On the convergence of *PatchMatch* and its variants: Supplementary material

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Abstract

In this supplementary material we detail here the proofs of the results presented in the paper.

2. Generic fast patch matching algorithm

2.3. Random search

In the following, we will say that $\eta \in \{U_z \geqslant \alpha\}$, η being a k-set of elements, if and only if $U_z(\eta) \geqslant \alpha$ (U_z being defined in Def. (2.1)).

Lemma 2.6 *For all $z \in \boldsymbol{A}$, $C(z, .)$ is a non-increasing function such that for $\varepsilon > 0$, $C(z,\varepsilon) \in [0,1[$ and $C(z,\varepsilon) = 1$ for $\varepsilon \leqslant 0$.*

Proof Let us recall the definition of C:

$$C(z,\alpha) = \sup_{\eta \in \{U_z \geqslant \alpha\}} Q(\eta, \{U_z \geqslant \alpha\}). \tag{22}$$

The function $C(z,\cdot)$ is a supremum of probabilities (in fact it is a maximum because the set $\{U_z \geqslant \varepsilon\}$ is finite). We remind the property for a candidate η, $Q(\eta, L) < 1$ if the set of acceptable list of candidates L is not empty. Thus $C(z,\varepsilon) \in [0,1[$, for all $z \in \mathbf{A}$ and $\varepsilon > 0$. Since $\inf U_z = 0$, the probability of transitioning to a negative energy is 0, and thus $C(z,\varepsilon) = 1$ for $\varepsilon \leqslant 0$. If $\varepsilon_1, \varepsilon_2 \in \mathbb{R}_+$ are such that $\varepsilon_1 \geqslant \varepsilon_2$, we have $\{U_z \geqslant \varepsilon_1\} \subseteq \{U_z \geqslant \varepsilon_2\}$. Therefore for all η a set of candidates of **B**, because Q is a stochastic kernel, $Q(\eta, \{U_z \geqslant \varepsilon_1\}) \leqslant Q(\eta, \{U_z \geqslant \varepsilon_2\})$. This implies that

$$\sup_{\eta \in \{U_z \geqslant \varepsilon_1\}} Q(\eta, \{U_z \geqslant \varepsilon_1\}) \leqslant \sup_{\eta \in \{U_z \geqslant \varepsilon_1\}} Q(\eta, \{U_z \geqslant \varepsilon_2\}) \leqslant \sup_{\eta \in \{U_z \geqslant \varepsilon_2\}} Q(\eta, \{U_z \geqslant \varepsilon_2\}). \tag{23}$$

and therefore $C(z,\varepsilon_1) \leqslant C(z,\varepsilon_2)$. ■

3. Convergence of the patch matching algorithms

For the proof of Lemma 3.1, we will use the following lemma and its subsequent corollary.

Lemma 3.0.1 *Let y, x be real random variables (x with pdf f according to Lebesgue measure μ) and E an event of the form $s_i \in]a_i, +\infty[$, $a_i \in \mathbb{R}$, for $i \in [\![1, N]\!]$. We assume that y is conditionally independent from E given x. Then for any Y, X:*

$$\mathbb{P}(y \in Y | x \in X, E) \leqslant \sup_{x \in X} \mathbb{P}(y \in Y | x). \tag{24}$$

Proof The proof follows from Bayes rule and the conditional independence between y and E:

$$\mathbb{P}(y \in Y | x \in X, E) = \frac{\mathbb{P}(y \in Y, x \in X | E)}{\mathbb{P}(x \in X | E)} = \frac{1}{\mathbb{P}(x \in X | E)} \int_{x \in X} \mathbb{P}(y \in Y | x, E) f(x) d\mu(x)$$
$$= \frac{1}{\mathbb{P}(x \in X | E)} \int_{x \in X} \mathbb{P}(y \in Y | x) f(x) d\mu(x) \leqslant \sup_{x \in X} \mathbb{P}(y \in Y | x) \frac{1}{\mathbb{P}(x \in X | E)} \int_{x \in X} f(x) d\mu(x). \tag{25}$$

■

Corollary 3.0.2 *Let $y_1, \ldots, y_n, x_1, \ldots, x_n$ be real random variables and E an arbitrary random event of the form defined in Lemma 3.0.1. We assume that for all $i \in [\![1, n]\!]$, given x_i, y_i is conditionally independent from E, x_j and y_j, for $j \neq i$. Then for any Y_i, X_i:*

$$\mathbb{P}\left(\forall i \in [\![i, n]\!], y_i \in Y_i | \forall i \in [\![1, n]\!], x_i \in X_i, E\right) \leqslant \prod_{i=1}^{n} \sup_{x \in X_i} \mathbb{P}(y_i \in Y_i | x). \tag{26}$$

Proof Using Bayes rule,

$$\mathbb{P}\left(\forall i \in [\![i, n]\!], y_i \in Y_i | \forall i \in [\![1, n]\!], x_i \in X_i, E\right) \tag{27}$$
$$= \prod_{i=1}^{n} \mathbb{P}\left(y_i \in Y_i | \forall j \in [\![1, n]\!], x_j \in X_j, E, \forall k \in [\![1, i-1]\!], y_k \in Y_k\right) \tag{28}$$
$$\leqslant \prod_{i=1}^{n} \sup_{x \in X_i} \mathbb{P}(y_i \in Y_i | x) \tag{29}$$

where the last inequality is given by the application of Lemma 3.0.1 on each term of the product. ■

3.1. Energy decay in the nth step

Lemma 3.1 (Constraints propagation) *Consider an assignment φ^{n+1} resulting from an iteration of Algorithm 1. Then for each pair of nodes $x, z \in \mathcal{V}$,*

$$U_x(\varphi_x^{n+1}) \geqslant \varepsilon \Rightarrow U_z(\varphi_z^{n+1}) \geqslant \varepsilon_{z,x}, \tag{30}$$

where the levels $\varepsilon_{z,x} \geqslant 0$ are as follows. For the ancestors of x (i.e. $\mathcal{P}(z, x) \neq \emptyset$) the levels $\varepsilon_{z,x}$ are defined via the following recursion starting from x and following the inverse propagation order:

$$\begin{cases} \varepsilon_{z,x} = \inf \left\{ U_z(\theta) \,\middle|\, \theta \in \bigcap_{y \text{ s.t. } z \sim y} A_{z,y}^{-1}(\{U_y \geqslant \varepsilon_{y,x}\}) \right\} \\ \varepsilon_{x,x} = \varepsilon. \end{cases} \tag{31}$$

For the rest of the nodes $\varepsilon_{z,x} = -1$.

Proof We start by observing the following property of the merge operator. Consider a node $y \in \mathcal{V}$ and two sets of candidates patches ξ, η. Then it is easy to show that

$$|\xi| \geqslant k \quad \text{and} \quad U_y(\xi) < \varepsilon \quad \Rightarrow \quad U_y(\text{merge}_y^k(\xi \cup \eta)) < \varepsilon. \tag{32}$$

The proof of the lemma is trivial for the nodes that are not ancestors of x. To prove it for the ancestors we proceed by induction starting from x and following the inverse propagation order. Let z, a node in the query image $\mathbf{A}$, be an ancestor of x. We assume that the statement holds for the nodes preceding z in the inverse propagation order (or equivalently those succeeding z with the propagation order).

In particular let y be a child of z, i.e. $z \sim y$, and assume that $\varphi_y^{n+1/2} \in \{U_y \geqslant \varepsilon_{y,x}\}$. The candidates $\varphi_y^{n+1/2}$ result from the propagation from the parents of y, among them is z. By (32) it follows that if $U_y(A_{z,y}\varphi_z^{n+1}) < \varepsilon_{y,x}$, then $U_y(\varphi_y^{n+1/2}) < \epsilon_{y,x}$, violating our assumption. Thus, necessarily

$$U_y(\varphi_y^{n+1/2}) \geqslant \varepsilon_{y,x} \quad \Rightarrow \quad U_z(A_{z,y}\varphi_z^{n+1}) \geqslant \varepsilon_{y,x} \quad \Rightarrow \quad \varphi_z^{n+1} \in A_{z,y}^{-1}\{U_y \geqslant \varepsilon_{y,x}\}. \tag{33}$$

Since this holds for all children y of z, we have that

$$\varphi_z^{n+1} \in \bigcap_{z \sim y} A_{z,y}^{-1}\{U_y \geqslant \varepsilon_{y,x}\} = L_{z,x}^{\varepsilon}. \tag{34}$$

Therefore

$$U_z(\varphi_z^{n+1}) \geqslant \inf U_z\left(L_{z,x}^{\varepsilon}\right) \quad \text{i.e.} \quad \varphi_z^{n+1} \in \{U_z \geqslant \varepsilon_{y,x}\}. \tag{35}$$

In the case of the nearest neighbor search, we can directly use the $L_{y,x}^{\varepsilon}$s instead of having to use the upper level sets defined by the $\varepsilon_{y,x}$s in the following proof for a tighter bound. ■

Theorem 3.2 (Point-wise convergence) *Consider the field of candidate matches at iteration n, φ^n. Define φ^{n+1} by applying an iteration of the Generic PatchMatch in Algorithm 1. Then, for all $\varepsilon > 0$, for all $x \in \mathbf{A}$, we have*

$$\mathbb{P}(U_x(\varphi_x^{n+1}) \geqslant \varepsilon) \leqslant \mathbb{P}(U_x(\varphi_x^{n}) \geqslant \varepsilon) \prod_{z \in \mathbf{A}} \left(C_2(z, \varepsilon_{z,x})^{\mu(z)} C_1(z, \varepsilon_{z,x})\right), \tag{36}$$

where $\mu(z)$ was defined in (5) as the number of parents of node z and C_i denotes the worst case transition probability for kernel Q_i, as in Eq. (8).

Proof We consider a topological ordering of the nodes in the query image. Given two nodes $z, y \in \mathbf{A}$, we use the notation $z < y$ if z precedes y. We denote by $y-1$ and $y+1$ the nodes before and after y in the ordering.

The proof consist on a recursion on the ordered set of nodes. For $y \in \mathbf{A}$ we define the following events:

$$\begin{aligned} \mathcal{S}_y \ &: \quad \forall z > y, \quad U_z(\varphi_z^n) \ \geqslant \varepsilon_{z,x} \\ \mathcal{P}_y \ &: \quad \forall z \leqslant y, \quad U_z(\varphi_z^{n+1}) \ \geqslant \varepsilon_{z,x}. \end{aligned} \tag{37}$$

The event $\mathcal{S}_y$ restricts the candidates at iteration n of the nodes *succeeding* y, whereas the event $\mathcal{P}_y$ considers the candidates at iteration $n+1$ of the nodes *preceding* y.

From Lemma 3.1 we have that: $U_x(\varphi_x^{n+1}) \geqslant \varepsilon \Rightarrow \forall z, U_z(\varphi_z^{n+1}) \geqslant \varepsilon_{z,x}$. Taking probabilities

$$\mathbb{P}(U_x(\varphi_x^{n+1}) \geqslant \varepsilon) \leqslant \mathbb{P}(\forall z, U_z(\varphi_z^{n+1}) \geqslant \varepsilon_{z,x}) \leqslant \mathbb{P}(\mathcal{P}_x \ ; \ \mathcal{S}_x). \tag{38}$$

The last equality holds because for $z > x$ as the level $\varepsilon_{z,x}$ is defined as -1. Therefore the conditions over φ_z^{n+1} or φ_z^n for such nodes are trivially satisfied.

We proceed by showing that the following recursive relation holds for any node y:

$$\mathbb{P}(\mathcal{P}_y \ ; \ \mathcal{S}_y) \leqslant \mathbb{P}(\mathcal{P}_{y-1} \ ; \ \mathcal{S}_{y-1}) C_1(y, \varepsilon_{y,x}) C_2(y, \varepsilon_{y,x})^{\mu(y)}. \tag{39}$$

The result then follows by applying this recursion backwards from x until the first node in the topological ordering. To show (39) we note that $\mathbb{P}(\mathcal{P}_y \ ; \ \mathcal{S}_y) = \mathbb{P}(\mathcal{P}_{y-1} \ ; \ \mathcal{S}_y \ ; \ U_y(\varphi_y^{n+1}) \geqslant \varepsilon_{y,x})$. Since φ_y^{n+1} results from a propagation step 7, we have that

$$\begin{aligned} U_y(\varphi_y^{n+1}) \geq \varepsilon_{y,x} \Rightarrow U_y(merge_y^k(\varphi_y^{n+1/2} \cup S_1 \varphi_y^{n+1/2})) \geqslant \varepsilon_{y,x} \\ \Rightarrow U_y(\varphi_y^{n+1/2}) \geqslant \varepsilon_{y,x} \text{ and } U_y(S_1 \varphi_y^{n+1/2}) \geqslant \varepsilon_{y,x}. \end{aligned} \tag{40}$$

Thus we have the following inequality:

$$\begin{aligned} \mathbb{P}(\mathcal{P}_y \ ; \ \mathcal{S}_y) &\leqslant \mathbb{P}(\mathcal{P}_{y-1} \ ; \ \mathcal{S}_y \ ; \ U_y(\varphi_y^{n+1/2}) \geqslant \varepsilon_{y,x} \ ; \ U_y(S_1 \varphi_y^{n+1/2}) \geqslant \varepsilon_{y,x}) \\ &\leqslant \mathbb{P}(U_y(S_1 \varphi_y^{n+1/2}) \geqslant \varepsilon_{y,x} \mid \mathcal{P}_{y-1} \ ; \ \mathcal{S}_y \ ; \ U_y(\varphi_y^{n+1/2}) \geqslant \varepsilon_{y,x}) \\ &\qquad \mathbb{P}(\mathcal{P}_{y-1} \ ; \ \mathcal{S}_y \ ; \ U_y(\varphi_y^{n+1/2}) \geqslant \varepsilon_{y,x}) \\ &\leqslant \sup \{Q_{1,y}(\eta, \{U_y \geqslant \varepsilon_{y,x}\}) \ : \ \eta \in \{U_y \geqslant \varepsilon_{y,x}\}\} \, \mathbb{P}(\mathcal{P}_{y-1} \ ; \ \mathcal{S}_y \ ; \ U_y(\varphi_y^{n+1/2}) \geqslant \varepsilon_{y,x}) \\ &\leqslant C_1(y, \varepsilon_{y,x}) \mathbb{P}(\mathcal{P}_{y-1} \ ; \ \mathcal{S}_y \ ; \ U_y(\varphi_y^{n+1/2}) \geqslant \varepsilon_{y,x}). \end{aligned} \tag{41}$$

The third inequality comes from the application of Lemma 3.0.1. The last step follows from the definition of C_1 in (8). We continue by noticing that,

$$\begin{aligned} U_y(\varphi_y^{n+1/2}) \geqslant \varepsilon_{y,x} \Rightarrow U_y(merge_y^k(\varphi_y^n \cup \bigcup_{z \sim y} A_{z,y} \varphi_z^{n+1} \cup \bigcup_{z \sim y} S_2 A_{z,y} \varphi_z^{n+1})) \geqslant \varepsilon_{y,x} \Rightarrow \\ U_y(\varphi_y^n) \geqslant \varepsilon_{y,x} \ ; \ \forall z \sim y, U_y(A_{z,y} \varphi_z^{n+1}) \geqslant \varepsilon_{y,x} \ ; \ \forall z \sim y, U_y(S_2 A_{z,y} \varphi_z^{n+1}) \geqslant \varepsilon_{y,x}. \end{aligned} \tag{42}$$

To simplify the notation, in the following we will drop the subscripts from $A_{z,y}$. The implication (42), yields the following

$$\begin{aligned} &\mathbb{P}(\mathcal{P}_{y-1} \ ; \ \mathcal{S}_y \ ; \ U_y(\varphi_y^{n+1/2}) \geqslant \varepsilon_{y,x}) \\ &\quad \leqslant \mathbb{P}(\mathcal{P}_{y-1} \ ; \ \mathcal{S}_y \ ; \ \forall z \sim y, \ U_y(A \varphi_z^{n+1}) \geqslant \varepsilon_{y,x}, \ U_y(S_2 A \varphi_z^{n+1}) \geqslant \varepsilon_{y,x} \ ; \ U_y(\varphi_y^n) \geqslant \varepsilon_{y,x}) \\ &\quad \leqslant \mathbb{P}(\mathcal{P}_{y-1} \ ; \ \mathcal{S}_{y-1} \ ; \ \forall z \sim y, \ U_y(A \varphi_z^{n+1}) \geqslant \varepsilon_{y,x}, \ U_y(S_2 A \varphi_z^{n+1}) \geqslant \varepsilon_{y,x}) \\ &\quad \leqslant \mathbb{P}(\forall z \sim y, \ U_y(S_2 A \varphi_z^{n+1}) \geqslant \varepsilon_{y,x} \mid \mathcal{P}_{y-1} \ ; \ \mathcal{S}_{y-1} \ ; \ \forall z \sim y, U_y(A \varphi_z^{n+1}) \geqslant \varepsilon_{y,x}) \\ &\qquad\qquad \mathbb{P}(\mathcal{P}_{y-1} \ ; \ \mathcal{S}_{y-1} \ ; \ \forall z \sim y \ , \ U_y(A \varphi_z^{n+1}) \geqslant \varepsilon_{y,x}) \\ &\quad \leqslant \prod_{z \sim y} \sup \{Q_{2,y}(\eta, \{U_y \geqslant \varepsilon_{y,x}\}) \ : \ \eta \in \{U_y \geqslant \varepsilon_{y,x}\}\} \\ &\qquad\qquad \mathbb{P}(\mathcal{P}_{y-1} \ ; \ \mathcal{S}_{y-1} \ ; \ \forall z \sim y \ , \ U_y(A \varphi_z^{n+1}) \geqslant \varepsilon_{y,x}) \\ &\quad \leqslant C_2(y, \{U_y \geqslant \varepsilon_{y,x}\})^{\mu(x)} \mathbb{P}(\mathcal{P}_{y-1} \ ; \ \mathcal{S}_{y-1} \ ; \ \forall z \sim y \ , \ U_y(A \varphi_z^{n+1}) \geqslant \varepsilon_{y,x}) \\ &\quad \leqslant C_2(y, \{U_y \geqslant \varepsilon_{y,x}\})^{\mu(x)} \mathbb{P}(\mathcal{P}_{y-1} \ ; \ \mathcal{S}_{y-1}). \end{aligned} \tag{43}$$

In the third step we have applied Corollary 3.0.2, whereas the forth step results from the definition of C_2.

The recursion (39) follows from (41) and (43). ■

Remark The proof is based on the fact that, due to the propagation, restricting the energy of the candidates at x implies constraints on the candidates of its ancestors (Lemma 3.1). We then bound the probability of all random samples drawn to satisfy those constraints. The more restrictive those constraints are, the smaller the probability of satisfying them. The Lemma in 3.1 establishes that $\varphi_z^{n+1} \in \{U_z \geqslant \varepsilon_{z,x}\}$. However, during the proof of Lemma 3.1 it is shown that the candidate sets φ_z^{n+1} belong to a set $L_{z,x}^{\varepsilon}$, which can be smaller than $\{U_z \geqslant \varepsilon_{z,x}\}$ (i.e. $L_{z,x}^{\varepsilon} \subseteq \{U_z \geqslant \varepsilon_{z,x}\}$. A tighter bound can be derived by considering this more restrictive constraint, and bounding the probability of $S_1\varphi_z^{n+1/2} \in L_{z,x}^{\varepsilon}$. In the cases of k-sets, the set $L_{z,x}^{\varepsilon}$ is a set of k-sets, and it is difficult to evaluate the probability of sampling a k-set in that set. This difficulty disappears for $k = 1$. Then $L_{z,x}^{\varepsilon} \subseteq \mathbf{B}$, and computing the probability of sampling in the allowed set becomes easier. Thus a tighter bound can be computed for $k = 1$.

Theorem 3.3 *Consider the field of candidate matches at iteration n, φ^n. Define φ^{n+1} by applying an iteration of the Generic PatchMatch in Algorithm 1. Then, for all $\varepsilon > 0$ we have*

$$\mathbb{P}(\|U_{\cdot}(\varphi_{\cdot}^{n+1})\|_{\infty} \geqslant \varepsilon) \leqslant \mathbb{P}(\|U_{\cdot}(\varphi_{\cdot}^{n})\|_{\infty} \geqslant \varepsilon) \prod_{z \in \mathbf{A}} \Big(C_2(z,\varepsilon)^{\mu(z)} C_1(z,\varepsilon) \Big) . \tag{44}$$

Proof The proof is similar to the one of Theorem 3.2 where instead of using Lemma 3.1 we use $\|U_{\cdot}(\varphi_{\cdot}^{n+1})\|_{\infty} \geqslant \varepsilon$, i.e. $\forall x \in \mathbf{A}, U_x(\varphi_x^{n+1}) \geqslant \varepsilon$. ■

Corollary 3.4 *Assume that for any pair (η, ξ) of sets of k candidate matches $Q_1(\eta, \xi) > 0$ (or $Q_2(\eta, \xi) > 0$). Let (φ^n) be a sequence defined by Algorithm 1. Then $\forall x \in \mathbf{A}, \mathbb{E}[U_x(\varphi_x^n)] \xrightarrow[n\to\infty]{} 0$ and $\mathbb{E}[\|U_{\cdot}(\varphi_{\cdot}^n)\|_{\infty}] \xrightarrow[n\to\infty]{} 0$.*

Proof We will show the convergence in the mean for a single node $x \in \mathbf{A}$, i.e. $\mathbb{E}[|U_x(\varphi_x^n)|] \xrightarrow[n\to\infty]{} 0$. We recall that for a non-negative random variable X, $\mathbb{E}[X] = \int_{x>0} \mathbb{P}(X \geqslant x)$. Then we have:

$$\begin{aligned}
\mathbb{E}[|U_x(\varphi_x^{n+1})|] = \mathbb{E}[U_x(\varphi_x^{n+1})] &= \int_{\varepsilon>0} \mathbb{P}(U_x(\varphi_x^{n+1}) \geqslant \varepsilon) && (45)\\
&\leqslant \int_{\varepsilon>0} \prod_{z \in \mathbf{A}} \Big(C_2(z,\varepsilon_{z,x})^{\mu(z)} C_1(z,\varepsilon_{z,x}) \Big) \mathbb{P}(U_x(\varphi_x^{n}) \geqslant \varepsilon) && (46)\\
&\leqslant \int_{\varepsilon>0} C_2(x,\varepsilon)^{\mu(x)} C_1(x,\varepsilon)\, \mathbb{P}(U_x(\varphi_x^{n}) \geqslant \varepsilon) && (47)\\
&\leqslant \int_{\varepsilon>0} \Big(\sup_{\alpha>0} C_2(x,\alpha)\Big)^{\mu(x)} \Big(\sup_{\alpha>0} C_1(x,\alpha)\Big) \mathbb{P}(U_x(\varphi_x^{n}) \geqslant \varepsilon) && (48)\\
&\leqslant \Big(\sup_{\alpha>0} C_2(x,\alpha)\Big)^{\mu(x)} \Big(\sup_{\alpha>0} C_1(x,\alpha)\Big) \int_{\varepsilon>0} \mathbb{P}(U_x(\varphi_x^{n}) \geqslant \varepsilon) && (49)\\
&\leqslant \Big(\sup_{\alpha>0} C_2(x,\alpha)\Big)^{\mu(x)} \Big(\sup_{\alpha>0} C_1(x,\alpha)\Big) \mathbb{E}[U_x(\varphi_x^{n})]. && (50)
\end{aligned}$$

Since $Q_1(\eta, \xi) > 0$, for all η, ξ, we have that $Q_1(\eta, \{U_x = 0\}) > 0$ for any η. In the discrete case (the one only one considered for this theorem) the sup is achieved and is not 1. Therefore $\sup_{\alpha>0} C_1(x,\alpha) = \max_{\alpha>0} C_1(x,\alpha) < 1$ and the convergence follows. With a similar derivation can be used to show the convergence of the L_{∞} norm of the whole NNF. ■

4. Specific *PatchMatch* algorithms

4.1. The original *PatchMatch* algorithm

Proposition 4.1 *The specific basic PatchMatch algorithm described in this section algorithm converges in probability to a NNF which minimizes the energy, namely*

$$\lim_{n\to\infty} \mathbb{P}(U_x(\varphi_x^n) \geqslant \varepsilon) = 0, \forall \varepsilon > 0, x \in \boldsymbol{A}, \tag{51}$$

with a geometric convergence rate.
Moreover for all $\varepsilon > 0$, for all $x \in \boldsymbol{A}$, we have that

$$\mathbb{P}(U_x(\varphi_x^{n+1}) \geqslant \varepsilon) \leqslant \mathbb{P}(U_x(\varphi_x^n) \geqslant \varepsilon) \prod_{z\in\mathbf{A}} \left(1 - (1 - C'(c_i, \varepsilon_{z,x}))^k\right), \tag{52}$$

with

$$C'(z, \alpha) := \sup_\eta Q'(\eta, \{U_z \geqslant a\}). \tag{53}$$

For $\alpha > 0$ we can guarantee that $C'(z, \alpha) < 1$.

Proof The bound in Theorem 3.2 applies. We now express C_1 in terms of the new C'. For that, we compute an upper bound for $\mathbb{P}(U_z(S_1\varphi_z^n) \geqslant a \mid U_z(\varphi_z^n) \geqslant a, E)$ with $a \geqslant 0$ and E any event, $\forall z \in \mathbf{A}$. We first remark that, if $S_1^i\varphi$ is the i^{th} sample generated from the random sampling around φ,

$$\mathbb{P}(U_z(S_1\varphi_z^n) \geqslant a \mid U_z(\varphi_z^n) \geqslant a, E) = 1 - \prod_{l=1}^{k} \mathbb{P}(U_z(S_1^l\varphi_z^n) \leqslant a \mid U_z(\varphi_z^n) \geqslant a, E). \tag{54}$$

For a candidate $\phi \in S_1\varphi_z^n$,

$$\mathbb{P}(U_z(\phi) \geqslant a \mid U_z(\varphi_z^n) \geqslant a, E) \leqslant \sup_{\eta\in\{U_z\geqslant a\}} Q(\eta, \{U_z \geqslant a\}) \tag{55}$$

Let us remind that η and $S\eta$ are k-sets, i.e. sets of k distinct candidates. The candidates in $S\eta$ are sampled centered at the best candidate in η (the one minimizing U_z). We know that $\eta \in \{U_z \geqslant a\}$, which only constrains the worst candidate in η, but says nothing about the best candidate. This is why for this proof there is no restriction on where the sample comes from. Therefore, with

$$\sup_\eta Q'(\eta, \{U_z \geqslant a\}) =: C'(z, a) \tag{56}$$

we have

$$\mathbb{P}(U_z(\phi) \leqslant a \mid U_z(\varphi_z^n) \geqslant a, E) \geqslant 1 - C'(z, a) \tag{57}$$

and

$$\mathbb{P}(U_z(S_1\varphi_z^n) \geqslant a \mid U_z(\varphi_z^n) \geqslant a, E) \leqslant 1 - (1 - C'(z, a))^k. \tag{58}$$

Since the support of the random search is the full image, it guarantees that $\forall z \in \mathbf{A}, \forall a > 0, C'(z, a) < 1$. This implies that the bound found is strictly inferior to one and therefore the convergence is insured. ■

Corollary 4.2 *In the case of the search of the nearest neighbor, the upper bound can be written as*

$$\mathbb{P}(U_x(\varphi_x^{n+1}) \geqslant \varepsilon) \leqslant \prod_{z \in \boldsymbol{A}} C'(z, \varepsilon_{z,x}) \mathbb{P}(U_x(\varphi_x^n) \geqslant \varepsilon). \tag{59}$$

with

$$C'(z, \alpha) := \sup_{\eta \in \{U_z \geqslant a\}} Q'(\eta, \{U_z \geqslant a\}). \tag{60}$$

This bound is actually tighter than the one derived in [1].

Proof The derivation of the tighter bound presented in Equation (59) is the same as in Proposition 4.1. The difference comes from the that in the case $k = 1$, the best current match is also the worst current match therefore the element used to sample is necessarily with an energy larger than $\varepsilon_{z,x}$.

The second part of the proof concerns the comparison with the bound of [1]. The difference between the bound (59) and the one in [1] lies in the levels $\varepsilon_{z,x}$ inside the factors C'. The levels used in [1] are of the form $\varepsilon - \ell_{z,x}$, where $\ell_{z,x}$ is defined as follows:

$$\ell_{z,x} = \min_{c \in \mathcal{P}(z,x)} \sum_{i=1}^{n} d_{c_i, c_{i-1}} \tag{61}$$

with $\mathcal{P}(z, x)$ being the set of all the paths from z to x in the graph and $d_{c_i,c_{i-1}} = \|U_{c_i} - U_{c_{i-1}} \circ A\|_\infty$ Therefore, due to the monotonicity of C' we just have to show that $\varepsilon_{z,x} \geqslant \varepsilon - \ell_{z,x}$.

For $z = x$, $\varepsilon_{z,x} = \varepsilon$ and $\ell_{z,x} = 0$ therefore the property is verified. Suppose now that the property is true for any y such that $y > z$. We will show that in this case the property stays true for z. Suppose that $\varepsilon_{z,x} < \varepsilon - \ell_{z,x}$. Because $\varepsilon_{z,x} = \inf U_z(L^\varepsilon_{z,x})$, this means that it exists $\eta \in L^\varepsilon_{z,x}$ such that $U_z(\eta) < \varepsilon - \ell_{z,x}$. Let y be the child of z such that $\ell_{z,x} = \ell_{y,x} + d_{z,y}$. In this case we have that

$$U_z(\eta) + d_{z,y} < \varepsilon - \ell_{z,x} + d_{z,y} \tag{62}$$

therefore

$$U_y(A\eta) \leqslant U_z(\eta) + d_{z,y} < \varepsilon - \ell_{y,x}. \tag{63}$$

We then have found $\eta \in L^\varepsilon_{y,x}$ *i.e.* $A\eta \in \{U_y \geqslant \varepsilon_{y,x}\}$ therefore $\varepsilon_{y,x} \leqslant U_y(A\eta) < \varepsilon - \ell_{y,x}$ which is contradictory with the hypothesis. The property is then also valid for z. ■

Remark The tighter bound for the specific case of $k = 1$ mentioned in the Remark after the proof of Theorem 12 applies in this case as well, yielding a bound for the original PatchMatch that is tighter than the one from Proposition 4.1. We compared both bounds and in practice the difference between them is small. For our experiments in the paper we have used a bound similar to the one of the Proposition 4.1.

4.2. The CSH algorithm

Lemma 4.4 *If $\mathcal{H}$ is (R, cR, p_1, p_2)-sensitive then an OR family $\mathcal{G}$ created using n functions from $\mathcal{H}$ is $(R, cR, 1 - (1 - p_1)^n, 1 - (1 - p_2)^n)$-sensitive.*

Proof Let p, q such that $\|p - q\| \leqslant R$ and $g \in \mathcal{G}$ generated by $h_1, \ldots, h_n \in \mathcal{H}$ with $\mathcal{H}$ (R, cR, p_1, p_2)-sensitive.

$$\mathbb{P}_{\mathcal{G}}(g(p) \neq g(q)) = \mathbb{P}_{\mathcal{H}}(h_1(p) \neq h_1(q), \ldots, h_n(p) \neq h_n(q)) \tag{64}$$

Because the h_is are independent,

$$\mathbb{P}_{\mathcal{H}}(h_1(p) \neq h_1(q), \ldots, h_n(p) \neq h_n(q)) = \prod_{i=1}^{n} \mathbb{P}_{\mathcal{H}}(h_i(p) \neq h_i(q)) \tag{65}$$

Using the (R, cR, p_1, p_2)-sensitive property of $\mathcal{H}$, for all i

$$\mathbb{P}_{\mathcal{H}}(h_i(p) \neq h_i(q)) = 1 - \mathbb{P}_{\mathcal{H}}(h_i(p) = h_i(q)) \tag{66}$$
$$\leqslant 1 - p_1 \tag{67}$$

Therefore

$$\mathbb{P}_{\mathcal{G}}(g(p) \neq g(q)) \leqslant (1 - p_1)^n \tag{68}$$

and

$$\mathbb{P}_{\mathcal{G}}(g(p) = g(q)) \geqslant 1 - (1 - p_1)^n. \tag{69}$$

Using similar derivation, the corresponding result can be proved for the second part of the sensitive definition. This result in $\mathcal{G}$ an OR family function being $(R, cR, 1-(1-p_1)^n, 1-(1-p_2)^n)$-sensitive. ■

Proposition 4.5 *For a (R, cR, p_1, p_2)-sensitive family of hashing functions such that $R \geqslant \max_{z \in \mathbf{A}} K_z$ (see Definition 2.1), the sequence (φ^n) defined by the CSH algorithm converges in probability to a minimizer of the total energy, in the sense that*

$$\lim_{n \to \infty} \mathbb{P}(U_x(\varphi_x^n) \geqslant \varepsilon) = 0, \forall \varepsilon > 0, x \in \mathbf{A}, \tag{70}$$

with a geometric convergence rate. Moreover for all $\varepsilon > 0$, for all $x \in \mathbf{A}$,

$$\mathbb{P}(U_x(\varphi_x^{n+1}) \geqslant \varepsilon) \leqslant \mathbb{P}(U_x(\varphi_x^n) \geqslant \varepsilon) \prod_{z \in \mathbf{A}} C_2(z, \{U_z \geqslant \varepsilon_{z,x}\})^{\mu(z)} f(p_1, \varepsilon_{z,x}\}), \tag{71}$$

where $f(\alpha, \beta) = (1 - \alpha^k)$ if $\varepsilon_{z,x} > 0$, 1 otherwise.

From a set of LSH hash functions $\mathcal{H}$, a OR family of function $\mathcal{G}$ can also be defined. The function $g \in \mathcal{G}$ is based on a set of n random functions $h_1, \ldots, h_n$ from $\mathcal{H}$ such that for all p, q, $g(p) = g(q)$ if and only if there exist $i \in [\![1, n]\!]$ such that $h_i(p) = h_i(q)$.

Proof Let $R \geqslant \max_{z \in \mathbf{A}} K_z$ and $\mathcal{H}$ an (R, cR, p_1, p_2)-sensitive family of functions. Consider also $\mathcal{G}$ an OR family function based on $\mathcal{H}$ so that a function from $\mathcal{G}$ is generated using at least k functions from $\mathcal{H}$. An upper bound for $\mathbb{P}(U_x(S_1\varphi_x^n) \geqslant a \mid E)$ with $a > 0$, $\forall x \in \mathbf{A}$ and E an undefined event will now be derived.
Firstly,

$$\forall p \in \mathbf{B}, \|p - x\| \leqslant R \Rightarrow \mathbb{P}(h(p) = h(x)) \geqslant p_1. \tag{72}$$

R is then chosen such that for all $x \in \mathbf{A}$, $\|x - N_k(x)\| \leqslant R$, where $N_k(x) \in \mathbf{B}$ is the k^{th} nearest neighbor of x. This property is true for the l^{th} nearest neighbor $N_l(x)$, with $l \leqslant k$. Therefore we have $\mathbb{P}(h(N_l(x)) = h(x)) \geqslant p_1$. Moreover the binning is independent from the current matching φ_x^n.

For a given $h \in \mathcal{H}$ and $\mathcal{H}' \subset \mathcal{H}$ such that $h \in \mathcal{H}'$, for a query q, for all $p \in \mathbf{B}$

$$h(q) = h(p) \Rightarrow \exists h' \in \mathcal{H}', h'(q) = h'(p) \tag{73}$$

Therefore for a list $H' = \{h_1, \ldots, h_k\} \subset \mathcal{H}$, for a query q, for all $p \in \mathbf{B}$

$$h_1(q) = h_1(p_1), \ldots, h_k(q) = h_k(p_n) \Rightarrow (\exists h \in \mathcal{H}', h(q) = h(p_1)), \ldots, (\exists h \in \mathcal{H}', h(q) = h(p_k)) \tag{74}$$

and

$$\mathbb{P}(h_1(q) = h_1(p_1), \ldots, h_k(q) = h_k(p_n)) \leqslant \mathbb{P}((\exists h \in \mathcal{H}', h(q) = h(p_1)), \ldots, (\exists h \in \mathcal{H}', h(q) = h(p_k))) \tag{75}$$

When working with the elements of the OR family $\mathcal{G}$, we can define $\mathcal{H}'$ as the set functions from $\mathcal{H}$ used to generate g ($\mathcal{H}'$ verifies the properties of the previously used $\mathcal{H}'$ because at least k functions are used to generate each element of $\mathcal{G}$),

$$\begin{aligned}\mathbb{P}_{\mathcal{G}}(g(q) = g(p_1), \ldots, g(q) = g(p_n)) &= \mathbb{P}_{\mathcal{H}}((\exists h \in \mathcal{H}', h(q) = h(p_1)), \ldots, (\exists h \in \mathcal{H}', h(q) = h(p_n))) \\ &\geqslant \mathbb{P}_{\mathcal{H}}(h_1(q) = h_1(p_1), \ldots, h_n(q) = h_n(p_n))\end{aligned} \tag{76}$$

which leads to

$$\mathbb{P}_{\mathcal{G}}(g(q) = g(p_1), \ldots, g(q) = g(p_k)) \geqslant \mathbb{P}_{\mathcal{H}}(h_1(q) = h_1(p_1), \ldots, h_k(q) = h_k(p_k)) \tag{77}$$

because the h_i are independent

$$\mathbb{P}_{\mathcal{G}}(g(q) = g(p_1), \ldots, g(q) = g(p_n)) \geqslant \prod_{i=1}^{k} \mathbb{P}_{\mathcal{H}}(h_i(q) = h_i(p_i)) \geqslant p_1^k \tag{78}$$

when using the original LSH property for $\mathcal{H}$. This implies that

$$\mathbb{P}(U_x(S_1\varphi_x^n) = 0) = \mathbb{P}_{\mathcal{G}}(g(q) = g(p_1), \ldots, g(q) = g(p_n)) \geqslant p_1^k \tag{79}$$

Finally, $\mathbb{P}(U_x(S_1\varphi_x^n) \geqslant a \mid E) \leqslant 1 - \mathbb{P}(U_x(S_1\varphi_x^n) \leqslant a \mid E) \leqslant 1 - \mathbb{P}(U_x(S_1\varphi_x^n) = 0 \mid E) \leqslant 1 - p_1^k$ Replacing $C_1(z, \varepsilon)$, for $z \in \mathbf{A}$ and $\varepsilon > 0$, by $(1 - p_1^k)$ in the proof of Theorem 3.2 gives the result in Proposition 4.5. ■

References

[1] P. Arias, V. Caselles, and G. Facciolo. Analysis of a variational framework for exemplar-based image inpainting. Multiscale Modeling & Simulation, 10(2):473–514, 2012. 7