{2,2}c_3. \quad (152)$$

C.5. Coefficients of curves at the infinity: Ellipsoid mirror ($A = 0$, $C = 0$)

Constraints for the Ellipsoidal mirror are:

$$\kappa_{20}^{0}[y, z] = (s_{1,2}s_{2,3} - s_{2,2}s_{1,3})a_1, \quad (153)$$

with:

$$a_1 = B^2 + 4c_3B, \quad (154)$$

and:

$$\kappa_{21}^{2}[y, z] = b_1y + b_2z^2 + b_3z + b_4, \quad (155)$$

with:

$$b_1 = B^2s_{2,1}s_{1,3} - B^2s_{1,1}s_{2,3} - 4Bs_{1,1}s_{2,3}c_3 + 4Bs_{2,1}s_{1,3}c_3 \quad (156)$$
$$b_2 = 4Bs_{2,1}s_{1,2} - 4Bs_{1,1}s_{2,2} \quad (157)$$
$$b_3 = 3B^2s_{1,1}s_{2,2} - 3B^2s_{2,1}s_{1,2} + 4Bs_{1,1}s_{2,2}c_3 - 4Bs_{2,1}s_{1,2}c_3 \quad (158)$$
$$b_4 = B^2s_{1,1}s_{2,2}c_3 - B^2s_{2,1}s_{1,2}c_3. \quad (159)$$

C.6. Coefficients of curves at the infinity: Conical mirror ($B = 0$, $C = 0$)

Constraints for the conical mirror are:

$$\kappa_{20}^{2}[y, z] = (s_{1,2}s_{2,3} - s_{2,2}s_{1,3})(a_1z^2 + a_2z), \quad (160)$$

with:

$$a_1 = 4A^2 - 4A \quad (161) \qquad a_2 = 8Ac_3, \quad (162)$$

and:

$$\kappa_{21}^{3}[y, z] = b_1yz^2 + b_2yz + b_3z^3 + b_4z^2, \quad (163)$$

with:

$$b_1 = 4As_{1,1}s_{2,3} - 4As_{2,1}s_{1,3} - 4A^2s_{1,1}s_{2,3} + 4A^2s_{2,1}s_{1,3} \quad (164)$$
$$b_2 = 8As_{2,1}s_{1,3}c_3 - 8As_{1,1}s_{2,3}c_3 \quad (165)$$
$$b_3 = 4As_{2,1}s_{1,2} - 4As_{1,1}s_{2,2} + 4A^2s_{1,1}s_{2,2} - 4A^2s_{2,1}s_{1,2} \quad (166)$$
$$b_4 = 4A^2s_{1,1}s_{2,2}c_3 - 4A^2s_{2,1}s_{1,2}c_3 + 4As_{1,1}s_{2,2}c_3 - 4As_{2,1}s_{1,2}c_3. \quad (167)$$

C.7. Coefficients of curves at the infinity: Cylindrical mirror ($A = 0$, $B = 0$)

Constraints for the cylindrical mirror are:

$$\kappa_{20}^{0}[y, z] = (s_{1,2}s_{2,3} - s_{2,2}s_{1,3})a_1, \quad (168)$$

with:

$$a_1 = -4C, \quad (169)$$

and:

$$\kappa_{21}^{1}[y, z] = b_1y + b_2z + b_3 \quad (170)$$

with:

$$b_1 = 4Cs_{1,1}s_{2,3} - 4Cs_{2,1}s_{1,3} \quad (171)$$
$$b_2 = 4Cs_{1,1}s_{2,2} - 4Cs_{2,1}s_{1,2} \quad (172)$$
$$b_3 = 4Cs_{2,1}s_{1,2}c_3 - 4Cs_{1,1}s_{2,2}c_3. \quad (173)$$

C.8. Solving Curves at the Infinity

In the general case, from Sec. C.1, we obtain the following expression for $\kappa^6_{27}[y,z]$:

$$\begin{aligned}\kappa^6_{27}[y,z] =& a_1y^4 + a_2y^3z^2 + a_3y^3z + a_4y^3 + a_5y^2z^4 + a_6y^2z^3 + a_7y^2z^2 + a_8y^2z + a_9y^2 + a_{10}yz^5 + \\ &+ a_{11}yz^4 + a_{12}yz^3 + a_{13}yz^2 + a_{14}yz + a_{15}y + a_{16}z^6 + a_{17}z^5 + a_{18}z^4 + a_{19}z^3 + a_{20}z^2 + \\ &+ a_{21}z + a_{22}\end{aligned} \quad (174)$$

where:

$$a_1 = 64n_1^2n_3^2c_2^2 + 64n_2^2n_3^2c_2^2 \quad (175)$$
$$a_2 = -16n_2n_3c_2(-4n_2n_3A^2 + 4n_2n_3A) - 16n_1^2n_3^2c_2(-4A^2 + 4A) \quad (176)$$
$$a_3 = 16n_2n_3c_2(8An_3^2c_2 + 4ABn_2n_3 + 8An_2n_3c_3) + 16n_1^2n_3^2c_2(8Ac_3 + 4AB) \quad (177)$$
$$a_4 = 16n_1^2n_3^2c_2(B^2 + 4c_3B - 4C) + 16n_2n_3c_2(n_2B^2n_3 + 4c_2Bn_3^2 + 4n_2c_3Bn_3 - 4Cn_2n_3) \quad (178)$$
$$a_5 = (-4n_2n_3A^2 + 4n_2n_3A)^2 + n_1^2n_3^2(-4A^2 + 4A)^2 \quad (179)$$
$$\begin{aligned}a_6 =& -2(-4n_2n_3A^2 + 4n_2n_3A)(8An_3^2c_2 + 4ABn_2n_3 + 8An_2n_3c_3) - 16n_2n_3c_2(-4A^2n_3^2 + 4An_3^2) + \\ &- 2n_1^2n_3^2(8Ac_3 + 4AB)(-4A^2 + 4A)\end{aligned} \quad (180)$$
$$\begin{aligned}a_7 =& (8An_3^2c_2 + 4ABn_2n_3 + 8An_2n_3c_3)^2 - 2(-4n_2n_3A^2 + 4n_2n_3A)(n_2B^2n_3 + 4c_2Bn_3^2 + 4n_2c_3Bn_3 + \\ &- 4Cn_2n_3) - n_1^2n_3^2(2(-4A^2 + 4A)(B^2 + 4c_3B - 4C) - (8Ac_3 + 4AB)^2) + 64An_1^2n_3^2c_2^2 + \\ &+ 16n_2n_3c_2(4A^2n_3^2c_3 - 4Bn_3^2 + 8ABn_3^2 + 4An_3^2c_3 - 4A^2n_2n_3c_2 + 4An_2n_3c_2)\end{aligned} \quad (181)$$
$$\begin{aligned}a_8 =& 2(8An_3^2c_2 + 4ABn_2n_3 + 8An_2n_3c_3)(n_2B^2n_3 + 4c_2Bn_3^2 + 4n_2c_3Bn_3 - 4Cn_2n_3) + 2n_1^2n_3^2(8Ac_3 + \\ &+ 4AB)(B^2 + 4c_3B - 4C) + 64Bn_1^2n_3^2c_2^2 + 16n_2n_3c_2(4Cn_3^2 + 3B^2n_3^2 - 8ACn_3^2 + 4Bn_3^2c_3 + \\ &+ 4ABn_3^2c_3 + 4Bn_2n_3c_2 - 4ABn_2n_3c_2)\end{aligned} \quad (182)$$
$$\begin{aligned}a_9 =& (n_2B^2n_3 + 4c_2Bn_3^2 + 4n_2c_3Bn_3 - 4Cn_2n_3)^2 + n_1^2n_3^2(B^2 + 4c_3B - 4C)^2 - 64Cn_1^2n_3^2c_2^2 + \\ &- 16n_2n_3c_2(-c_3B^2n_3^2 + n_2c_2B^2n_3 + 4CBn_3^2 + 4Cc_3n_3^2 + 4Cn_2c_2n_3)\end{aligned} \quad (183)$$
$$a_{10} = 2(-4A^2n_3^2 + 4An_3^2)(-4n_2n_3A^2 + 4n_2n_3A) \quad (184)$$
$$\begin{aligned}a_{11} =& -2(-4A^2n_3^2 + 4An_3^2)(8An_3^2c_2 + 4ABn_2n_3 + 8An_2n_3c_3) - 2(-4n_2n_3A^2 + 4n_2n_3A)(4A^2n_3^2c_3 + \\ &- 4Bn_3^2 + 8ABn_3^2 + 4An_3^2c_3 - 4A^2n_2n_3c_2 + 4An_2n_3c_2) - 16An_1^2n_3^2c_2(-4A^2 + 4A)\end{aligned} \quad (185)$$
$$\begin{aligned}a_{12} =& 2(8An_3^2c_2 + 4ABn_2n_3 + 8An_2n_3c_3)(4A^2n_3^2c_3 - 4Bn_3^2 + 8ABn_3^2 + 4An_3^2c_3 - 4A^2n_2n_3c_2 + \\ &+ 4An_2n_3c_2) - 2(-4n_2n_3A^2 + 4n_2n_3A)(4Cn_3^2 + 3B^2n_3^2 - 8ACn_3^2 + 4Bn_3^2c_3 + 4ABn_3^2c_3 + \\ &+ 4Bn_2n_3c_2 - 4ABn_2n_3c_2) - 2(-4A^2n_3^2 + 4An_3^2)(n_2B^2n_3 + 4c_2Bn_3^2 + 4n_2c_3Bn_3 + \\ &- 4Cn_2n_3) - 16Bn_1^2n_3^2c_2(-4A^2 + 4A) + 16An_1^2n_3^2c_2(8Ac_3 + 4AB)\end{aligned} \quad (186)$$

$$
\begin{aligned}
a_{13} =&2(-4n_2n_3A^2 + 4n_2n_3A)(-c_3B^2n_3^2 + n_2c_2B^2n_3 + 4CBn_3^2 + 4Cc_3n_3^2 + 4Cn_2c_2n_3) + 2(8An_3^2c_2 + \\
&+ 4ABn_2n_3 + 8An_2n_3c_3)(4Cn_3^2 + 3B^2n_3^2 - 8ACn_3^2 + 4Bn_3^2c_3 + 4ABn_3^2c_3 + 4Bn_2n_3c_2 + \\
&- 4ABn_2n_3c_2) + 2(n_2B^2n_3 + 4c_2Bn_3^2 + 4n_2c_3Bn_3 - 4Cn_2n_3)(4A^2n_3^2c_3 - 4Bn_3^2 + 8ABn_3^2 + \\
&+ 4An_3^2c_3 - 4A^2n_2n_3c_2 + 4An_2n_3c_2) + 16An_1^2n_3^2c_2(B^2 + 4c_3B - 4C) + 16Cn_1^2n_3^2c_2(-4A^2 + 4A) + \\
&+ 16Bn_1^2n_3^2c_2(8Ac_3 + 4AB) \qquad (187)
\end{aligned}
$$

$$
\begin{aligned}
a_{14} =&2(n_2B^2n_3 + 4c_2Bn_3^2 + 4n_2c_3Bn_3 - 4Cn_2n_3)(4Cn_3^2 + 3B^2n_3^2 - 8ACn_3^2 + 4Bn_3^2c_3 + 4ABn_3^2c_3 + \\
&+ 4Bn_2n_3c_2 - 4ABn_2n_3c_2) - 2(8An_3^2c_2 + 4ABn_2n_3 + 8An_2n_3c_3)(-c_3B^2n_3^2 + n_2c_2B^2n_3 + \\
&+ 4CBn_3^2 + 4Cc_3n_3^2 + 4Cn_2c_2n_3) + 16Bn_1^2n_3^2c_2(B^2 + +4c_3B - 4C) - 16Cn_1^2n_3^2c_2(8Ac_3 + 4AB) \\
& \qquad (188)
\end{aligned}
$$

$$
\begin{aligned}
a_{15} =& - 2(n_2B^2n_3 + 4c_2Bn_3^2 + 4n_2c_3Bn_3 - 4Cn_2n_3)(-c_3B^2n_3^2 + n_2c_2B^2n_3 + 4CBn_3^2 + \\
&+ 4Cc_3n_3^2 + 4Cn_2c_2n_3) - 16Cn_1^2n_3^2c_2(B^2 + 4c_3B - 4C) \qquad (189)
\end{aligned}
$$

$$
a_{16} =(-4A^2n_3^2 + 4An_3^2)^2 + An_1^2n_3^2(-4A^2 + 4A)^2 \qquad (190)
$$

$$
\begin{aligned}
a_{17} =&Bn_1^2n_3^2(-4A^2 + 4A)^2 - 2(-4A^2n_3^2 + 4An_3^2)(4A^2n_3^2c_3 - 4Bn_3^2 + 8ABn_3^2 + 4An_3^2c_3 + \\
&- 4A^2n_2n_3c_2 + 4An_2n_3c_2) - 2An_1^2n_3^2(8Ac_3 + 4AB)(-4A^2 + 4A) \qquad (191)
\end{aligned}
$$

$$
\begin{aligned}
a_{18} =&(4A^2n_3^2c_3 - 4Bn_3^2 + 8ABn_3^2 + 4An_3^2c_3 - 4A^2n_2n_3c_2 + 4An_2n_3c_2)^2 - 2(-4A^2n_3^2 + \\
&+ 4An_3^2)(4Cn_3^2 + 3B^2n_3^2 - 8ACn_3^2 + 4Bn_3^2c_3 + 4ABn_3^2c_3 + 4Bn_2n_3c_2 - 4ABn_2n_3c_2) + \\
&- Cn_1^2n_3^2(-4A^2 + 4A)^2 - An_1^2n_3^2(2(-4A^2 + 4A)(B^2 + 4c_3B - 4C) - (8Ac_3 + 4AB)^2) + \\
&- 2Bn_1^2n_3^2(8Ac_3 + 4AB)(-4A^2 + 4A) \qquad (192)
\end{aligned}
$$

$$
\begin{aligned}
a_{19} =&2(-4A^2n_3^2 + 4An_3^2)(-c_3B^2n_3^2 + n_2c_2B^2n_3 + 4CBn_3^2 + 4Cc_3n_3^2 + 4Cn_2c_2n_3) + \\
&+ 2(4A^2n_3^2c_3 - 4Bn_3^2 + 8ABn_3^2 + 4An_3^2c_3 - 4A^2n_2n_3c_2 + 4An_2n_3c_2)(4Cn_3^2 + 3B^2n_3^2 - 8ACn_3^2 + \\
&+ 4Bn_3^2c_3 + 4ABn_3^2c_3 + 4Bn_2n_3c_2 - 4ABn_2n_3c_2) - Bn_1^2n_3^2(2(-4A^2 + 4A)(B^2 + 4c_3B - 4C) + \\
&- (8Ac_3 + 4AB)^2) + 2An_1^2n_3^2(8Ac_3 + 4AB)(B^2 + 4c_3B - 4C) + 2Cn_1^2n_3^2(8Ac_3 + \\
&+ 4AB)(-4A^2 + 4A) \qquad (193)
\end{aligned}
$$

$$
\begin{aligned}
a_{20} =&(4Cn_3^2 + 3B^2n_3^2 - 8ACn_3^2 + 4Bn_3^2c_3 + 4ABn_3^2c_3 + 4Bn_2n_3c_2 - 4ABn_2n_3c_2)^2 + \\
&- 2(-c_3B^2n_3^2 + n_2c_2B^2n_3 + 4CBn_3^2 + 4Cc_3n_3^2 + 4Cn_2c_2n_3)(4A^2n_3^2c_3 - 4Bn_3^2 + 8ABn_3^2 + \\
&+ 4An_3^2c_3 - 4A^2n_2n_3c_2 + 4An_2n_3c_2) + An_1^2n_3^2(B^2 + 4c_3B - 4C)^2 + Cn_1^2n_3^2(2(-4A^2 + \\
&+ 4A)(B^2 + 4c_3B - 4C) - (8Ac_3 + 4AB)^2) + 2Bn_1^2n_3^2(8Ac_3 + 4AB)(B^2 + 4c_3B - 4C) \qquad (194)
\end{aligned}
$$

$$
\begin{aligned}
a_{21} =&Bn_1^2n_3^2(B^2 + 4c_3B - 4C)^2 - 2(-c_3B^2n_3^2 + n_2c_2B^2n_3 + 4CBn_3^2 + 4Cc_3n_3^2 + 4Cn_2c_2n_3)(4Cn_3^2 + \\
&+ 3B^2n_3^2 - 8ACn_3^2 + 4Bn_3^2c_3 + 4ABn_3^2c_3 + 4Bn_2n_3c_2 - 4ABn_2n_3c_2) - 2Cn_1^2n_3^2(8Ac_3 + \\
&+ 4AB)(B^2 + 4c_3B - 4C) \qquad (195)
\end{aligned}
$$

$$
a_{22} =(-c_3B^2n_3^2 + n_2c_2B^2n_3 + 4CBn_3^2 + 4Cc_3n_3^2 + 4Cn_2c_2n_3)^2 - Cn_1^2n_3^2(B^2 + 4c_3B - 4C)^2 \qquad (196)
$$

To conclude, we can obtain all the points in curve at the infinity by traversing either y or z in the range desired and obtaining the other from $\kappa_{27}^6[y, z] = 0$. Afterwards, we can get the x for each individual point by using the fact that the point must be on the mirror surface ($\Omega(\mathbf{r}) = 0$).

D. Vanishing Points in Central Unified Spherical Model

In this section we propose a simpler solution to the analytical modeling of vanishing points in central catadioptric camera systems. For that purpose, we considering the central unified model proposed in [2, 1]. Next, we present the projection of 3D straight lines using the sphere model and present a solution to the modeling of vanishing points.

D.1. Line Projection

Wen considering the central unified sphere model, 3D lines are projected into the sphere, by considering that the line is on a plane that constrains the line and the origin of the catadioptric system. Let us denote this plane as:

$$\Pi = \begin{bmatrix} l_1 & l_2 & l_3 & 0 \end{bmatrix}^T, \tag{197}$$

This line is then projected onto the sphere in a circle (intersection between the plane Π and the unit sphere), which is the projected into the conic $\mathbf{C}$ in the canonical image plane[1], using the implicit relation:

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}^T \underbrace{\begin{bmatrix} l_1^2(1-\xi^2) - l_3^2\xi^2 & l_1 l_2(1-\xi^2) & l_1 l_3 \\ l_1 l_2(1-\xi^2) & l_2^2(1-\xi^2) - l_3^2\xi^2 & l_2 l_3 \\ l_1 l_2 & l_2 l_3 & l_3^2 \end{bmatrix}}_{\mathbf{C}} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0 \Rightarrow$$

$$e_1 x^2 + e_2 xy + e_3 x + e_4 y^2 + e_5 y + e_6 = 0 \tag{198}$$

for some coefficients e_i (for $i = 1, \dots 6$) and unknowns x and y.

D.2. Modeling Vanishing Points using the Central Unified Model

Now that we have defined the projection of the curves into the canonical image plane, the estimation of the vanishing points is given by the intersection points of the projection of two parallel 3D lines into the canonical image plane. By knowing two parallel lines, we have two 2-degree polynomial equations as shown in (198):

$$e_1 x^2 + e_2 xy + e_3 x + e_4 y^2 + e_5 y + e_6 = 0, \tag{199}$$

$$f_1 x^2 + f_2 xy + f_3 x + f_4 y^2 + f_5 y + f_6 = 0. \tag{200}$$

Then, the intersection of these two polynomial equations (which represents the intersection of two quadratic curves) can be estimated by solving one of the functions in terms of x (lets say (199)) and substituting the result in the second. After some simplifications, we get:

$$x = \frac{e_3 + e_2 y \pm \sqrt{e_2^2 y^2 + 2 e_2 e_3 y + e_3^2 - 4 e_1 e_4 y^2 - 4 e_1 e_5 y - 4 e_1 e_6}}{-2 e_1}, \text{ and} \tag{201}$$

$$g_1 y^4 + g_2 y^3 + g_3 y^2 + g_4 y + g_5 = 0. \tag{202}$$

which can be solved by estimating the roots of the four degree (202) and back substituting the resulting y in (201).

E. Numerical Example of the Absolute Pose Estimation using Minimal Data

[1] Notice that this is an intermediate projection, it does not represent directly the image pixels.

Although it was not mention in the paper, we illustrate the procedure used to estimate the camera pose (Sec. 4.2), by considering a numerical example. For this purpose, we consider the minimal data case, i.e. consider two know directions in the world coordinate system: $\breve{\mathbf{d}}_1 = [0, 1, 0]$, $\breve{\mathbf{d}}_2 = [0.7071, 0.7071, 0]$ and $\breve{\mathbf{d}}_3 = \breve{\mathbf{d}}_1 \times \breve{\mathbf{d}}_2 = [0, 0, 1]$, and the image shown in Fig. E.2. By using the correspondent vanishing points, we can determine $\mathbf{d}_1$, $\mathbf{d}_2$ and $\mathbf{d}_3$ to be:

$$\mathbf{d}_1 = [-0.7071, 0.7071, 0] \tag{203}$$

$$\mathbf{d}_2 = [0, 1, 0] \tag{204}$$

$$\mathbf{d}_3 = \mathbf{d}_1 \times \mathbf{d}_2 = [0, 0, -0.7071]\,, \tag{205}$$

and obtain:

$$\mathbf{R} = \begin{bmatrix} 0.7071 & -0.7071 & 0 \\ 0.7071 & 0.7071 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{206}$$

using the *Procrustes* problem.

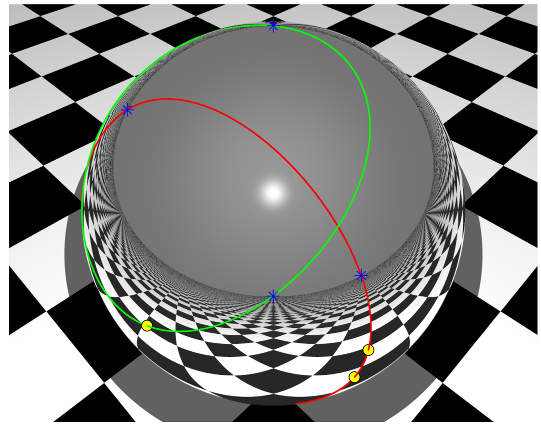

Figure E.2: Image of a spherical catadioptric camera, with two parallel lines, their respective vanishing points, and some points from each line marked in yellow.

Since we now know the rotation matrix $\mathbf{R}$, we can use it to estimate the translation parameters as explained also in Sec. 4.2. Considering that the two lines in Fig. E.2 are represented in the 3D world by $\breve{\mathbf{l}}_1 = (\breve{\mathbf{s}}_1, \breve{\mathbf{m}}_1) = (0, 1, 0, 1, 0, 0)$ and $\breve{\mathbf{l}}_2 = (\breve{\mathbf{s}}_2, \breve{\mathbf{m}}_2) = (0.7071, 0.7071, 0, 0.7071, -0.7071, 2.8284)$, we can obtain a linear system of three equations (using the three yellow points in Fig. E.2, $j = 1, 2, 3$), as shown in Sec. 4.2. Then, one can estimate the translation parameters as:

$$\mathbf{A} = \begin{bmatrix} -0.6587 & -0.6587 & 0.1965 \\ -0.5306 & -0.5306 & 0.1299 \\ -0.6676 & 0 & -0.7087 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -0.1965 \\ -0.1299 \\ 0.7087 \end{bmatrix}. \tag{207}$$

and:

$$\mathbf{t} = \mathbf{A}^{\dagger}\mathbf{b} = \begin{bmatrix} 0 & 0 & -1 \end{bmatrix}^T. \tag{208}$$

Both estimated values of rotation and translation parameters correspond to the ones introduced in this simulated environment.

References

[1] J. P. Barreto and H. Araujo. Issues on the geometry of central catadioptric image formation. In *IEEE Computer Vision and Pattern Recognition (CVPR)*, volume 2, pages 422–427, 2001.

[2] C. Geyer and K. Daniilidis. A unifying theory for central panoramic systems and practical implications. In *European Conf. Computer Vision (ECCV)*, pages 445–461, 2000.