A functional approach to rotation equivariant non-linearities for Tensor Field Networks (Supplementary material).

Adrien Poulenard
Stanford University
padrien@stanford.edu

Leonidas J. Guibas
Stanford University
guibas@cs.stanford.edu

We provide a proof of theorem (3.1), we show the equivariance of the exact form of the non linearity defined in equation (16). More precisely, it commutes with the Wigner matrix.

Theorem (3.1). *Our continuous non linearity $\xi(\bullet)$ is $\mathrm{SO}(3)$ equivariant. That is, for any rotation $R \in \mathrm{SO}(3)$ we have $\xi(R.f)^{\ell}_{i:c} = D^{\ell}(R)\xi(f)^{\ell}_{i:c}$.*

Proof. We have:

$$
\begin{aligned}
\xi(R.f)^{\ell}_{i:c} &:= \int_{\mathcal{S}_2} \xi(\mathcal{F}^{+}(R.f)_{ic}(x))Y^{\ell}(x)\mathrm{d}x \\
&= \int_{\mathcal{S}_2} \xi\left(\sum_{\ell=0}^{\ell_{\max}} \langle D^{\ell}(R)f^{\ell}_{i:c}, Y^{\ell}(x)\rangle\right) Y^{\ell}(x)\mathrm{d}x \\
&= \int_{\mathcal{S}_2} \xi\left(\sum_{\ell=0}^{\ell_{\max}} \langle f^{\ell}_{i:c}, D^{\ell}(R)^{\top} Y^{\ell}(x)\rangle\right) Y^{\ell}_{m}(x)\mathrm{d}x \\
&= \int_{\mathcal{S}_2} \xi\left(\sum_{\ell=0}^{\ell_{\max}} \langle f^{\ell}_{i:c}, Y^{\ell}(R^{-1}x)\rangle\right) Y^{\ell}(x)\mathrm{d}x \\
&\underset{t=R^{-1}x}{=} \int_{\mathcal{S}_2} \xi\left(\sum_{\ell=0}^{\ell_{\max}} \langle f^{\ell}_{i:c}, Y^{\ell}(t)\rangle\right) Y^{\ell}(Rt)\mathrm{d}t \\
&= \int_{\mathcal{S}_2} \xi\left(\sum_{\ell=0}^{\ell_{\max}} \langle f^{\ell}_{i:c}, Y^{\ell}(t)\rangle\right) D^{\ell}(R)Y^{\ell}(t)\mathrm{d}t \\
&= D^{\ell}(R)\xi[f]^{\ell}_{i:c}
\end{aligned}
$$

which concludes the proof. □

We provide a proof of theorem (3.2), we show that the discrete version of our non linearity defined in equation (19) is equivariant w.r.t. the rotation group of the sampling.

Theorem (3.2). *Our discrete non linearity $\xi[\bullet]$ is equivariant w.r.t. the symmetry group of the sampling. That is, for any rotation $R \in \mathrm{SO}(3)$ in the symmetry group of the sampling P we have $\xi[R.f]^{\ell}_{i:c} = D^{\ell}(R)\xi[f]^{\ell}_{i:c}$.*

Proof. Let $P = \{p_1, \dots, p_k\} \subset \mathcal{S}_2$ be our discrete sampling of the unit sphere and $S = \{R_1, \dots, R_m\} \subset \mathrm{SO}(3)$ its symmetry group. The action of S on P is given by permutations, that is, for all j there exist a permutation $\sigma_j \in \mathfrak{S}_k$ such that for all i, $R_j p_i = p_{\sigma_j(i)}$. We have:

$$
\begin{aligned}
\mathcal{F}^{+}(R_j.f)_{vic} &:= \mathcal{F}^{+}(R_j.f_{v:c})(p_i) \\
&= \sum_{\ell,m} \langle D^{\ell}(R_j)f^{\ell}_{v:c}, Y^{\ell}(p_i)\rangle \\
&= \sum_{\ell,m} \langle f^{\ell}_{v:c}, D^{\ell}(R_j)^{\top} Y^{\ell}(p_i)\rangle \\
&= \sum_{\ell,m} \langle f^{\ell}_{v:c}, D^{\ell}(R_j)^{\top} Y^{\ell}(p_i)\rangle \\
&= \sum_{\ell,m} \langle f^{\ell}_{v:c}, Y^{\ell}(p_{\sigma^{-1}(i)})\rangle \\
&= \mathcal{F}^{+}(f)_{v,\sigma_j^{-1}(i),c}
\end{aligned}
$$

Replacing $\mathcal{F}^{+}(R_j.f)_{vic}$ by $\mathcal{F}^{+}(f)_{v,\sigma_j^{-1}(i),c}$ in the expression of $\xi[R_j.f]^{\ell}_{i:c}$ we obtain:

$$
\begin{aligned}
&\xi[R_j.f]^{\ell}_{i:c} \\
&= \frac{4\pi}{k}\sum_{i=1}^{k} \xi \circ \mathcal{F}^{+}(R_j.f))_{vic} Y^{\ell}(p_i) \\
&= \frac{4\pi}{k}\sum_{i=1}^{k} \mathcal{F}(\xi \circ \mathcal{F}^{+}(f))_{v,\sigma_j^{-1}(i),c} Y^{\ell}(p_i) \\
&= \frac{4\pi}{k}\sum_{i=1}^{k} \mathcal{F}(\xi \circ \mathcal{F}^{+}(f))_{vic} Y^{\ell}(p_{\sigma_j(i)}) \\
&= \frac{4\pi}{k}\sum_{i=1}^{k} \mathcal{F}(\xi \circ \mathcal{F}^{+}(f))_{vic} Y^{\ell}(R_j p_i) \\
&= \frac{4\pi}{k}\sum_{i=1}^{k} \mathcal{F}(\xi \circ \mathcal{F}^{+}(f))_{vic} D^{\ell}(R_j) Y^{\ell}\binom{p}{i} \\
&= D^{\ell}(R_j)\xi[f]^{\ell}_{i:c}
\end{aligned}
$$

which concludes the proof. □

We propose to measure the impact of the number of samples of the Fibonacci sampling on the equivariance of our non linearity for different types of features. For any function f over the sphere expressed in the SH basis we can compute the standard deviation of the non discretized nonlinearity ξ applied to f under rotations of the sphere sampling. Ideally we would like to average this standard deviation over the unit norm functions but, this infeasible as it would require sampling a high dimensional space. Instead, for each ℓ we compute the average standard deviation of the image of the degree ℓ spherical harmonics. For any $k \in \mathbb{N}^*$ we denote by $p^k = \{p_1^k, \dots, p_k^k\} \subset \mathcal{S}_2$ the Fibonacci sampling of the unit sphere with k samples. For each type ℓ and $m \in [\![-\ell, \ell]\!]$ and for any rotation matrix R we define:

$$f_m^\ell(R, p^k) = \bigoplus_{q=0}^{\ell_{\max}} \frac{4\pi}{k} \sum_{i=1}^{k} \xi(Y_m^\ell(Rp_i^k))Y^k(Rp_i)$$

ideally $f_m^\ell(R, p)$ should be invariant w.r.t. R. We compute its standard deviation w.r.t. R to measure the equivariance error:

$$\mathbb{E}_R[f_m^\ell(R, p)] := \int_{\mathrm{SO}(3)} f_m^\ell(R, p)\mathrm{d}R$$

$$\mathrm{Var}(\ell, k) := \frac{1}{2\ell + 1} \sum_{m=-\ell}^{\ell} \mathrm{Var}_R(f_m^\ell(R, p^k)_m)$$

$$:= \frac{1}{2\ell + 1} \sum_{m=-\ell}^{\ell} \int_{\mathrm{SO}(3)} \|f_m^\ell(R, p) - \mathbb{E}_R[f_m^\ell(R, p)]\|_2^2 \mathrm{d}R$$

In practice we approximate the integrals by averaging over 10000 random rotations. We report the standard variation σ w.r.t. the degree of spherical harmonics ℓ and number of samples k in Figure (1). As expected, we observe that the error is decreasing with the number of samples. Also higher frequency spherical harmonics produces higher error but, we observe a sharper decrease of the error initially for higher frequencies as we increase the number of samples.

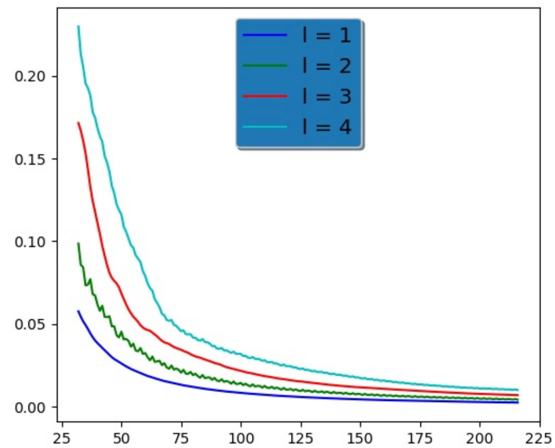

Figure 1. Average standard deviation $\sigma(\ell, k)$ (y-axis) of our non linearity applied to degree ℓ spherical harmonics under rotation of the Fibonacci sampling w.r.t. the number k of samples (x-axis). We used $\ell_{\max} = 4$.