# Supplementary Material

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## 1 Fractional-order total variation

In this section, we consider $\mathcal{C}_0^\alpha(\Omega, \mathbb{R}^d)$, where $\alpha > 0$, as the space comprising functions that are continuously differentiable and $\alpha$-compactly supported. Then the $\alpha$-order total variation is defined as

$$TV^\alpha(u) = \sup_{\Phi} \int_\Omega (u \operatorname{div}^\alpha \Phi)\, dx, \tag{1}$$

where $\Phi \in \mathcal{C}_0^\alpha(\Omega, \mathbb{R}^d)$, satisfying the constraint $|\phi(x)| \leq 1$ for all $x \in \Omega$.

In the case of $\Phi \in \mathcal{C}_0^\alpha(\Omega, \mathbb{R}^2)$, the $\alpha$-order divergence is defined as

$$\operatorname{div}^\alpha \Phi = \frac{\partial^\alpha \phi_1}{\partial x_1^\alpha} + \frac{\partial^\alpha \phi_2}{\partial x_2^\alpha}, \tag{2}$$

where $\frac{\partial^\alpha \phi_i}{\partial x_i^\alpha}$ represents the fractional $\alpha$-order derivative of $\phi_i$ along the $x_i$ direction, with $i = 1, 2$.

We note that the expression $TV^\alpha(u)$ remains invariant across different definitions of $\frac{\partial^\alpha \phi_i}{\partial x_i^\alpha}$, due to the adherence of $\Phi$ to equivalence conditions. Nonetheless, the fractional derivative $\frac{\partial^\alpha u}{\partial x_i^\alpha}$ varies with different definitions of fractional derivatives. We specifically focus on three classical definitions: Riemann-Liouville (R-L), Grünwald-Letnikov (G-L), and Caputo fractional-order derivatives.

Based on the semi-norm presented in Equation 1, we introduce the space of functions characterized by bounded $\alpha$-order total variation, designated as

$$BV^\alpha(\Omega) := \left\{ u \in L^1(\Omega) : TV^\alpha(u) < \infty \right\}. \tag{3}$$

Analogous to the space of functions with bounded variations, $BV^\alpha(\Omega)$ can be demonstrated to be a Banach space. Furthermore, $TV^\alpha(u)$ in $BV^\alpha(\Omega)$ space exhibits the property of lower semi-continuity.

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