Supplementary Materials
Consensus Maximisation Using Influences of Monotone Boolean Functions

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In this supplementary material we provided additional material on 1) The theoretical analysis of the influence function under "ideal" case (Section 1), 2) Influence calculation Algorithm (Section 2), 3) Influences of restricted Boolean functions (Section 3) and 4) Additional Results on synthetic and real data (Section 4).

1. Influence Function in the Ideal Case

In this section we provide the proofs of the two theorems (Theorem 3.1 and 3.2) presented in the main paper. For completeness we have replicated some definitions here which also appear in the main paper.

1.1. Ideal single structure case

We first formally define an ideal structure in data. Let $L_k := \{\mathbf{x} \in \{0,1\}^n : \|\mathbf{x}\|_1 = k\}$ be the level k in the n-dimensional Boolean cube and, $L_{\leq k}$ the levels below $k+1$.

Definition 3.1. *Given a monotone Boolean function f, for $\mathbf{x}^k \in L_k$ ($p < k \leq n$), f is called ideal with respect to $\mathbf{x}^k$ if*

$$f(\mathbf{x}) = \begin{cases} 0 & \forall \mathbf{x} \in B_{\mathbf{x}^k} \cup L_{\leq p} \\ 1 & \textit{others} \end{cases} \tag{1}$$

where $B_{\mathbf{x}^k} = \left\{\mathbf{x} \in \{0,1\}^n : d\left(\mathbf{x}, \mathbf{x}^k\right) = l, \mathbf{x} \in L_{k-l}\right\}$ for all $0 \leq l \leq k-p-1$, is the Boolean sub-cube determined by $\mathbf{x}^k$. Here, $d(\cdot,\cdot)$ is the hamming distance.

Theorem 3.1. *If a monotone Boolean function f is ideal with respect to $\mathbf{x}^k \in L_k$, then*

$$\mathrm{Inf}_i[f] = \begin{cases} \frac{C_p^{n-1} - C_p^{k-1}}{2^n} & \textit{if } i \textit{ is inlier} \\ \frac{C_p^{n-1} + \sum_{l=p+1}^{k} C_l^k}{2^n} & \textit{if } i \textit{ is outlier} \end{cases} \tag{2}$$

Proof. At level l ($p+1 \leqslant l \leqslant k$), by the definition of the Boolean sub-cube $B_{\mathbf{x}^k}$, boundary edges (pointing from feasible area to infeasible area) only come from flipping outliers (changing from 0 to 1), which have $C_l^k(n-k)$ edges.

At level p, boundary edges are composed by three parts: (1) edges pointing from $B_{\mathbf{x}^k} \cap L_p$ to $L_{p+1} \setminus B_{\mathbf{x}^k}$ by flipping outliers, (2) edges pointing from $L_p \setminus B_{\mathbf{x}^k}$ to $L_{p+1} \setminus B_{\mathbf{x}^k}$ by flipping inliers, and (3) edges pointing from $L_p \setminus B_{\mathbf{x}^k}$ to $L_{p+1} \setminus B_{\mathbf{x}^k}$ by flipping outliers. The first part has $C_p^k(n-k)$ edges. Since

$$2 \leqslant d(x,y) \leqslant \min\{2p, 2(n-k)\}$$

for any $x \in L_p \setminus B_{x^k}$ and $y \in L_p \cap B_{x^k}$, then the second part has $\sum_{l=\max\{0,p+k-n\}}^{p-1} C_l^k C_{p-l}^{n-k}(k-l)$ edges and the third part has $\sum_{l=\max\{0,p+k-n\}}^{p-1} C_l^k C_{p-l}^{n-k}(n-k-(p-l))$ edges. Since all inliers (or outliers) have the same influences, then we have

$$\begin{aligned} 2^n \mathrm{Inf}_i[f] &= \sum_{l=\max\{0,p+k-n\}}^{p-1} C_l^k C_{p-l}^{n-k}\left(1 - \frac{l}{k}\right) \\ &= \sum_{l=\max\{0,p+k-n\}}^{p-1} \left(C_l^k C_{p-l}^{n-k} - C_{l-1}^{k-1} C_{p-l}^{n-k}\right) \\ &= C_p^n - C_p^k - \left(C_{p-1}^{n-1} - C_{p-1}^{k-1}\right) \\ &= C_p^{n-1} - C_p^{k-1} \end{aligned}$$

and

$$\begin{aligned} 2^n \mathrm{Inf}_j[f] &= \sum_{l=p}^{k} C_l^k + \\ &\quad \sum_{l=\max\{0,p+k-n\}}^{p-1} C_l^k C_{p-l}^{n-k}\left(1 - \frac{p-l}{n-k}\right) \\ &= \sum_{l=p}^{k} C_l^k + \\ &\quad \sum_{l=\max\{0,p+k-n\}}^{p-1} \left(C_l^k C_{p-l}^{n-k} - C_l^k C_{p-l-1}^{n-k-1}\right) \\ &= \sum_{l=p}^{k} C_l^k + C_p^n - C_p^k - C_{p-1}^{n-1} \\ &= C_p^{n-1} + \sum_{l=p+1}^{k} C_l^k. \end{aligned}$$

□

1.2. Ideal K-structure case

In what follows, we generalize the above result to ideal K-structure ($K \geq 1$), namely there are K upper zeros only and the Boolean sub-cubes determined by these upper zeros are disjoint above level p.

Definition 3.1. *Let f be a monotone Boolean function and $\left\{\mathbf{x}^{k_r}\right\}_{r=1}^{K}$ upper zeros, where $p < k_1 \leq k_2 \leq \cdots \leq k_K \leq n$, then f is called K-ideal with respect to $\left\{\mathbf{x}^{k_r}\right\}_{r=1}^{K}$ if*

1) $d\left(B_{\mathbf{x}^{k_i}} \setminus L_{\leq p}, B_{\mathbf{x}^{k_j}} \setminus L_{\leq p}\right) > 0 \quad \forall k_i \neq k_j$

2) $$f\left(\mathbf{x}\right) = \begin{cases} 0 & \forall \mathbf{x} \in \bigcup_{r=1}^{K} B_{\mathbf{x}^{k_r}} \cup L_{\leq p} \\ 1 & others \end{cases} \tag{3}$$

Let $\mathcal{S}_{\mathbf{x}^{k_r}}^{c}$ be the set of inlier (if $c = 1$) or the set of outliers (if $c = 0$) with respect to $\mathbf{x}^{k_r}$. Then we can define

$$\mathcal{S}_{c_1 c_2 \cdots c_K} = \bigcap_{r=1}^{K} \mathcal{S}_{\mathbf{x}^{k_r}}^{c_r} \quad c_r \in \{0, 1\} \tag{4}$$

which represent the index set of inlier to structures where bit string $c_1 c_2 \cdots c_K$ is one. For example, $\mathcal{S}_{11\cdots1}$ is the index set of points that are inliers with respect to all $\mathbf{x}^{k_r}$, and $\mathcal{S}_{00\cdots0}$ is the index set of points that are outliers with respect to all $\mathbf{x}^{k_r}$.

Theorem 3.2.

$$2^n \mathrm{Inf}_{\mathcal{S}_{c_1 \cdots c_K}}\left[f\right] = C_p^{n-1} + \sum_{\substack{c_r=0 \\ 1 \leq r \leq K}} \sum_{l=p+1}^{k_r} C_l^{k_r} - \sum_{\substack{c_r=1 \\ 1 \leq r \leq K}} C_p^{k_r-1} \tag{5}$$

where $\mathrm{Inf}_{\mathcal{S}_{c_1 \cdots c_K}}\left[f\right]$ denote the influence $\mathrm{Inf}_i\left[f\right]$ of $i \in \mathcal{S}_{c_1 c_2 \cdots c_K}$.

Proof. We prove the theorem by induction on K. When $K = 1$, equation (5) holds by Theorem 3.1.

Suppose equation (5) holds for $1, 2, \cdots, K-1$, namely,

$$2^n \mathrm{Inf}_{\mathcal{S}_{c_1 \cdots c_{K-1}}}\left[f\right] = C_p^{n-1} + \sum_{\substack{c_r=0 \\ 1 \leq r \leq K-1}} \sum_{l=p+1}^{k_r} C_l^{k_r} - \sum_{\substack{c_r=1 \\ 1 \leq r \leq K-1}} C_p^{k_r-1}$$

now we only have to prove

$$2^n \mathrm{Inf}_{\mathcal{S}_{c_1 \cdots c_K}}\left[f\right] = \begin{cases} 2^n \mathrm{Inf}_{\mathcal{S}_{c_1 \cdots c_{K-1}}}\left[f\right] + \sum_{l=p+1}^{k_K} C_l^{k_K}, & j_K = 0, \\ 2^n \mathrm{Inf}_{\mathcal{S}_{c_1 \cdots c_{K-1}}}\left[f\right] - C_p^{k_K-1}, & j_K = 1. \end{cases}$$

Since $\mathcal{S}_{c_1 \cdots c_K} = \mathcal{S}_{c_1 \cdots c_{K-1}} \cap \mathcal{S}_{\mathbf{x}^{k_K}}^{c_K}$, we consider four types of sets: $\mathcal{S}_{\mathbf{x}^{k_K}}^{1} \setminus (\mathcal{S}_{c_1 \cdots c_{K-1}} \cap \mathcal{S}_{\mathbf{x}^{k_K}}^{1})$, $\mathcal{S}_{c_1 \cdots c_{K-1}} \cap \mathcal{S}_{\mathbf{x}^{k_K}}^{1}$, $\mathcal{S}_{c_1 \cdots c_{K-1}} \cap \mathcal{S}_{\mathbf{x}^{k_K}}^{0}$, $\mathcal{S}_{\mathbf{x}^{k_K}}^{0} \setminus (\mathcal{S}_{c_1 \cdots c_{K-1}} \cap \mathcal{S}_{\mathbf{x}^{k_K}}^{0})$. When adding one more upper zero $\mathbf{x}^{k_K}$, the increased boundary edges by flipping outliers with respect to $\mathbf{x}^{k_K}$ (changing from 0 to 1 in $\mathcal{S}_{\mathbf{x}^{k_K}}^{0}$ from level $p+1$ to k_K) have

$$\sum_{l=p+1}^{k_K} \sum_{s=\max\{0, l-(k_K - |\mathcal{S}_{c_1 \cdots c_{K-1}} \cap \mathcal{S}_{x^{k_K}}^{1}|)\}}^{|\mathcal{S}_{c_1 \cdots c_{K-1}} \cap \mathcal{S}_{x^{k_K}}^{1}|} C_{l-s}^{k_K - |\mathcal{S}_{c_1 \cdots c_{K-1}} \cap \mathcal{S}_{x^{k_K}}^{1}|} C_s^{|\mathcal{S}_{c_1 \cdots c_{K-1}} \cap \mathcal{S}_{x^{k_K}}^{1}|} = \sum_{l=p+1}^{k_K} C_l^{k_K}.$$

While boundary edges coming from flipping inliers with respect to x^{k_K} (changing from 0 to 1 and pointing from $L_p \setminus \mathcal{B}_{x^{k_K}}$ to $L_{p+1} \setminus \mathcal{B}_{x^{k_K}}$) decrease, which have

$$\begin{aligned} &\sum_{l=0}^{|\mathcal{S}_{c_1 \cdots c_{K-1}} \cap \mathcal{S}_{x^{k_K}}^{1}|} C_l^{|\mathcal{S}_{c_1 \cdots c_{K-1}} \cap \mathcal{S}_{x^{k_K}}^{1}|} C_{p-l}^{k_K - |\mathcal{S}_{c_1 \cdots c_{K-1}} \cap \mathcal{S}_{x^{k_K}}^{1}|} \left(1 - \frac{p-l}{k_K - |\mathcal{S}_{c_1 \cdots c_{K-1}} \cap \mathcal{S}_{x^{k_K}}^{1}|}\right) \\ &= C_p^{k_K} - \sum_{l=0}^{|\mathcal{S}_{c_1 \cdots c_{K-1}} \cap \mathcal{S}_{x^{k_K}}^{1}|} C_l^{|\mathcal{S}_{c_1 \cdots c_{K-1}} \cap \mathcal{S}_{x^{k_K}}^{1}|} C_{p-l-1}^{k_K - |\mathcal{S}_{c_1 \cdots c_{K-1}} \cap \mathcal{S}_{x^{k_K}}^{1}| - 1} \\ &= C_p^{k_K} - C_{p-1}^{k_K-1} \\ &= C_p^{k_K-1}, \end{aligned}$$

which completes the proof. □

2. Influence calculation algorithm

In our experiments, the influences are computed using the following equation:

$$\mathrm{Inf}_i^{(q)}\left[f\right] = \mathop{\mathbb{E}}_{\mathbf{x} \sim \mu_q(\mathbf{x})} \delta\left[f\left(\mathbf{x}\right) \neq f\left(\mathbf{x}^{\oplus i}\right)\right]. \tag{6}$$

One can use the monotonic nature of f to save some computations in the above estimation. In a MBF, the function value before the bit flip has some information regarding the value after: If the output is infeasible and the bit is flipped from $0 \rightarrow 1$ then the new output should be one or, If the output is feasible and the bit is flipped from $1 \rightarrow 0$ then the new output should be zero. The algorithm we use to estimate the influences is available in algorithm 1.

Algorithm 1 Estimating the influences

Require: Indices for which the influences are to be estimated $\mathcal{B}$, $\{\mathbf{p}_i\}_{j=1}^{n}$, m, q.

$\mathbf{I} \leftarrow \text{zeros}\,(m, |\mathcal{B}|)$
for $j \in \{1, 2, \ldots, m\}$ **do**
  $\mathbf{x} \sim \mu_q(\mathbf{x})$ ▷ Sample an $\mathbf{x}$ from $\mu_q(\mathbf{x})$
  $y_{\mathbf{x}} \leftarrow f\,(\mathbf{x})$ ▷ Evaluate feasibility
  for $i \in \mathcal{B}$ **do**
    if $(x_i = 1 \wedge y_{\mathbf{x}} = 0) \vee (x_i = 0 \wedge y_{\mathbf{x}} = 1)$ **then**
      $y_{\mathbf{x}^{\oplus i}} \leftarrow y_{\mathbf{x}}$
    else
      $y_{\mathbf{x}^{\oplus i}} \leftarrow f\left(\mathbf{x}^{\oplus i}\right)$
    end if
    $\mathbf{I}\,(j, i) \leftarrow \delta\,[y_{\mathbf{x}} \neq y_{\mathbf{x}^{\oplus i}}]$
  end for
end for
return Average $\mathbf{I}$ across axis 1.

3. Influences of restricted Boolean functions

Any real valued Boolean function $f : \{0,1\}^n \to \{0,1\}$ can be expressed in terms of its Fourier expansion [2]:

$$f\,(\mathbf{x}) = \sum_{s \in \{0,1\}^n} \hat{f}\,(s)\,\chi_s\,(\mathbf{x})\,. \tag{7}$$

Here $\chi_s : \{0,1\}^n \to \mathbb{R}$ is the Fourier basis which consists of 2^n functionals and, $\hat{f}\,(s) \in \mathbb{R}$ are the Fourier coefficients. The above definition of the Boolean function leads to the basis: $\chi_s\,(\mathbf{x}) = (-1)^{1+\sum_{i \in s} x_i}$.

For a MBF, the influence of the i'th coordinate is equal to the degree-1 Fourier coefficient [2].

$$\text{Inf}_i\,[f] = \hat{f}\,(\{i\}) \tag{8}$$

If $\hat{f}_{(t)}\,(\{i\})$ is the degree-1 Fourier coefficient at iteration t of the algorithm 1 in the main paper, then $\hat{f}_{(t)}\,(\{i\}) \neq \hat{f}_{(t-1)}\,(\{i\})$. Here the function at level t is a restricted version of the function at level $t-1$ (some bits in the function are fixed to certain bit values). In algorithm 1, we fix one of the bits to zeros (exclude a point) when going from one iteration to the other. The degree-1 Fourier coefficient at iteration t of the algorithm can be written as:

$$\hat{f}_{(t)}\,(\{i\}) = \hat{f}_{(t-1)}\,(\{i\}) + \hat{f}_{(t-1)}\,(\{i, e\}) \tag{9}$$

where e is the data point, removed at iteration $t-1$. From equations (8) and (9) we can see that: If $\text{Inf}_i^{[t]}\,[f]$ is the influence of point i at iteration t of the algorithm, then $\text{Inf}_i^{[t]}\,[f] \neq \text{Inf}_i^{[t-1]}\,[f]$.

4. Additional Results

4.1. Controlled experiments with synthetic data

To study the computational time behaviour of the proposed algorithm under higher percentage of outliers, we extended the synthetic data experiments presented in Section 4.2 in the main paper to include up to 40% outliers (the size of each synthetic dataset is 200 data points). The results in figure 1, inline with the results presented in the main paper, show that the computational time of the proposed algorithms increase linearly with the number of outliers.

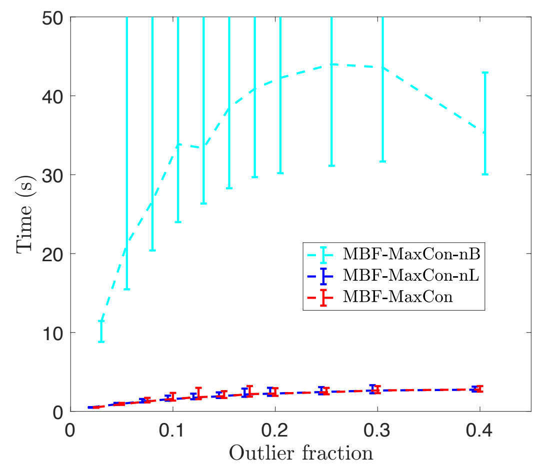

Figure 1. Variation of computational time with the fraction of outliers in 8-dimensional robust linear regression with synthetic data. The experiments were repeated 100 times and the error-bars indicate the 0.05-th and0.95-th percentile.

4.2. Affects of local expansion step on Linearized fundamental matrix estimation

To evaluate how the local expansion step (Algorithm 2 in the main paper) in the proposed algorithm affect the overall performance on Linearized fundamental matrix estimation, we compared the performance of the algorithm with (MBF-MaxCon) and without (MBF-MaxCon-nL) the local expansion step. We used the first five crossroads image pairs from the sequence "00" of the KITTI Odometry dataset [1]. The difference between the Number of inliers returned by each method ($|\mathcal{I}_{\bullet}|$) and ASTAR-NAPA-DIPB ($|\mathcal{I}_{A^*}|$) and the computation times for each algorithm are shown in Figure 2. The results reported for the probabilistic methods are the mean (max, min) over 100 random runs. The results follow the same trend observed on synthetic data (Figure 5 in the main paper).

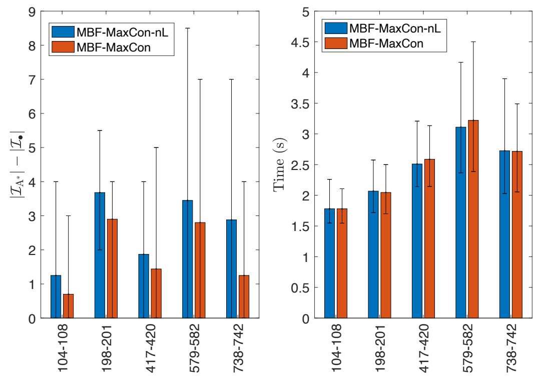

Figure 2. Linearized fundamental matrix estimation result with (MBF-MaxCon) and without (MBF-MaxCon-nL) the local expansion step. Error in the number of inliers compared to ASTAR-NAPA-DIPB is in the left and the computation times in the right. The experiments were repeated 100 times and the error-bars indicate the 0.05-th and0.95-th percentile.

References

[1] Andreas Geiger, Philip Lenz, and Raquel Urtasun. Are we ready for autonomous driving? the kitti vision benchmark suite. In *2012 IEEE Conference on Computer Vision and Pattern Recognition*, pages 3354–3361. IEEE, 2012. 3

[2] Ryan O'Donnell. *Analysis of Boolean Functions*. Cambridge University Press, 2014. 3