Supplementary Material of A Dual Iterative Refinement Method for Non-rigid Shape Matching

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A. The Effect of Correspondence Error on Spectral Alignment

Proof of Theorem 1 We first show that a necessary condition for $\|C_a - C_T\|_2 = 0$ is that permutation σ is an involution, i.e. symmetric matrix.

Since $\Phi_{\mathcal{M}_2} = \Phi_{\mathcal{M}_1} C_T$, we have $Y_i = X_i C_T, i = 1, 2$. It is clear that $\Phi_{\mathcal{M}_1}^\top \Pi \Phi_{\mathcal{M}_2} = (X_1^\top X_1 + X_2^\top \sigma X_2) C_T$. Let us consider the SVD decomposition $X_1^\top X_1 + X_2^\top \sigma X_2 = U \Sigma V^\top$. Then we have a SVD decomposition $\Phi_{\mathcal{M}_1}^\top \Pi \Phi_{\mathcal{M}_2} = U \Sigma V^\top C_T$ because C_T is an orthonormal matrix. Therefore, $C_a = \mathrm{Proj}_{\mathfrak{R}}(\Phi_{\mathcal{M}_1}^\top \Pi \Phi_{\mathcal{M}_2}) = U V^\top C_T$. This yields

$$\|C_a - C_T\|_2 = \|U V^\top C_T - C_T\|_2 = \|U V^\top - \mathrm{Id}\|_2$$

Thus, $\|C_a - C_T\|_2 = 0$ implies $U = V$. This leads to σ being symmetric. From the fact that there are $n_2!$ permutation matrix of size $n_2 \times n_2$ and there are $n_2! \displaystyle\sum_{j=0}^{\lfloor n_2/2 \rfloor} \frac{1}{2^j j! (n_2 - 2j)!}$ symmetric permutation matrix of size $n_2 \times n_2$ [1]. This concludes the proof.

One obvious observation is that $\eta = \displaystyle\sum_{j=0}^{\lfloor n_2/2 \rfloor} \frac{1}{2^j j! (n_2 - 2j)!}$ decreases rapidly as n_2 grows. For example, when $n_2 = 25$, $\eta \approx 10^{-12}$. However, to give a more quantitative characterization of the perturbation for an arbitrary shuffling is difficult since it depends not only on σ but also on X_1 and X_2. Instead, we conduct a few numerical experiments to demonstrate how an inaccurate correspondence Π will affect the correlation matrix $\Phi_{\mathcal{M}_1}^\top \Pi \Phi_{\mathcal{M}_2}$, which is the essential information to align spectral bases, e.g., functional maps, between two shapes.

In our experiments, $\Phi_{\mathcal{M}_1}, \Phi_{\mathcal{M}_2} \in \mathbb{R}^{n \times k}$, where n is the total number of points and k is the spectral dimension. We map a human shape with 12,500 vertices to itself, which is a perfect isometric shape correspondence problem with identity map as the ground truth. Theoretically, $0 \leq \|C_a - C_T\|_2 \leq 2$, since both C_a and C_T are orthonormal. The first experiment tests the behavior of $||C_T - C_a||_2$ with respect to the ratio of $\frac{k}{n_2}$ for two fixed spectral dimensions, $k = 50$ and $k = 100$; the second experiment tests the behavior of $||\mathcal{C}_T - \mathcal{C}_a||_2$ with respect to n_2 for two fixed $\frac{k}{n_2}$ ratio. A random permutation is imposed on Y_2 to compute C_a, and we independently run 50 trails for each parameter combination. Box-plots are used to illustrate the statistics of our computation in Figure 1. The experiments indicate that using inaccurate correspondences at the current step will introduce error to the estimation of functional map at the next step and the error can be quite significant (as big as the worst case, $\|C_a - C_T\|_2 = 2$). This could cause a failure or slow convergence for an iterative refinement strategy.

We further test a real example on one pair of shapes from TOSCA data set using ZoomOut, which is a simple iterative refinement method based on functional maps in spectral domain. A fixed increment of spectral dimension is prespecified for each iteration. Due to the use of current correspondence at all points to construct the functional map, the following iterative refinements may not lead to a satisfactory result in the end. In this test, we use 1000 LB eigenfunctions for ZoomOut (and our method) with two different initial setups: (1) correspondence provided by SHOT, or (2) 4 given landmarks. For ZoomOut, we first compute a functional map for the first 4 spectral modes from the initial correspondence and then use the code provided by the authors on GitHub to run the experiment. Test results are plotted in Figure 2. These experiment further verify our analysis on possible issues using simple iterative spectral refinement without filtering out bad correspondences.

References

[1] S Skiena. Involutions. *Implementing Discrete Mathematics: Combinatorics and Graph Theory with Mathematica*, pages 32–33, 1990.

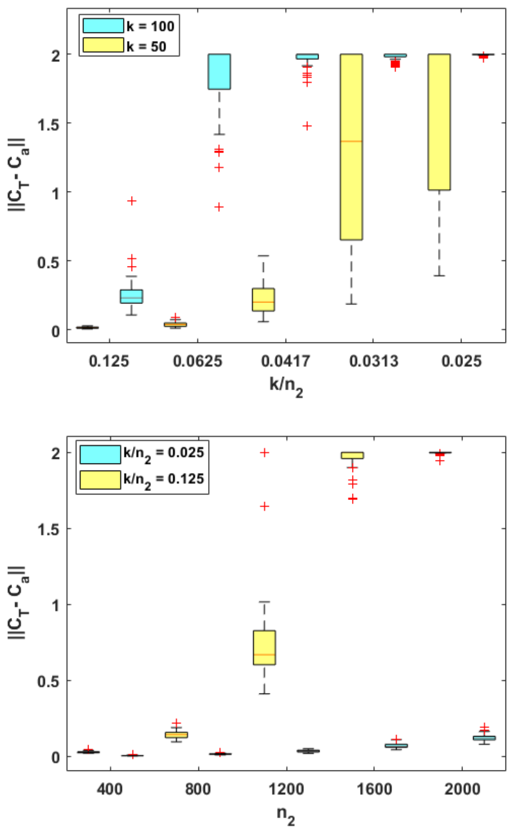

Figure 1. *Top:* Error statistics for the relation between functional map error and $\frac{k}{n_2}$. *Bottom:* Error statistics for the relation between functional map error and number of points in X_2.

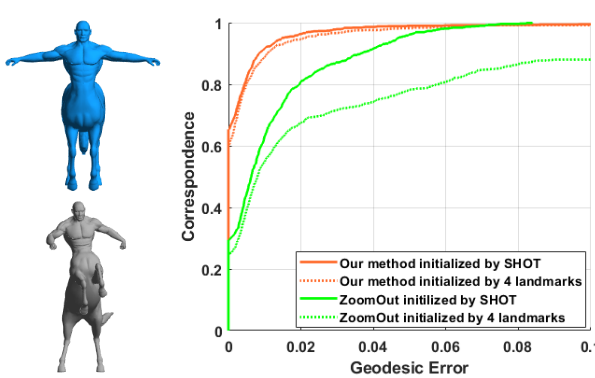

Figure 2. *Left:* The blue centaur is mapped to the gray centaur from TOSCA data set. *Right:* Geodesic error of our method and ZoomOut using different initial guess.