On the Instability of Relative Pose Estimation and RANSAC's Role: Supplementary Material

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These supplementary materials have four sections in total, giving the additional information not covered in the main body of the paper. Section S1 provides proofs for the contents of Section 4.1 in the main paper, and we also display explicit Jacobian matrices there. Section S2 gives the proofs of Theorems 1 and 2. Section S3 gives the proofs of Theorems 3 and 4. In Section S3, we also describe the steps for computing the degenerate X.5-point curves based on solving polynomial systems by homotopy continuation. Last, additional experimental results are given in Section S4. For reproducibility, code for this paper is available at https://github.com/HongyiFan/minimalInstability.

S1. Proofs for "Section 4.1: Condition Number Formulas"

In this section, we prove Propositions 1 and 2 from the main body, and we display explicit Jacobian matrices.

S1.1. Preliminaries on Tangent Spaces, Inner Products and Orthonormal Bases

First we collect together basic facts about the relevant Riemannian manifolds.

• **Special orthogonal group.** Consider $\mathrm{SO}(3)$. By linearizing the equations $\mathbf{R}\mathbf{R}^\top = \mathbf{R}^\top\mathbf{R} = I$,

$$T(\mathrm{SO}(3), \mathbf{R}) = \{\delta\mathbf{R} \in \mathbb{R}^{3\times3} : (\delta\mathbf{R})\mathbf{R}^\top + \mathbf{R}(\delta\mathbf{R})^\top = \mathbf{R}^\top(\delta\mathbf{R}) + (\delta\mathbf{R})^\top\mathbf{R} = 0\} \subseteq \mathbf{R}^{3\times3}.$$

This tangent space may be parameterized as R multiplied by skew-symmetric matrices:

$$T(\mathrm{SO}(3), \mathbf{R}) = \{[s]_\times\mathbf{R} : s \in \mathbb{R}^3\} \qquad \text{or} \qquad T(\mathrm{SO}(3), \mathbf{R}) = \{\mathbf{R}[s]_\times : s \in \mathbb{R}^3\}, \tag{S1}$$

where $[s]_\times := \begin{pmatrix} 0 & -s_3 & s_2 \\ s_3 & 0 & -s_1 \\ -s_2 & s_1 & 0 \end{pmatrix}$ for $s = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}$. The Riemannian metric's inner product on the tangent space is the restriction of the Frobenius inner product on $\mathbb{R}^{3\times3}$,

$$\langle [s]_\times\mathbf{R}, [\tilde{s}]_\times\mathbf{R}\rangle := \mathrm{trace}(([s]_\times\mathbf{R})^\top[\tilde{s}]_\times\mathbf{R}) = \mathrm{trace}([s]_\times^\top[\tilde{s}]_\times) = 2\langle s, \tilde{s}\rangle,$$

where the rightmost inner product is the standard one on $\mathbb{R}^3$. An orthonormal basis for $T(\mathrm{SO}(3), \mathbf{R})$ is

$$\frac{1}{\sqrt{2}}[e_1]_\times\mathbf{R},\, [e_2]_\times\frac{1}{\sqrt{2}}\mathbf{R},\, \frac{1}{\sqrt{2}}[e_3]_\times\mathbf{R}, \tag{S2}$$

where e_1, e_2, e_3 is the standard basis on $\mathbb{R}^3$.

• **Unit sphere.** Consider the two-dimensional unit sphere $\mathbb{S}^2$. Its tangent space are the perpendicular spaces:

$$T(\mathbb{S}^2, \hat{\mathbf{T}}) = \hat{\mathbf{T}}^\perp := \{\tilde{\mathbf{T}} \in \mathbb{R}^3 : \langle\hat{\mathbf{T}}, \tilde{\mathbf{T}}\rangle = 0\} \subseteq \mathbb{R}^3.$$

The Riemannian metric's inner product arises by restricting of the Euclidean inner product on $\mathbb{R}^3$. We fix

$$\hat{\mathbf{T}}_1^\perp, \hat{\mathbf{T}}_2^\perp \in \mathbb{R}^3 \tag{S3}$$

to be an orthonormal basis for $T(\mathbb{S}^2, \hat{\mathbf{T}})$.

• **Projective space.** Consider the real projective space of 3×3 matrices, $\mathbb{P}(\mathbb{R}^{3\times 3})$. The map

$$\mathbb{S}^8 = \{M \in \mathbb{R}^{3\times 3} : \|M\|_F = 1\} \longrightarrow \mathbb{P}(\mathbb{R}^{3\times 3}), \quad M \mapsto [M] \tag{S4}$$

witnesses $\mathbb{P}(\mathbb{R}^{3\times 3})$ as a quotient of $\mathbb{S}^8$ by $\mathbb{Z}/2\mathbb{Z}$ acting via a sign flip. By [6, Exam. 2.34 and Prop. 2.32], this induces the structure of a Riemannian manifold on $\mathbb{P}(\mathbb{R}^{3\times 3})$ such that (S4) is locally an isometry. At a given point in $\mathbb{P}(\mathbb{R}^{3\times 3})$ we can choose a representative $M \in \mathbb{S}^8$ and the tangent space can be identified as follows:

$$T(\mathbb{P}(\mathbb{R}^{3\times 3}), [M]) \cong T(\mathbb{S}^8, M) = M^\perp = \{\tilde{M} \in \mathbb{R}^{3\times 3} : \langle M, \tilde{M}\rangle = 0\} \subseteq \mathbb{R}^{3\times 3}. \tag{S5}$$

The Riemannian metric's inner product is the Frobenius inner product on $M^\perp$.

• **Essential matrices.** Consider the manifold of real essential matrices,

$$\mathcal{E} \subseteq \mathbb{P}(\mathbb{R}^{3\times 3}).$$

(This departs from the notation in the main body.) It is known that $\mathcal{E}$ is a compact smooth real manifold of dimension 5.

Lemma 1 *At each point in* $\mathrm{SO}(3) \times \mathbb{S}^2$*, the differential of the map*

$$\mathrm{SO}(3) \times \mathbb{S}^2 \to \mathcal{E}, \;\; (\mathbf{R}, \hat{\mathbf{T}}) \mapsto [\hat{\mathbf{T}}]_\times \mathbf{R}$$

has rank 5*. Thus the map is a submersion onto the manifold of real essential matrices* $\mathcal{E} \subseteq \mathbb{P}(\mathbb{R}^{3\times 3})$*.*

Proof: The map is linear separately in $\mathbf{R}$ and $\hat{\mathbf{T}}$. So by the product rule, at $(\delta\mathbf{R}, \delta\hat{\mathbf{T}}) \in T(\mathrm{SO}(3), \mathbf{R}) \times T(\mathbb{S}^2, \hat{\mathbf{T}})$ its differential evaluates to $\frac{1}{\sqrt{2}}[\hat{\mathbf{T}}]_\times(\delta\mathbf{R}) + \frac{1}{\sqrt{2}}[\delta\hat{\mathbf{T}}]_\times\mathbf{R} \in T(\mathcal{E}, [\hat{\mathbf{T}}]_\times\mathbf{R}) \subseteq T(\mathbb{P}(\mathbb{R}^{3\times 3})), [\hat{\mathbf{T}}]_\times\mathbf{R} = T(\mathbb{S}^8, \frac{1}{\sqrt{2}}[\hat{\mathbf{T}}]_\times\mathbf{R}) = (R[\hat{\mathbf{T}}]_\times)^\perp \subseteq \mathbb{R}^{3\times 3}$, where we used (S5). We need to show that this quantity equals 0 only if $\delta\mathbf{R} = 0$ and $\delta\hat{\mathbf{T}} = 0$. By (S1), $\delta\mathbf{R} = \mathbf{R}[s]_\times$ for some $s \in \mathbb{R}^3$ and $\delta\hat{\mathbf{T}}$ is perpendicular to $\hat{\mathbf{T}}$. Substituting these in gives the condition

$$\tfrac{1}{\sqrt{2}}[\hat{\mathbf{T}}]_\times[s]_\times\mathbf{R} + \tfrac{1}{\sqrt{2}}[\delta\hat{\mathbf{T}}]_\times\mathbf{R} = 0.$$

Right-multiplying by $\sqrt{2}\mathbf{R}^\top$, this is equivalent to

$$[\hat{\mathbf{T}}]_\times[s]_\times + [\delta\hat{\mathbf{T}}]_\times = 0. \tag{S6}$$

If we multiply on the left by $\hat{\mathbf{T}}$, it follows that $\hat{\mathbf{T}}[\delta\hat{\mathbf{T}}]_\times = 0$. But if $\delta\hat{\mathbf{T}} \neq 0$, then $[\delta\hat{\mathbf{T}}]_\times$ is rank-2 with kernel spanned by $\delta\hat{\mathbf{T}}$ which is perpendicular to $\hat{\mathbf{T}}$. The last two sentences give a contradiction. Thus we must have $\delta\hat{\mathbf{T}} = 0$. So now (S6) reads

$$[\hat{\mathbf{T}}]_\times[s]_\times = 0. \tag{S7}$$

Assume $s \neq 0$. Then $[s]_\times$ is a rank 2 matrix of size 3×3. Since $[\hat{\mathbf{T}}]_\times$ is rank 2 and 3×3 as well (recall $\hat{\mathbf{T}} \in \mathbb{S}^2$ so that $\hat{\mathbf{T}} \neq 0$), the product $[\hat{\mathbf{T}}]_\times[s]_\times$ must have rank at least 1. This contradicts (S7), so $s = 0$, and the lemma follows. □

Lemma 1 lets us write down tangent spaces to the essential matrices:

$$T(\mathcal{E}, [\hat{\mathbf{T}}]_\times\mathbf{R}) \;=\; \{\tfrac{1}{\sqrt{2}}[\hat{\mathbf{T}}]_\times[s]_\times\mathbf{R} + \tfrac{1}{\sqrt{2}}[\delta\hat{\mathbf{T}}]_\times\mathbf{R} : s \in \mathbb{R}^3, \delta\hat{\mathbf{T}} \in \mathbb{R}^3, \langle\delta\hat{\mathbf{T}}, \hat{\mathbf{T}}\rangle = 0\} \;\subseteq\; \mathbb{R}^{3\times 3}.$$

The Riemannian metric's inner product is the restriction of the Frobenius inner product on $\mathbb{R}^{3\times 3}$. We get an orthonormal basis for the tangent space by orthonormalizing the image of (S2) and (S3), *i.e.*, by orthonormalizing

$$\frac{1}{2}[\hat{\mathbf{T}}]_\times[e_1]_\times\mathbf{R}, \quad \frac{1}{2}[\hat{\mathbf{T}}]_\times[e_2]_\times\mathbf{R}, \quad \frac{1}{2}[\hat{\mathbf{T}}]_\times[e_3]_\times\mathbf{R}, \quad \frac{1}{\sqrt{2}}[\hat{\mathbf{T}}_1^\perp]_\times\mathbf{R}, \quad \frac{1}{\sqrt{2}}[\hat{\mathbf{T}}_2^\perp]_\times\mathbf{R}. \tag{S8}$$

Elementary linear algebra implies that if $\alpha \in \mathbb{R}^5$ expresses an element of $T(\mathcal{E}, [\hat{\mathbf{T}}]_\times\mathbf{R})$ in terms of the basis (S8) then $G^{1/2}\alpha$ expresses the same tangent vector in terms of an orthonormal basis for $T(\mathcal{E}, [\hat{\mathbf{T}}]_\times\mathbf{R})$, where G is the Grammian matrix for the matrices in (S8) with respect to the Frobenius inner product. Explicitly, G equals

$$G = \begin{pmatrix} A & B \\ B^\top & C \end{pmatrix} \tag{S9}$$

where

$$A = \begin{pmatrix} \frac{1}{2}\hat{\mathbf{T}}_1^2 + \frac{1}{4}\hat{\mathbf{T}}_2^2 + \frac{1}{4}\hat{\mathbf{T}}_3^2 & \frac{1}{4}\hat{\mathbf{T}}_1\hat{\mathbf{T}}_2 & \frac{1}{4}\hat{\mathbf{T}}_1\hat{\mathbf{T}}_3 \\ \frac{1}{4}\hat{\mathbf{T}}_1\hat{\mathbf{T}}_2 & \frac{1}{4}\hat{\mathbf{T}}_1^2 + \frac{1}{2}\hat{\mathbf{T}}_2^2 + \frac{1}{4}\hat{\mathbf{T}}_3^2 & \frac{1}{4}\hat{\mathbf{T}}_2\hat{\mathbf{T}}_3 \\ \frac{1}{4}\hat{\mathbf{T}}_1\hat{\mathbf{T}}_3 & \frac{1}{4}\hat{\mathbf{T}}_2\hat{\mathbf{T}}_3 & \frac{1}{4}\hat{\mathbf{T}}_1^2 + \frac{1}{4}\hat{\mathbf{T}}_2^2 + \frac{1}{2}\hat{\mathbf{T}}_3^2 \end{pmatrix} \tag{S10}$$

$$B = \begin{pmatrix} \frac{1}{2\sqrt{2}}\hat{\mathbf{T}}_3(\hat{\mathbf{T}}_1^\perp)_2 - \frac{1}{2\sqrt{2}}\hat{\mathbf{T}}_2(\hat{\mathbf{T}}_1^\perp)_3 & \frac{1}{2\sqrt{2}}\hat{\mathbf{T}}_3(\hat{\mathbf{T}}_2^\perp)_2 - \frac{1}{2\sqrt{2}}\hat{\mathbf{T}}_2(\hat{\mathbf{T}}_2^\perp)_3 \\ -\frac{1}{2\sqrt{2}}\hat{\mathbf{T}}_3(\hat{\mathbf{T}}_1^\perp)_1 + \frac{1}{2\sqrt{2}}\hat{\mathbf{T}}_1(\hat{\mathbf{T}}_1^\perp)_3 & -\frac{1}{2\sqrt{2}}\hat{\mathbf{T}}_3(\hat{\mathbf{T}}_2^\perp)_1 + \frac{1}{2\sqrt{2}}\hat{\mathbf{T}}_1(\hat{\mathbf{T}}_2^\perp)_3 \\ \frac{1}{2\sqrt{2}}\hat{\mathbf{T}}_2(\hat{\mathbf{T}}_1^\perp)_1 - \frac{1}{2\sqrt{2}}\hat{\mathbf{T}}_1(\hat{\mathbf{T}}_1^\perp)_2 & \frac{1}{2\sqrt{2}}\hat{\mathbf{T}}_2(\hat{\mathbf{T}}_2^\perp)_1 - \frac{1}{2\sqrt{2}}\hat{\mathbf{T}}_1(\hat{\mathbf{T}}_2^\perp)_2 \end{pmatrix} \tag{S11}$$

$$C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{S12}$$

• **Projection Matrices.** For the uncalibrated cameras, as described in the main text, the world scene space $\mathcal{W}$ is defined as

$$\mathcal{W} = \mathbb{R}^{3\times 4} \times \mathbb{R}^{3\times 4} \times (\mathbb{R}^3)^{\times 7} = \{\mathcal{P}, \bar{\mathcal{P}}, \Gamma_1, \ldots, \Gamma_7)\}, \tag{S13}$$

where $\mathcal{P}$ and $\bar{\mathcal{P}}$ are 3×4 projection matrix of the cameras. For a pinhole camera model, the projection matrix is computed as

$$\mathcal{P} = K \begin{bmatrix} \mathbf{R} & \mathbf{T} \end{bmatrix} \tag{S14}$$

where K is a 3×3 intrinsic matrix of the camera; $\mathbf{R}$ and $\mathbf{T}$ are the absolute rotation and translation. Our analysis needs to represent the relative pose in an (almost everywhere) one-to-one way using a minimal number of parameters. However, it turns out that our main results are independent of the coordinate system choice we make for $\mathcal{W}$, thus we will represent the relative pose by the open dense subset of $\mathbb{R}^7 = \{b = (b_1, \ldots, b_7)\}$ where

$$M(b) := \begin{pmatrix} 1 & b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 & b_7 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{S15}$$

has rank 3. In the supplementary materials, we prove that this set gives a normal form for almost all uncalibrated relative poses, *i.e.* for an open dense subset of pairs of uncalibrated camera matrices we can uniquely bring the projection matrices $\{\mathcal{P}, \bar{\mathcal{P}}\}$ to the form $[I\ 0] \in \mathbb{R}^{3\times 4}$ and $M(b) \in \mathbb{R}^{3\times 4}$ by multiplying the pair on right by an appropriate projective world transformation in $\mathrm{PGL}(4) = \{g \in \mathbb{P}(\mathbb{R}^{4\times 4}) : \det(g) \neq 0\}$.

Here we justify the claim that for $(\mathcal{P}_1, \mathcal{P}_2)$ lying in a certain open dense subset $\mathcal{U}$ of the set of pairs of uncalibrated cameras:

$$\mathcal{C} = \{((\mathcal{P}, \bar{\mathcal{P}}) \in \mathbb{P}(\mathbb{R}^{3\times 4})^{\times 2} : \mathrm{rank}(\mathcal{P}_1) = \mathrm{rank}(\bar{\mathcal{P}}) = 3\},$$

there exists a unique world transformation $g \in \mathrm{PGL}(4)$ and vector of parameters $b \in \mathbb{R}^7$ such that

$$((\mathcal{P}g, \bar{\mathcal{P}}g) = ([I\ 0], M(b)) \in \mathbb{P}(\mathbb{R}^{3\times 4})^{\times 2} \tag{S16}$$

where $M(b) \in \mathbb{P}(\mathbb{R}^{3\times 4})$ is as defined in Eq. 17 of the main text. Specifically, we claim that we can take the set to be

$$\mathcal{U} = \{((\mathcal{P}, \bar{\mathcal{P}}) \in \mathcal{C} \ : \ \det[(\mathcal{P}, \bar{\mathcal{P}}(3,:)] \neq 0, \quad B(1,:)([(\mathcal{P}; \bar{\mathcal{P}}(3,:)]^{-1}(:,1)) \neq 0\}, \tag{S17}$$

where we are using Matlab notation to denote submatrices and matrix concatenations.

Firstly, we note that the conditions in Eq. (S17) are independent of the choice of scales in $\mathcal{P}$ and $\bar{\mathcal{P}}$, so they describe a well-defined subset of projective space. Indeed if λ and μ are nonzero scalars, then

$$\begin{aligned} &\det[\lambda\mathcal{P}; (\mu\bar{\mathcal{P}})(3,:)] = \lambda^3\mu \det[\mathcal{P}; \bar{\mathcal{P}}(3,:)], \\ &(\mu\bar{\mathcal{P}})(1,:)([\lambda\mathcal{P}; (\mu\bar{\mathcal{P}})(3,:)]^{-1}(:,1)) = \mu\lambda^{-1}\bar{\mathcal{P}}(1,:)([\mathcal{P}; \bar{\mathcal{P}}(3,:)]^{-1}(:,1)). \end{aligned} \tag{S18}$$

Next, let $(\mathcal{P}, \bar{\mathcal{P}}) \in \mathcal{U}$. Note that Eq. (S16) holds if and only there exist scales for $\mathcal{P}, \bar{\mathcal{P}}, g$ such that in affine space we have

$$[\mathcal{P}; \bar{\mathcal{P}}]g = [[I\ 0]; M(b)] \in \mathbb{R}^{6\times 4}. \tag{S19}$$

Comparing rows $1, 2, 3, 6$ in Eq. (S19), we must have $g = [\mathcal{P}; \bar{\mathcal{P}}(3,:)]^{-1}$. Then $(\bar{\mathcal{P}}g)(1,1) \neq 0$ by (S18), and we can choose scales for $\mathcal{P}, \bar{\mathcal{P}}$ so that $(\bar{\mathcal{P}}g)(1,1) = 1$ by Eq. (S18).

• **Fundamental matrices.** Consider the manifold of real fundamental matrices,

$$\mathcal{F} \subseteq \mathbb{P}(\mathbb{R}^{3\times 3}).$$

It is known that $\mathcal{F}$ is a non-compact smooth real manifold of dimension 7. One can build the fundamental matrix with the $\mathbb{R}^7$ parameterization b as [5, Eq. 17.3]:

$$F_{ji} := (-1)^{i+j} \det \begin{pmatrix} [I \ \ 0] \text{ with row } i \text{ omitted} \\ M(b) \text{ with row } j \text{ omitted} \end{pmatrix}. \tag{S20}$$

We will work with $\mathcal{F}$ using the parameterization from $\mathbb{R}^7$ given by Eq. (S20). This sends $b \in \mathbb{R}^7$ to

$$F(b) := \begin{pmatrix} b_4 & b_5 & b_6 \\ -1 & -b_1 & -b_2 \\ -b_3 b_4 + b_7 & -b_3 b_5 + b_1 b_7 & -b_3 b_6 + b_2 b_7 \end{pmatrix}. \tag{S21}$$

Lemma 2 *At each point $b \in \mathbb{R}^7$ where the camera matrix $M(b)$ in (S15) has full rank, the differential of the map $F : \mathbb{R}^7 \dashrightarrow \mathcal{F}$ has rank 7. Thus F is a submersion on the open set where it is defined.*

Proof: The differential of F at b evaluated at $\delta b \in \mathbb{R}^7$ equals

$$\begin{pmatrix} (\delta b)_4 & (\delta b)_5 & (\delta b)_6 \\ 0 & -(\delta b)_1 & -(\delta b)_2 \\ -(\delta b)_3 b_4 - b_3 (\delta b)_4 + (\delta b)_7 & -(\delta b)_3 b_5 - b_3 (\delta b)_5 + (\delta b)_1 b_7 + b_1 (\delta b)_7 & -(\delta b)_3 b_6 - b_3 (\delta b)_6 + (\delta b)_2 b_7 + b_2 (\delta b)_7 \end{pmatrix}. \tag{S22}$$

Equating this with 0, the first two rows show that $0 = (\delta b)_1 = (\delta b)_2 = (\delta b)_4 = (\delta b)_5 = (\delta b)_6$. Then the last row reads:

$$\begin{pmatrix} -b_4 & 1 \\ -b_5 & b_1 \\ -b_6 & b_2 \end{pmatrix} \begin{pmatrix} (\delta b)_3 \\ (\delta b)_7 \end{pmatrix} = 0. \tag{S23}$$

The coefficient matrix in (S23) consists of the first two rows of $F(b)$ transposed and negated. However the first two rows of $F(b)$ span the row space of $F(b)$, since the third row of $F(b)$ is $-b_3$ times the first row added to $-b_7$ times the second row. Because $F(b)$ has rank 2, $(\delta b)_3 = (\delta b)_7 = 0$. All together, $\delta b = 0$ whence $DF(b)$ is injective. $\square$

Lemma 2 lets us write down the tangent spaces to fundamental matrices. They are spanned by the matrices (S22) as δb ranges over a standard basis $e_1, \ldots, e_7$ for $\mathbb{R}^7$. The Riemannian metric's inner product is the restriction of the Frobenius inner product. We get an orthonormal basis for $T(\mathcal{F}, F(b))$ by orthonormalizing

$$\frac{1}{\|F(b)\|_F} \frac{\partial F(b)}{\partial b_1}, \quad \ldots \quad, \frac{1}{\|F(b)\|_F} \frac{\partial F(b)}{\partial b_7}. \tag{S24}$$

Elementary linear algebra implies that if $\alpha \in \mathbb{R}^7$ expresses an element of $T(\mathcal{F}, F(b))$ in terms of the basis (S24) then $G^{1/2}\alpha$ expresses the same tangent vector in terms of an orthonormal basis for $T(\mathcal{F}, F(b))$, where G is the Grammian matrix for the matrices in (S24) with respect to the Frobenius inner. Explicitly, G equals

$$\frac{1}{\|F(b)\|_F^2} \begin{pmatrix} b_7^2+1 & 0 & -b_5 b_7 & 0 & -b_3 b_7 & 0 & b_1 b_7 \\ 0 & b_7^2+1 & -b_6 b_7 & 0 & 0 & -b_3 b_7 & b_2 b_7 \\ -b_5 b_7 & -b_6 b_7 & b_4^2+b_5^2+b_6^2 & b_3 b_4 & b_3 b_5 & b_3 b_6 & -b_1 b_5 - b_2 b_6 - b_4 \\ 0 & 0 & b_3 b_4 & b_3^2+1 & 0 & 0 & -b_3 \\ -b_3 b_7 & 0 & b_3 b_5 & 0 & b_3^2+1 & 0 & -b_1 b_3 \\ 0 & -b_3 b_7 & b_3 b_6 & 0 & 0 & b_3^2+1 & -b_2 b_3 \\ b_1 b_7 & b_2 b_7 & -b_1 b_5 - b_2 b_6 - b_4 & -b_3 & -b_1 b_3 & -b_2 b_3 & b_1^2+b_2^2+1 \end{pmatrix}. \tag{S25}$$

S1.2. Proof of Proposition 1

Proof: Uniqueness of the reconstruction map is by Lemma 1 (which is a restatement of the inverse function theorem). This is because we are assuming that the world scene $w \in \mathrm{SO}(3) \times \mathbb{S}^2 \times (\mathbb{R}^3)^{\times 5}$ is not ill-posed. Eq. 18 in the main paper expresses the condition number of $\mathbf{S}$ as the largest singular value of the product of a 5×20 matrix and the inverse of a 20×20 matrix:

$$\|D\Psi(w) \circ D\Phi(w)^{-1}\|.$$

We need to make this formula explicit. Here the forward map is given by Eq. 3 in the main body:

$$\Phi(\mathbf{R}, \hat{\mathbf{T}}, \Gamma_1, \ldots, \Gamma_5) = ((\pi(\Gamma_1), \pi(\mathbf{R}\Gamma_1 + \hat{\mathbf{T}})), \ldots (\pi(\Gamma_5), \pi(\mathbf{R}\Gamma_5 + \hat{\mathbf{T}}))), \tag{S26}$$

where $\pi : \mathbb{R}^3 \dashrightarrow \mathbb{R}^2$ is the projection $\pi(a_1, a_2, a_3) = (a_1/a_3, a_2/a_3)$ defined whenever $a_3 \neq 0$. It is natural to factor $\Phi = \Phi_2 \circ \Phi_1$ as the composition of a map $\Phi_1 : \mathrm{SO}(3) \times \mathbb{S}^2 \times (\mathbb{R}^3)^{\times 5} \to (\mathbb{R}^3 \times \mathbb{R}^3)^{\times 5}$ given by

$$\Phi_1(\mathbf{R}, \hat{\mathbf{T}}, \Gamma_1, \ldots, \Gamma_5) := ((\Gamma_1, \mathbf{R}\Gamma_1 + \hat{\mathbf{T}}), \ldots, (\Gamma_5, \mathbf{R}\Gamma_5 + \hat{\mathbf{T}})),$$

followed by an almost-everywhere-defined map $\Phi_2 : (\mathbb{R}^3 \times \mathbb{R}^3)^{\times 5} \dashrightarrow (\mathbb{R}^2 \times \mathbb{R}^2)^{\times 5}$ given by

$$\Phi_2((Z_1, \tilde{Z}_1), \ldots, (Z_5, \tilde{Z}_5)) := ((\pi(Z_1), \pi(\tilde{Z}_1)), \ldots, (\pi(Z_5), \pi(\tilde{Z}_5))).$$

where $Z_i = \Gamma_i$ and $\tilde{Z}_i = \mathbf{R}\Gamma_i + \hat{\mathbf{T}}$.

By the chain rule, $D\Phi(w) = D\Phi_2(\Phi(w)) \circ D\Phi_1(w)$. This writes the forward Jacobian matrix as the product of a 20×30 matrix multiplied by a 30×20 matrix. Let us explicitly write down $D\Phi_1(w)$ in terms of the orthonormal bases for the tangent spaces from the previous section, with columns ordered according to $\delta\Gamma_1, \ldots, \delta\Gamma_5, \delta\mathbf{R}, \delta\hat{\mathbf{T}}$ (corresponding to an orthonormal basis for $T(\mathrm{SO}(3) \times \mathbb{S}^2 \times (\mathbb{R}^3)^{\times 5}, w)$), and rows ordered according to $\delta Z_1, \ldots, \delta Z_5, \delta\tilde{Z}_1, \ldots, \delta\tilde{Z}_5$ (corresponding to a standard basis on $(\mathbb{R}^3 \times \mathbb{R}^3)^{\times 5}$). Since Φ_1 is separately linear in $\Gamma_1, \ldots, \Gamma_5, \mathbf{R}, \hat{\mathbf{T}}$, we can compute the following block form:

$$D\Phi_1(w) = \left(\begin{array}{ccc|ccccc} & I_{15\times 15} & & & & 0_{15\times 5} & & \\ \hline \mathbf{R} & & & \frac{1}{\sqrt{2}}\big([e_1]_\times \mathbf{R}\Gamma_1 & [e_2]_\times \mathbf{R}\Gamma_1 & [e_3]_\times \mathbf{R}\Gamma_1\big) & \hat{\mathbf{T}}_1^\perp & \hat{\mathbf{T}}_2^\perp \\ & \ddots & & & \vdots & & \vdots & \vdots \\ & & \mathbf{R} & \frac{1}{\sqrt{2}}\big([e_1]_\times \mathbf{R}\Gamma_5 & [e_2]_\times \mathbf{R}\Gamma_5 & [e_3]_\times \mathbf{R}\Gamma_5\big) & \hat{\mathbf{T}}_1^\perp & \hat{\mathbf{T}}_2^\perp \end{array}\right)_{30\times 20}. \tag{S27}$$

The Jacobian matrix $D\Phi_2$ has the following block-diagonal form with respect to the standard bases:

$$D\Phi_2(\Phi_1(w)) = \begin{pmatrix} \frac{\partial\pi}{\partial Z_1} & & & & & \\ & \ddots & & & & \\ & & \frac{\partial\pi}{\partial Z_5} & & & \\ & & & \frac{\partial\pi}{\partial \tilde{Z}_1} & & \\ & & & & \ddots & \\ & & & & & \frac{\partial\pi}{\partial \tilde{Z}_5} \end{pmatrix}_{20\times 30}. \tag{S28}$$

Here, *e.g.* $\frac{\partial\pi}{\partial Z_1}$ denotes the 2×3 Jacobian matrix of $\pi(Z_1) = \begin{pmatrix} \frac{(Z_1)_1}{(Z_1)_3} & \frac{(Z_1)_2}{(Z_1)_3} \end{pmatrix}^\top$ with respect to Z_1. Explicitly,

$$\frac{\partial\pi}{\partial Z_1} = \begin{pmatrix} \frac{1}{(Z_1)_3} & 0 & \frac{-(Z_1)_1}{(Z_1)_3^2} \\ 0 & \frac{1}{(Z_1)_3} & \frac{-(Z_1)_2}{(Z_1)_3^2} \end{pmatrix}$$

and likewise for the other blocks. In (S28), the Jacobian is evaluated at $\Phi(w)$, *i.e.* $Z_1 = \Gamma_1, \ldots, Z_5 = \Gamma_5$ and $\tilde{\Gamma}_1 = \mathbf{R}\Gamma_1 + \hat{\mathbf{T}}, \ldots, \tilde{Z}_5 = \mathbf{R}\Gamma_5 + \hat{\mathbf{T}}$. Multiplying (S27) with (S28) and then inverting gives the 20×20 matrix $(D\Phi(w))^{-1}$.

Next consider differential of the epipolar map, *i.e.* the 5×20 matrix $D\Psi(w)$. Here Ψ factors as the coordinate projection $(\Gamma_1, \ldots, \Gamma_5, \mathbf{R}, \hat{\mathbf{T}}) \mapsto (\mathbf{R}, \hat{\mathbf{T}})$ followed by the map $(\mathbf{R}, \hat{\mathbf{T}}) \mapsto [\hat{\mathbf{T}}]_\times \mathbf{R}$. Of course, the Jacobian of the projection is

$$\begin{pmatrix} 0_{5\times 15} & I_{5\times 5} \end{pmatrix}.$$

As for $(\mathbf{R}, \hat{\mathbf{T}}) \mapsto [\hat{\mathbf{T}}]_\times \mathbf{R}$, if we express its Jacobian so that the rows correspond to the non-orthonormal basis (S8) for the tangent space $T(\mathcal{E}, [\hat{\mathbf{T}}]_\times \mathbf{R})$, then we simply get I_5. Then re-expressing this in terms of an orthonormal basis for the tangent space, we need to multiply by a positive-definite square root $G^{1/2}$ for the 5×5 Grammian matrix in (S9).

All together, the product $D\Psi(w) \circ (D\Phi(w))^{-1}$ is computed by multiplying (S28) with (S27) (in that order); inverting the product; selecting the last 5 rows of the inverse; and finally multiplying on the left by $G^{1/2}$. The condition number of the solution map is the largest singular value of the resulting 5×20 matrix. This finishes Proposition 1. ☐

Before proceeding, we record an easy fact that will be useful in Section S2.

Remark 1 *The kernel of the 2×3 matrices $\frac{\partial \pi}{\partial Z_i}$ and $\frac{\partial \pi}{\partial \tilde{Z}_i}$ in (S28) are spanned by Z_i and $\tilde{Z}_i$ respectively.*

S1.3. Proof of Proposition 2

Proof: This is very similar to Proposition 1. Uniqueness of the reconstruction map is by Lemma 1. We obtain explicit Jacobian formulas by first factoring $\Phi = \Phi_2 \circ \Phi_1$ where $\Phi_1 : \mathbb{R}^7 \times (\mathbb{R}^3)^{\times 7} \to (\mathbb{R}^3 \times \mathbb{R}^3)^{\times 7}$ is given by

$$\Phi_1(b, \Gamma_1, \ldots, \Gamma_7) = ((\Gamma_1, M(b)\begin{pmatrix}\Gamma_1 \\ 1\end{pmatrix}), \ldots, (\Gamma_7, M(b)\begin{pmatrix}\Gamma_7 \\ 1\end{pmatrix}))$$

and $\Phi_2 : (\mathbb{R}^3 \times \mathbb{R}^3)^{\times 7} \dashrightarrow (\mathbb{R}^2 \times \mathbb{R}^2)^7$ is given by

$$\Phi_2((Z_1, \tilde{Z}_1), \ldots, (Z_7, \tilde{Z}_7)) = ((\pi(Z_1), \pi(\tilde{Z}_1)), \ldots, (\pi(Z_7), \pi(\tilde{Z}_7))).$$

The chain rule gives $D\Phi(w) = D\Phi_2(w) \circ D\Phi_1(w)$. Here all spaces involved in the forward map are Euclidean spaces, so we use the standard orthonormal bases to write down the matrices.

The first matrix $D\Phi_1(w)$ is 42×28. Ordering its columns according to $\delta\Gamma_1, \ldots, \delta\Gamma_7, \delta b$ and its rows according to $\delta Z_1, \ldots, \delta Z_7, \delta\tilde{Z}_1, \ldots, \delta\tilde{Z}_7$, it reads

$$D\Phi_1(w) = \left(\begin{array}{ccc|ccc} & I_{21\times 21} & & & 0_{21\times 7} & \\ \hline M(b)(1:3,1:3) & & & \frac{\partial M(b)}{\partial b_1}\begin{pmatrix}\Gamma_1 \\ 1\end{pmatrix} & \cdots & \frac{\partial M(b)}{\partial b_7}\begin{pmatrix}\Gamma_1 \\ 1\end{pmatrix} \\ & \ddots & & \vdots & & \vdots \\ & & M(b)(1:3,1:3) & \frac{\partial M(b)}{\partial b_1}\begin{pmatrix}\Gamma_7 \\ 1\end{pmatrix} & \cdots & \frac{\partial M(b)}{\partial b_7}\begin{pmatrix}\Gamma_7 \\ 1\end{pmatrix} \end{array}\right)_{42\times 28} \tag{S29}$$

The bottom-left 21×21 submatrix is block-diagonal with seven 3×3 blocks, each of which is $M(b)(1:3,1:3)$ denoting the first three columns of $M(b)$. In the bottom-right 21×7 submatrix, note that each matrix $\frac{\partial M(b)}{\partial b_i}$ is zero in all but one entry where it takes the value of 1.

The second Jacobian matrix $D\Phi_2(\Phi_1(w))$ is 28×42. It is block-diagonal with fourteen blocks each of size 3×2, analogously to (S28) with Remark 1 still applying:

$$D\Phi_2(\Phi_1(w)) \;=\; \begin{pmatrix} \frac{\partial \pi}{\partial Z_1} & & & & & \\ & \ddots & & & & \\ & & \frac{\partial \pi}{\partial Z_7} & & & \\ & & & \frac{\partial \pi}{\partial \tilde{Z}_1} & & \\ & & & & \ddots & \\ & & & & & \frac{\partial \pi}{\partial \tilde{Z}_7} \end{pmatrix}_{28\times 42}. \tag{S30}$$

Multiplying (S29) with (S30) and inverting the product gives the 28×28 matrix $(D\Phi(w))^{-1}$.

Next we consider the differential of the epipolar map, *i.e.* the 7×28 matrix $D\Psi(w)$. Here Ψ factors as the coordinate projection $(\Gamma_1, \ldots, \Gamma_7, b) \mapsto b$ followed by the map $b \mapsto F(b)$ given by (S21). Of course, the Jacobian of the projection is

$$\begin{pmatrix} 0_{7\times 21} & I_{7\times 7} \end{pmatrix}.$$

The Jacobian matrix of $b \mapsto F(b)$ is simply I_7, if we express it with respect to bases so that the rows correspond to the non-orthonormal basis (S24) for the tangent space $T(\mathcal{F}, F(b))$. Re-expressing it in terms of an orthonormal basis for the tangent space, we need to multiply by a positive-definite square root $G^{1/2}$ for the 7×7 Grammian matrix in (S25).

All together, the product $D\Psi(w) \circ (D\Phi(w))^{-1}$ is computed by multiplying (S29) with (S30) (in that order); inverting the product; selecting the last 7 rows of the inverse; and finally multiplying on the left by $G^{1/2}$. The condition number of the solution map is the largest singular value of the resulting 7×18 matrix. This finishes Proposition 2. $\square$

S2. Proofs for "Section 4.2: Ill-Posed World Scenes"

In this section, we characterize the degenerate world scenes for the 5-point and 7-point minimal problems in terms of quadric surfaces in $\mathbb{R}^3$.

Remark 2 *Our definition of "quadric surface" given in the main body in Eq. 22 includes the case of affine planes (which occur when the top-left 3×3 submatrix of Q in Eq. 22 is zero). This choice is deliberate, and needed for full accuracy in Theorems 1 and 2. Likewise, by "circle" in the statement of Theorem 1 we mean a plane conic defined by*

$$\left\{ (a_1, a_2) \in \mathbb{R}^2 : \begin{pmatrix} a & 1 \end{pmatrix} Q \begin{pmatrix} a \\ 1 \end{pmatrix} = 0 \right\}, \tag{S31}$$

for some symmetric matrix $Q \in \mathbb{R}^{3\times 3}$ such that $Q_{11} = Q_{22}$ and $Q_{12} = Q_{21} = 0$. Eq. (S31) includes the cases of affine lines and points, interpreted as circles of radius ∞ and 0 respectively.

S2.1. Proof of Theorem 1

Proof: The assumption that the forward map Φ is defined at the world scene w implies that the points Γ_i and $\mathbf{R}\Gamma_i + \hat{\mathbf{T}}$ in $\mathbb{R}^3$ do not have a vanishing third coordinate for each $i = 1, \ldots, 5$.

Let $\Delta := \begin{pmatrix} \delta\Gamma_1 & \ldots & \delta\Gamma_5 & \delta r_1 & \delta r_2 & \delta r_3 & \delta\hat{\mathbf{T}}_1 & \delta\hat{\mathbf{T}}_2 \end{pmatrix}^\top \in \mathbb{R}^{20}$. Our task is characterize for which scenes w does there a nonzero solution to the linear system $D\Phi(w)\Delta = 0$, where the variable is Δ. Let us massage this equation repeatedly.

Firstly using the factorization $D\Phi = D\Phi_2 \circ D\Phi_1$ from the previous section, the explicit Jacobian matrix expressions (S27) and (S28), and Remark 1 characterizing the kernel of $D\Phi_2$, we equivalently have the system of equations

$$\begin{cases} \delta\Gamma_i \;\propto\; \Gamma_i \quad \text{for all } i, \\ \mathbf{R}(\delta\Gamma_i) + [s]_\times \mathbf{R}\Gamma_i + \hat{\mathbf{T}}_*^\perp \;\propto\; \mathbf{R}\Gamma_i + \hat{\mathbf{T}} \quad \text{for all } i. \end{cases} \tag{S32}$$

Here '$\propto$' indicates a proportionality, $s := \frac{1}{\sqrt{2}} \begin{pmatrix} \delta r_1 & \delta r_2 & \delta r_3 \end{pmatrix}^\top \in \mathbb{R}^3$ and $\hat{\mathbf{T}}_*^\perp := \delta\hat{\mathbf{T}}_1 \hat{\mathbf{T}}_1^\perp + \delta\hat{\mathbf{T}}_2 \hat{\mathbf{T}}_2^\perp \in \mathbb{R}^3$. We need to characterize when (S32) admits a nonzero solution in the variables $\delta\Gamma_1, \ldots, \delta\Gamma_5, s, \hat{\mathbf{T}}_*^\perp$.

Let $\lambda_i \in \mathbb{R}$ denote the proportionality constants in the first line of (S32), and likewise $\mu_i \in \mathbb{R}$ for the second line. Then the first line of (S32) reads $\delta\Gamma_i = \lambda_i\Gamma_i$. Substituting this into the second line of (S32) gives

$$\lambda_i \mathbf{R}\Gamma_i + [s]_\times \mathbf{R}\Gamma_i + \hat{\mathbf{T}}_*^\perp = \mu_i \mathbf{R}\Gamma_i + \mu_i \hat{\mathbf{T}} \quad \text{for all } i. \tag{S33}$$

We need to characterize when (S33) admits a solution in $\lambda_1, \ldots, \lambda_5, \mu_1, \ldots, \mu_5, s, \hat{\mathbf{T}}_*^\perp$ nonzero in $\lambda_1, \ldots, \lambda_5, s, \hat{\mathbf{T}}_*^\perp$. (Note $\lambda_i \neq 0 \Leftrightarrow \delta\Gamma_i \neq 0$ since $(\Gamma_i) \neq 0$.) It is the same to ask the solution to (S33) be not all-zero in $\lambda_1, \ldots, \lambda_5, \mu_1, \ldots, \mu_5, s, \hat{\mathbf{T}}_*^\perp$ (with μ's included), for if $\lambda_1, \ldots, \lambda_5, s, t$ are all zero then (S33) implies $\mu_i = 0$, since $(\mathbf{R}\Gamma_i + \hat{\mathbf{T}})_3 \neq 0$.

We can simplify Eq. (S33) by changing notation as follows:

$$\begin{cases} \lambda_i & \longleftarrow \quad \lambda_i - \mu_i \\ \mu_i & \longleftarrow \quad \mu_i \\ \hat{\mathbf{T}}_*^\perp & \longleftarrow \quad -\hat{\mathbf{T}}_*^\perp \\ [s]_\times & \longleftarrow \mathbf{R}[s]_\times \mathbf{R}^\top \\ \Gamma_i & \longleftarrow \quad \mathbf{R}\Gamma_i \\ \mathbf{R} & \longleftarrow \quad I_3. \end{cases} \tag{S34}$$

The first four lines in (S34) describe an invertible linear change of variables for (S33). This does not affect whether there exists a nonzero solution to (S33). The last lines in (S34) rotate the world points $\Gamma_1, \ldots, \Gamma_5$, and this operation does not affect whether there exists a quadric surface in $\mathbb{R}^3$ satisfying the claimed condition in Theorem 1. So indeed, the transformation (S34) is without loss of generality. In updated notation, (S33) reads

$$\lambda_i \Gamma_i + [s]_\times \Gamma_i = \mu_i \hat{\mathbf{T}} + \hat{\mathbf{T}}_*^\perp \quad \text{for all } i. \tag{S35}$$

Applying a further rotation in $\mathbb{R}^3$, we can assume that $\hat{\mathbf{T}} = e_3$ and $\hat{\mathbf{T}}_1^\perp = e_1$ and $\hat{\mathbf{T}}_2^\perp = e_2$ in (S35) without loss of generality. Since $\hat{\mathbf{T}}$ and $\hat{\mathbf{T}}_*^\perp$ are perpendicular, we can eliminate μ_i from (S35), because it is equivalent to equate the first two coordinates of both sides of (S35):

$$\lambda_i \begin{pmatrix} (\Gamma_i)_1 \\ (\Gamma_i)_2 \end{pmatrix} + \begin{pmatrix} -s_3(\Gamma_i)_2 + s_2(\Gamma_i)_3 \\ s_3(\Gamma_i)_1 - s_1(\Gamma_i)_3 \end{pmatrix} = \begin{pmatrix} \delta\hat{\mathbf{T}}_1 \\ \delta\hat{\mathbf{T}}_2 \end{pmatrix} \quad \text{for all } i. \tag{S36}$$

We need to characterize when the system (S36) has a nonzero solution in $\lambda_1, \ldots, \lambda_5, s_1, s_2, s_3, \delta\hat{\mathbf{T}}_1, \delta\hat{\mathbf{T}}_2$.

Rewrite (S36) as follows:

$$\lambda_i \begin{pmatrix} (\Gamma_i)_1 \\ (\Gamma_i)_2 \end{pmatrix} + s_3 \begin{pmatrix} -(\Gamma_i)_2 \\ (\Gamma_i)_1 \end{pmatrix} + (\Gamma_i)_3 \begin{pmatrix} s_2 \\ -s_1 \end{pmatrix} - \begin{pmatrix} \delta\hat{\mathbf{T}}_1 \\ \delta\hat{\mathbf{T}}_2 \end{pmatrix} = 0 \quad \text{for all } i. \tag{S37}$$

Now eliminate λ_i from (S37). Indeed, we claim that (S37) admits a nonzero solution in $\lambda_1, \ldots, \lambda_5, s_1, s_2, s_3, \delta\hat{\mathbf{T}}_1, \delta\hat{\mathbf{T}}_2$ if and only if the system obtained by multiplying (S37) on the left by $\begin{pmatrix} -(\Gamma_i)_2 & (\Gamma_i)_1 \end{pmatrix}$ (each i) admits a nonzero solution in $s_1, s_2, s_3, \delta\hat{\mathbf{T}}_1, \delta\hat{\mathbf{T}}_2$. That is, we claim we can reduce to:

$$s_3((\Gamma_i)_1^2 + (\Gamma_i)_2^2) - s_2(\Gamma_i)_2(\Gamma_i)_3 - s_1(\Gamma_i)_1(\Gamma_i)_3 + \delta\hat{\mathbf{T}}_1(\Gamma_i)_2 - \delta\hat{\mathbf{T}}_2(\Gamma_i)_1 = 0 \quad \text{for all } i. \tag{S38}$$

To justify this, note that if $\begin{pmatrix} (\Gamma_i)_1 \\ (\Gamma_i)_2 \end{pmatrix} \neq 0$ for each i, then the vectors $\begin{pmatrix} (\Gamma_i)_1 \\ (\Gamma_i)_2 \end{pmatrix}$ and $\begin{pmatrix} -(\Gamma_i)_2 \\ (\Gamma_i)_1 \end{pmatrix}$ give an orthogonal basis for $\mathbb{R}^2$ for each i. In this case, changing to this basis from the standard basis, (S37) becomes (S38) together with

$$\lambda_i((\Gamma_i)_1^2 + (\Gamma_i)_2^2) + s_2(\Gamma_i)_1(\Gamma_i)_3 - s_1(\Gamma_i)_2(\Gamma_i)_3 - \delta\hat{\mathbf{T}}_1(\Gamma_i)_1 - \delta\hat{\mathbf{T}}_2(\Gamma_i)_2 = 0 \quad \text{for all } i. \tag{S39}$$

Clearly (S39) determines λ_i in terms of $s_1, \ldots, \delta\hat{\mathbf{T}}_2$, so (S38) and (S39) have a nonzero solution in $\lambda_1, \ldots, \delta\hat{\mathbf{T}}_2$ if and only if (S38) does in $s_1, \ldots, \delta\hat{\mathbf{T}}_2$. Meanwhile, if $\begin{pmatrix} (\Gamma_i)_1 \\ (\Gamma_i)_2 \end{pmatrix} = 0$ for some i, then both (S37) and (S38) admit nonzero solutions: for (S37), we can explicitly set $\lambda_i = 1$ and all other nine variables equal to 0; for (S38), once we remove the i-th equation (which is trivial) this leaves an undetermined linear system of four equations in five unknowns, which must have a nonzero solution. Thus, we need to characterize when (S38) admits a nonzero solution in $s_1, s_2, s_3, \delta\hat{\mathbf{T}}_1, \delta\hat{\mathbf{T}}_2$.

To complete the proof, we argue that we simply need to geometrically reinterpret (S38). Letting z_1, z_2, z_3 be variables on $\mathbb{R}^3$, consider the following linear subspace of inhomogeneous quadratic polynomials:

$$\operatorname{span}\{z_1^2 + z_2^2,\ z_2 z_3,\ z_1 z_3,\ z_2,\ z_1\} \ \subseteq\ \mathbb{R}[z_1, z_2, z_3] \tag{S40}$$

Then (S38) states that there exists a quadric surface $\mathcal{Q} \subseteq \mathbb{R}^3$, cut out by some nonzero polynomial in (S40), passing through the points $\Gamma_1, \ldots, \Gamma_5 \in \mathbb{R}^3$. However, (S40) precisely describes the quadric surfaces in $\mathbb{R}^3$ that contain the baseline $\operatorname{Span}(-R^\top \hat{\mathbf{T}}) = \operatorname{Span}(e_3) \subseteq \mathbb{R}^3$, and are such that intersecting the quadric with any affine plane in $\mathbb{R}^3$ which is perpendicular to the baseline results in a circle (with the caveats of Remark 2 applying). Indeed (S40) exactly corresponds to the subspace of 4×4 real symmetric matrices of the following form:

$$Q = \begin{pmatrix} q_1 & 0 & q_2 & q_3 \\ 0 & q_1 & q_4 & q_5 \\ q_2 & q_4 & 0 & 0 \\ q_3 & q_5 & 0 & 0 \end{pmatrix} \quad \text{for some } q_1, \ldots, q_5 \in \mathbb{R}.$$

Precisely such matrices give quadrics containing $\operatorname{Span}(\mathbb{R}^3)$ (because of the zero bottom-right 2×2 submatrix), and also intersecting planes parallel to $\operatorname{Span}(e_1, e_2)$ in circles (because of the top-right 2×2 submatrix). This finishes Theorem 1. $\square$

S2.2. Proof of Theorem 2

Proof: This argument is similar to the proof of Theorem 1, although somewhat more computational. Here the forward map Φ is given by Eq. 7, and the assumption that Φ is defined at w implies that the points Γ_i and $M(b)\begin{pmatrix} X_i \\ 1 \end{pmatrix}$ in $\mathbb{R}^3$ do not have a vanishing third coordinate for each $i = 1, \ldots, 7$. Let $\Delta := \begin{pmatrix} \delta\Gamma_1 & \ldots & \delta\Gamma_7 & \delta b_1 & \ldots & \delta b_7 \end{pmatrix}^\top \in \mathbb{R}^{28}$. Our task is characterize for which scenes w does there a nonzero solution to the linear system $D\Phi(w)\Delta = 0$, where the variable is Δ.

Using the factorization $D\Phi = D\Phi_2 \circ D\Phi_1$ from the previous section, the explicit Jacobian matrix expressions (S29) and (S30), and Remark 1 characterizing the kernel of $D\Phi_2$, we equivalently have the system

$$\begin{cases} \delta\Gamma_i \ \propto\ \Gamma_i \quad \text{for all } i \\ \begin{pmatrix} 1 & b_1 & b_2 \\ b_4 & b_5 & b_6 \\ 0 & 0 & 0 \end{pmatrix} \delta\Gamma_i + \begin{pmatrix} 0 & \delta b_1 & \delta b_2 & \delta b_3 \\ \delta b_4 & \delta b_5 & \delta b_6 & \delta b_7 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} X_i \\ 1 \end{pmatrix} \propto \begin{pmatrix} 1 & b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 & b_7 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X_i \\ 1 \end{pmatrix} \quad \text{for all } i. \end{cases} \tag{S41}$$

Comparing the third coordinate of both sides, we see that the proportionality constant in the second line of (S41) must be 0 for each i. Let $\lambda_i \in \mathbb{R}$ be the proportionality constant in the first line of (S41) for each i. Rewrite (S41) as

$$\begin{cases} \delta\Gamma_i \ =\ \lambda_i \Gamma_i \quad \text{for all } i, \\ \begin{pmatrix} 1 & b_1 & b_2 \\ b_4 & b_5 & b_6 \end{pmatrix} \delta\Gamma_i + \begin{pmatrix} 0 & \delta b_1 & \delta b_2 & \delta b_3 \\ \delta b_4 & \delta b_5 & \delta b_6 & \delta b_7 \end{pmatrix} \begin{pmatrix} \Gamma_i \\ 1 \end{pmatrix} \ =\ 0 \quad \text{for all } i. \end{cases} \tag{S42}$$

In (S42), we substitute the first line into the first term of the second line and we rearrange the second term in the second line:

$$\lambda_i \begin{pmatrix} 1 & b_1 & b_2 \\ b_4 & b_5 & b_6 \end{pmatrix} \Gamma_i + \begin{pmatrix} (\Gamma_i)_2 & (\Gamma_i)_3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (\Gamma_i)_1 & (\Gamma_i)_2 & (\Gamma_i)_3 & 1 \end{pmatrix} \delta b \ =\ 0 \quad \text{for all } i. \tag{S43}$$

We need to characterize when the system (S43) has a nonzero solution in $\lambda_1, \ldots, \lambda_7, \delta b_1, \ldots, \delta b_7$. (That this is equivalent to the characterization for (S41) uses that $\delta\Gamma_i \neq 0 \Leftrightarrow \lambda_i \neq 0$, because $\delta\Gamma_i = \lambda_i \Gamma_i$ and $(\Gamma_i)_3 \neq 0$.)

Now we eliminate λ_i from (S43), following what we did for (S37) above. Here it is equivalent to multiply (S43) on the left by the transpose of

$$\begin{pmatrix} -b_4(\Gamma_i)_1 - b_5(\Gamma_i)_2 - b_6(\Gamma_i)_3 \\ (\Gamma_i)_1 + b_1(\Gamma_i)_2 + b_2(\Gamma_i)_3 \end{pmatrix},$$

which is normal to $\begin{pmatrix} 1 & b_1 & b_2 \\ b_4 & b_5 & b_6 \end{pmatrix} \Gamma_i$. Then we need to characterize when the resulting 7×7 linear system has a nonzero solution in $\delta b_1, \ldots, \delta b_7$. So we reduce to:

$$\begin{pmatrix} -b_4(\Gamma_i)_1(\Gamma_i)_2 - b_5(\Gamma_i)_2^2 - b_6(\Gamma_i)_2(\Gamma_i)_3 \\ -b_4(\Gamma_i)_1(\Gamma_i)_3 - b_5(\Gamma_i)_2(\Gamma_i)_3 - b_6(\Gamma_i)_3^2 \\ -b_4(\Gamma_i)_1 - b_5(\Gamma_i)_2 - b_6(\Gamma_i)_3 \\ (\Gamma_i)_1^2 + b_1(\Gamma_i)_1(\Gamma_i)_2 + b_2(\Gamma_i)_1(\Gamma_i)_3 \\ (\Gamma_i)_1(\Gamma_i)_2 + b_1(\Gamma_i)_2^2 + b_2(\Gamma_i)_2(\Gamma_i)_3 \\ (\Gamma_i)_1(\Gamma_i)_3 + b_1(\Gamma_i)_2(\Gamma_i)_3 + b_2(\Gamma_i)_3^2 \\ (\Gamma_i)_1 + b_1(\Gamma_i)_2 + b_2(\Gamma_i)_3 \end{pmatrix}^{\top} \begin{pmatrix} \delta b_1 \\ \delta b_2 \\ \delta b_3 \\ \delta b_4 \\ \delta b_5 \\ \delta b_6 \\ \delta b_7 \end{pmatrix} = 0 \quad \text{for each } i. \tag{S44}$$

To complete the proof, we only need to reinterpret (S44) geometrically. Let z_1, z_2, z_3 be variables on $\mathbb{R}^3$. Then (S44) states that there exists a quadric surface $\mathcal{Q} \subseteq \mathbb{R}^3$ (with the caveats of Remark 2), passing through $X_1, \ldots, X_7 \in \mathbb{R}^3$ and cut out by a nonzero element of the following subspace of quadratic polynomials:

$$\begin{aligned} &\operatorname{span}\{b_4z_1z_2 + b_5z_2^2 + b_6z_2z_3, b_4z_1z_3 + b_5z_2z_3 + b_6z_3^2, b_4z_1 + b_5z_2 + b_6z_3, z_1^2 + b_1z_1z_2 + b_2z_1z_3, \\ &\qquad z_1z_2 + b_2z_2^2 + b_2z_2z_3, z_1z_3 + b_1z_2z_3 + b_2z_3^2, z_1 + b_1z_2 + b_2z_3\} \quad \subseteq \quad \mathbb{R}[z_1, z_2, z_3]. \end{aligned} \tag{S45}$$

We just need to verify the subspace (S45) consists precisely of the inhomogeneous quadratic polynomials vanishing on all of the baseline. Let check this by direct calculation over the next three paragraphs.

The center of the first camera $\begin{pmatrix} I_3 & 0 \end{pmatrix}$ is $\begin{pmatrix} 0 & 0 & 0 \end{pmatrix}^{\top} \in \mathbb{R}^3$. By Cramer's rule, the center of the second camera $M(b)$ is the following point at infinity:

$$\begin{aligned} &\left(-\det \begin{pmatrix} b_1 & b_2 & b_3 \\ b_5 & b_6 & b_7 \\ 0 & 0 & 1 \end{pmatrix} \quad \det \begin{pmatrix} 1 & b_2 & b_3 \\ b_4 & b_6 & b_7 \\ 0 & 0 & 1 \end{pmatrix} \quad -\det \begin{pmatrix} 1 & b_1 & b_3 \\ b_4 & b_5 & b_7 \\ 0 & 0 & 1 \end{pmatrix} \quad \det \begin{pmatrix} 1 & b_1 & b_2 \\ b_4 & b_5 & b_6 \\ 0 & 0 & 0 \end{pmatrix} \right)^{\top} \\ &\quad = \begin{pmatrix} b_2b_5 - b_1b_6 & -b_2b_4 + b_6 & b_1b_4 - b_5 & 0 \end{pmatrix}^{\top} \quad \in \quad \mathbb{P}^3. \end{aligned}$$

Thus the baseline is

$$\{\lambda \begin{pmatrix} b_2b_5 - b_1b_6 & -b_2b_4 + b_6 & b_1b_4 - b_5 \end{pmatrix}^{\top} : \lambda \in \mathbb{R}\} \quad \subseteq \quad \mathbb{R}^3. \tag{S46}$$

Substituting (S46) into (S45) shows that all these polynomials indeed vanish identically on the baseline.

Next, note that the seven polynomials in (S45) are linearly independent in $\mathbb{R}[z_1, z_2, z_3]$. Indeed, suppose $\alpha \in \mathbb{R}^7$ satisfies

$$\begin{aligned} &\alpha_1 \left(b_4z_1z_2 + b_5z_2^2 + b_6z_2z_3\right) + \alpha_2 \left(b_4z_1z_3 + b_5z_2z_3 + b_6z_3^2\right) + \alpha_3 \left(b_4z_1 + b_5z_2 + b_6z_3\right) + \alpha_4 \left(z_1^2 + b_1z_1z_2 + b_2z_1z_3\right) \\ &+ \alpha_5 \left(z_1z_2 + b_2z_2^2 + b_2z_2z_3\right) + \alpha_6 \left(z_1z_3 + b_1z_2z_3 + b_2z_3^2\right) + \alpha_7 \left(z_1 + b_1z_2 + b_2z_3\right) \;=\; 0 \;\in\; \mathbb{R}[z_1, z_2, z_3]. \end{aligned} \tag{S47}$$

From the coefficient of z_1^2, we see $\alpha_4 = 0$. Since we are assuming that $M(b)$ has rank 3, it follows that $\begin{pmatrix} 1 & b_1 & b_2 \\ b_4 & b_5 & b_6 \end{pmatrix}$ has rank 2. So implies that third and seventh polynomials in (S47) are linearly independent, and since their monomial support is disjoint from that of the other polynomials in (S47), we have $\alpha_3 = \alpha_7 = 0$. This leaves the first, second, fifth and sixth polynomials in (S47). Writing out what remains in terms of the monomials $z_1z_2, z_1z_3, z_2^2, z_2z_3, z_3^2$ gives

$$\begin{pmatrix} b_4 & 0 & 1 & 0 \\ 0 & b_4 & 0 & 1 \\ b_5 & 0 & b_1 & 0 \\ b_6 & b_5 & b_2 & b_1 \\ 0 & b_6 & 0 & b_2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_5 \\ \alpha_6 \end{pmatrix} = 0. \tag{S48}$$

Actually, assumption that $\operatorname{rank}(M(b)) = 3$ implies that the coefficient matrix in (S48) has rank 4. Indeed, one verifies using computer algebra, *e.g.* Macaulay2 [4], that in the ring $\mathbb{R}[z_1, z_2, z_3]$ the radical of the ideal generated by the 4×4 minors of the matrix in (S48) equals the ideal generated by the 3×3 minors of $M(b)$. This forces $\alpha_1 = \alpha_2 = \alpha_5 = \alpha_6 = 0$.

Last, notice that requiring a quadric in $\mathbb{R}^3$ to contain a given line is a codimension 3 condition on the quadric. Indeed by projective symmetry, the codimension is independent of the specific choice of fixed line; and if we choose the z_3-axis, this amounts to requiring the vanishing of the bottom-left 2×2 submatrix of the quadric's 4×4 symmetric coefficient matrix.

Combining the last three pagragraphs, (S45) consists of the quadratic polynomials vanishing on the baseline as desired. $\square$

S3. Proofs for "Section 4.3: Ill-Posed Image Data"

In this section, we describe the locus of ill-posed image data for the 5-point and 7-point minimal problems in terms of the X.5-point curves. The logic is to use the classical epipolar relations in multiview geometry to relate these minimal problems to the task of intersecting a fixed complex projective algebraic variety with a varying linear subspace of complementary dimension. Then we apply tools from computational algebraic geometry, which were developed to analyze this task [2, 9].

S3.1. Background from algebraic geometry

Consider complex projective space $\mathbb{P}^n_{\mathbb{C}}$. The set of subspaces of $\mathbb{P}^n_{\mathbb{C}}$ of codimension d is naturally an irreducible projective algebraic variety, called the *Grassmannian*:

$$\mathrm{Gr}(\mathbb{P}^{n-d}_{\mathbb{C}}, \mathbb{P}^n_{\mathbb{C}}) = \{L \subseteq \mathbb{P}^n_{\mathbb{C}} : \dim(L) = n - d\}.$$

We use two classic coordinate systems for the Grassmannian. If a point $L \in \mathrm{Gr}(\mathbb{P}^{n-d}_{\mathbb{C}}, \mathbb{P}^n_{\mathbb{C}})$ is written as the kernel of a full-rank matrix $M \in \mathbb{C}^{d \times (n+1)}$, then the *primal Plücker coordinates* for L are defined to be the maximal minors of M:

$$p(L) \;=\; \left(p_{\mathcal{I}}(L) = \det(M(:,\mathcal{I}))\; : \; \mathcal{I} \in \binom{[n+1]}{d}\right) \tag{S49}$$

This gives a well-defined point in $\mathbb{P}_{\mathbb{C}}^{\binom{n+1}{d}-1}$ independent of the choice of M. Meanwhile, if we write L as the row span of a full-rank matrix $N \in \mathbb{C}^{(n-d+1)\times(n+1)}$, then the *dual Plücker coordinates* for L are defined to be the maximal minors of N:

$$q(L) \;=\; \left(q_{\mathcal{J}}(L) = \det(N(:,\mathcal{J}))\; : \; \mathcal{J} \in \binom{[n+1]}{n+1-d}\right). \tag{S50}$$

Again this gives a well-defined point $\mathbb{P}_{\mathbb{C}}^{\binom{n+1}{d}-1}$ independent of the choice of N. The primal and dual coordinates agree up to permutation and sign flips, namely for each $L \in \mathrm{Gr}(\mathbb{P}^{n-d}_{\mathbb{C}}, \mathbb{P}^n_{\mathbb{C}})$ it holds

$$\left(p_{\mathcal{I}}(L) : \mathcal{I} \in \binom{[n+1]}{d}\right) = \left((-1)^{n+|\mathcal{I}|} q_{[n+1]\setminus\mathcal{I}}(L) : \mathcal{I} \in \binom{[n+1]}{d}\right), \tag{S51}$$

where $|\mathcal{I}| := \sum_{i \in \mathcal{I}} i$.

Next let $X \subseteq \mathbb{P}^n_{\mathbb{C}}$ be an irreducible complex projective algebraic variety of dimension d. There exists a positive integer p, called the degree of X, such that for Zariski-generic subspaces of complementary dimension, $L \in \mathrm{Gr}(\mathbb{P}^{n-d}_{\mathbb{C}}, \mathbb{P}^n_{\mathbb{C}})$, the intersection of $L \cap X$ consists precisely p reduced (complex) intersection points. Here, one says that an intersection point $x \in L \cap X$ is reduced if $L \cap T_x X$ consists of one point, where $T_x X$ is the Zariski tangent space to X at x given by

$$T_x X \;=\; \left\{ v \in \mathbb{P}^n_{\mathbb{C}} \; : \; \left(\frac{\partial f_i(x)}{\partial x_j}\right)_{\substack{i=1,\dots,t \\ j=0,\dots,n}} v \;=\; 0 \right\} \;\subseteq\; \mathbb{P}^n_{\mathbb{C}}$$

for generators $f_1, \dots, f_t \in \mathbb{C}[x_0, \dots, x_n]$ of the prime ideal of X.

The *Hurwitz form* of X is defined to be the set of linear subspaces which are exceptional with respect to the property in the preceding paragraph. More precisely, it is

$$\mathcal{H}_X \;=\; \left\{ L \in \mathrm{Gr}(\mathbb{P}^{n-d}_{\mathbb{C}}, \mathbb{P}^n_{\mathbb{C}}) \; : \; L \cap X \text{ does not consist of } p \text{ reduced intersection points} \right\} \;\subseteq\; \mathrm{Gr}(\mathbb{P}^{n-d}_{\mathbb{C}}, \mathbb{P}^n_{\mathbb{C}}).$$

We will use the following result in the proofs of Theorems 3 and 4.

Theorem 1 [9, Thm. 1.1] *Let X be an irreducible subvariety of $\mathbb{P}^n_{\mathbb{C}}$ with dimension d, degree p and sectional genus g. Assume that X is not a linear subspace. Then $\mathcal{H}_X$ is an irreducible hypersurface in $\mathrm{Gr}(\mathbb{P}^{n-d}_{\mathbb{C}}, \mathbb{P}^n_{\mathbb{C}})$, and there exists a homogeneous polynomial Hu_X in the (primal) Plücker coordinates for $L \in \mathrm{Gr}(\mathbb{P}^{n-d}_{\mathbb{C}}, \mathbb{P}^n_{\mathbb{C}})$ such that*

$$L \in \mathcal{H}_X \quad \Leftrightarrow \quad \mathrm{Hu}_X\,(p(L)) = 0.$$

Further if the singular locus of X has codimension at least 2, then the degree of Hu_X in Plücker coordinates is $2p + 2g - 2$.

S3.2. Proof of Theorem 3

Proof: Let $\mathcal{E}_{\mathbb{C}} \subseteq \mathbb{P}^8_{\mathbb{C}}$ be the Zariski closure of the set of real essential matrices $\mathcal{E}$ inside complex projective space. It is known that $\mathcal{E}_{\mathbb{C}}$ is an irreducible complex projective variety, and its prime ideal is minimally generated by the ten cubic polynomials in Eq. 4 in the main paper. By a computer algebra calculation, $\mathcal{E}_{\mathbb{C}}$ has complex dimension $d = 5$, degree $p = 10$ and sectional genus $g = 6$. By [3, Prop. 2(i)], the singular locus of $\mathcal{E}_{\mathbb{C}}$ is a surface isomorphic to $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ with no real points, and in particular has codimension 3 in $\mathcal{E}_{\mathbb{C}}$. Therefore Theorem 1 applies, and tells us that the Hurwitz form $\mathcal{H}_{\mathcal{E}_{\mathbb{C}}}$ is a hypersurface in the Grassmannian $\mathrm{Gr}(\mathbb{P}^3_{\mathbb{C}}, \mathbb{P}^8_{\mathbb{C}})$ cut out by a polynomial $\mathrm{Hu}_{\mathcal{E}_{\mathbb{C}}}$ which is degree $2 \cdot 10 + 2 \cdot 6 - 2 = 30$ in Plücker coordinates.

For each $x = ((\gamma_1, \bar{\gamma}_1), \ldots, (\gamma_5, \bar{\gamma}_5)) \in (\mathbb{R}^2 \times \mathbb{R}^2)^{\times 5}$, we define the subspace

$$L(x) \;:=\; \mathrm{kernel} \begin{pmatrix} (\gamma_1)_1(\bar{\gamma}_1)_1 & (\gamma_1)_2(\bar{\gamma}_1)_1 & (\bar{\gamma}_1)_1 & (\gamma_1)_1(\bar{\gamma}_1)_2 & (\gamma_1)_2(\bar{\gamma}_1)_2 & (\bar{\gamma}_1)_2 & (\gamma_1)_1 & (\gamma_1)_2 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (\gamma_5)_1(\bar{\gamma}_5)_1 & (\gamma_5)_2(\bar{\gamma}_5)_1 & (\bar{\gamma}_5)_1 & (\gamma_5)_1(\bar{\gamma}_5)_2 & (\gamma_5)_2(\bar{\gamma}_5)_2 & (\bar{\gamma}_5)_2 & (\gamma_5)_1 & (\gamma_5)_2 & 1 \end{pmatrix}_{5\times 9} \subseteq \mathbb{P}^8_{\mathbb{C}}. \tag{S52}$$

Then we define $\mathbf{P}$ as follows:

$$\mathbf{P}(\gamma_1, \ldots, \bar{\gamma}_5) \;:=\; \mathrm{Hu}_{\mathcal{E}_{\mathbb{C}}}(p(L(x))),$$

where $p(L(x))$ are the primal Plücker coordinates of $L(x)$. In other words, $\mathbf{P}$ is obtained by substituting the $\binom{9}{5} = 126$ maximal minors of the matrix in (S52) into $\mathrm{Hu}_{\mathcal{E}_{\mathbb{C}}}$. Note $\mathbf{P}$ has degree 30 separately in each of the ten points $\gamma_1, \ldots, \bar{\gamma}_5$, because $\mathrm{Hu}_{\mathcal{E}_{\mathbb{C}}}$ has degree 30 in Plücker coordinates and each Plücker coordinate is separately linear in each x_i and each y_i.

We shall verify $\mathbf{P}$ has the property in the third sentence of the theorem statement. For the remainder of the proof, fix image data $x = ((\gamma_1, \bar{\gamma}_1), \ldots, (\gamma_5, \bar{\gamma}_5)) \in (\mathbb{R}^2 \times \mathbb{R}^2)^{\times 5}$ such that $\mathbf{P}(\gamma_1, \ldots, \bar{\gamma}_5) \neq 0$. We need to show that at every world scene that is compatible with x the forward Jacobian is invertible.

We show this using the epipolar constraints for two-view geometry [5, Part II]. We consider the following system in w:

$$\begin{cases} \Phi(w) = x \\ w \in \mathcal{W}, \end{cases} \tag{S53}$$

and the following system in E:

$$\begin{cases} \begin{pmatrix} \bar{\gamma}_i \\ 1 \end{pmatrix}^{\top} E \begin{pmatrix} \gamma_i \\ 1 \end{pmatrix} \;=\; 0 \quad \forall\, i = 1, \ldots, 5 \\ E \in \mathcal{E}. \end{cases} \tag{S54}$$

Each solution to (S54) corresponds to four solutions to (S53) via Ψ, and there are no other solutions to (S53). Moreover w depends smoothly on E, see [5, Result 9.19].

However solutions to (S54) are the real intersection points in

$$L(x) \cap \mathcal{E}_{\mathbb{C}} \;\subseteq\; \mathbb{P}^8_{\mathbb{C}}, \tag{S55}$$

because $\mathcal{E} = \mathcal{E}_{\mathbb{C}} \cap \mathbb{P}^8_{\mathbb{R}}$ (see [3, Sec. 2.1]). But we know (S55) consists of 10 reduced intersection points, by definition of $\mathbf{P}$ and the Hurwitz form. Denote these $\{E_1, \ldots, E_{10}\}$ where the real intersection points are $E_1, \ldots, E_a$. Using an appropriate version of the implicit function theorem (see [8, App. A]), there exists an open neighborhood $\mathcal{U}$ of x in $\mathcal{X}$ and differentiable functions $\widetilde{E}_1, \ldots, \widetilde{E}_{10} : \mathcal{U} \to \mathcal{E}_{\mathbb{C}}$ such that: *(i)* $\widetilde{E}_i(x) = E_i$ for each $i = 1, \ldots, 10$; *(ii)* for each $x' \in \mathcal{U}$ we have $L(x') \cap \mathcal{E}_{\mathbb{C}} = \{\widetilde{E}_1(x'), \ldots, \widetilde{E}_{10}(x')\}$, these intersection points are all reduced, and the only the first a points are real.

Combining the last two paragraphs, there exists differentiable functions $\widetilde{w}_1, \ldots, \widetilde{w}_{4a} : \mathcal{U} \to \mathcal{W}$ such that for each $x' \in \mathcal{U}$,

$$\{w \in \mathcal{W} : \Phi(w) = x'\} = \{\widetilde{w}_1(x'), \ldots, \widetilde{w}_{4a}(x')\}. \tag{S56}$$

Therefore $\Phi \circ \widetilde{w}_i = \mathrm{id}_{\mathcal{U}}$ for each i. Differentiating this and evaluating at $x \in \mathcal{X}$ gives

$$(D\Phi)(\widetilde{w}_i(x)) \circ (D\widetilde{w}_i)(x) = I_{20},$$

so that, in particular, $(D\Phi)(\widetilde{w}_i(x))$ is invertible for each i.

This proves that every world scene compatible with x has an invertible forward Jacobian as we needed. For a discussion of how to plot the 4.5-point curve using homotopy continuation, see "Numerical Computation of the X.5-Point Curves". □

S3.3. Proof of Theorem 4

Proof: This is similar to the proof of Theorem 3, although somewhat easier. Let $\mathcal{F}_{\mathbb{C}} \subseteq \mathbb{P}^8_{\mathbb{C}}$ be the Zariski closure of the set of real fundamental matrices $\mathcal{F}$ inside complex projective space. Then $\mathcal{F}_{\mathbb{C}}$ consists of all rank-deficient 3×3 matrices. So $\mathcal{F}_{\mathbb{C}}$ is an irreducible complex projective hypersurface, defined by the determinantal cubic equation. It has dimension $d = 7$, degree $p = 3$ and sectional genus $g = 1$. The singular locus of $\mathcal{F}_{\mathbb{C}}$ consists of all rank 1 matrices, and in particular has codimension 3. Therefore Theorem 1 applies, and tells us that the Hurwitz form $\mathcal{H}_{\mathcal{F}_{\mathbb{C}}}$ is a hypersurface in the Grassmannian $\mathrm{Gr}(\mathbb{P}^1_{\mathbb{C}}, \mathbb{P}^8_{\mathbb{C}})$ cut out by a polynomial $\mathrm{Hu}_{\mathcal{F}_{\mathbb{C}}}$ which is degree $2 \cdot 3 + 2 \cdot 1 - 2 = 6$ in Plücker coordinates.

For each $x = ((\gamma_1, \bar{\gamma}_1), \ldots, (\gamma_7, \bar{\gamma}_7)) \in (\mathbb{R}^2 \times \mathbb{R}^2)^{\times 7}$, we define the subspace

$$L(x) := \mathrm{kernel} \begin{pmatrix} (\gamma_1)_1(\bar{\gamma}_1)_1 & (\gamma_1)_2(\bar{\gamma}_1)_1 & (\bar{\gamma}_1)_1 & (\gamma_1)_1(\bar{\gamma}_1)_2 & (\gamma_1)_2(\bar{\gamma}_1)_2 & (\bar{\gamma}_1)_2 & (\gamma_1)_1 & (\gamma_1)_2 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (\gamma_7)_1(\bar{\gamma}_7)_1 & (\gamma_7)_2(\bar{\gamma}_7)_1 & (\bar{\gamma}_7)_1 & (\gamma_7)_1(\bar{\gamma}_7)_2 & (\gamma_7)_2(\bar{\gamma}_7)_2 & (\bar{\gamma}_7)_2 & (\gamma_7)_1 & (\gamma_7)_2 & 1 \end{pmatrix}_{7 \times 9} \subseteq \mathbb{P}^8_{\mathbb{C}}. \tag{S57}$$

Then we define $\mathbf{P}$ as follows:

$$\mathbf{P}(\gamma_1, \ldots, \bar{\gamma}_7) := \mathrm{Hu}_{\mathcal{F}_{\mathbb{C}}}(p(L(x))),$$

where $p(L(x))$ are the primal Plücker coordinates of $L(x)$. In other words, $\mathbf{P}$ is obtained by substituting the $\binom{9}{7} = 36$ maximal minors of the matrix in (S57) into $\mathrm{Hu}_{\mathcal{F}_{\mathbb{C}}}$. Note $\mathbf{P}$ has degree 6 separately in each of the fourteen points $x_1, \ldots, y_7$, because $\mathrm{Hu}_{\mathcal{F}_{\mathbb{C}}}$ has degree 6 in Plücker coordinates and each Plücker coordinate is separately linear in each x_i and each y_i.

The argument that $\mathbf{P}$ does the job is analogous to that for the calibrated case. One uses the epipolar constraints for the fundamental matrix, and the correspondence between fundamental matrices and world scenes (which this time is 1-1 due to [5, Sec. 9.5.2]). Given $x \in \mathcal{X}$ such that $\mathbf{P}(x) \neq 0$, each compatible fundamental matrix $F \in \mathcal{F}$ is a locally defined smooth function of x, by our choice of $\mathbf{P}$ and the definition of the Hurwitz form. The corresponding world scene $w \in \mathcal{W}$ is smooth as a function of F, and therefore a locally defined smooth function of x. Then we conclude with the chain rule again.

□

S3.4. Numerical Computation of the X.5-Point Curves

In the body of the paper, we specified that for the 5-point problem and 7-point problem, if we fix "4.5" and "6.5" correspondences as described in the body, then we can find a degree 30 and 6 curve on the second image indicating the ill-posed positions for the last image point. In this section, the generation of the **X.5-point curves** will be introduced in depth.

Suppose we have two images I and $\bar{I}$. Consider the essential matrix E and fundamental matrix F representing their relative pose in calibrated and uncalibrated cases respectively.

For the **uncalibrated case**, to generate the 6.5-point curve, six known point correspondences $(\gamma_i^\top = [(\gamma_i)_1, (\gamma_i)_2], \bar{\gamma}_i^\top = [(\bar{\gamma}_i)_1, (\bar{\gamma}_i)_2]), i = 1, \ldots, 6$, are needed. These point correspondences satisfy

$$\begin{pmatrix} \bar{\gamma}_i \\ 1 \end{pmatrix}^\top F \begin{pmatrix} \gamma_i \\ 1 \end{pmatrix} = 0, \qquad i = 1, \ldots, 6.$$

This equation can then be rewritten as a linear system:

$$\begin{cases} f^\top = (F_{11}, F_{12}, F_{13}, F_{21}, F_{22}, F_{23}, F_{31}, F_{32}, F_{33}) \\ w_i^T f = 0, \qquad i = 1, \dots, 6 \\ w_i^\top = ((\gamma_i)_1(\bar{\gamma}_i)_1, (\gamma_i)_2(\bar{\gamma}_i)_1, (\bar{\gamma}_i)_1, (\gamma_i)_1(\bar{\gamma}_i)_2, (\gamma_i)_2(\bar{\gamma}_i)_2, (\bar{\gamma}_i)_2, (\gamma_i)_1, (\gamma_i)_2, 1)\,. \end{cases} \tag{S58}$$

Note that the equations $w_i^T \tilde{F} = 0, i = 1, \dots, 6$ can then be rearranged into the form $Wf = 0$, where W is a 6×9 matrix. We extract a basis for the null space of this linear system, by computing three right singular vectors of W with 0 singular value. So the fundamental matrix can be reconstructed as

$$F = \alpha_1 F_1 + \alpha_2 F_2 + F_3, \tag{S59}$$

where F_1, F_2 and F_3 are the basis of the nullspace reshaped into 3×3 matrices; and α_1 and α_2 are free parameters in building the fundamental matrix F. From [7, Thm. 1], the fundamental matrix should satisfy

$$\det(F) = 0,$$

so that we have a polynomial with respect to α_1 and α_2:

$$\det(\alpha_1 F_1 + \alpha_2 F_2 + F_3) = 0. \tag{S60}$$

Now consider the final seventh "0.5" point correspondence between the images, $(\gamma_7 = ((\gamma_7)_1, (\gamma_7)_2)\,, \bar{\gamma}_7^\top = ((\bar{\gamma}_7)_1, (\bar{\gamma}_7)_2))$, where γ_7 is known but $\bar{\gamma}_7$ is not. We seek the values of $\bar{\gamma}_7$ such that all of the image data becomes ill-posed. To cut down on the number of variables, our strategy is to fix various numerical values for $(\bar{\gamma}_7)_1$ and then determine the corresponding degenerate values for the last coordinate $(\bar{\gamma}_7)_2$. Firstly we have the additional linear constraint

$$\begin{pmatrix} \bar{\gamma}_7 \\ 1 \end{pmatrix}^\top (\alpha_1 F_1 + \alpha_2 F_2 + F_3) \begin{pmatrix} \gamma_7 \\ 1 \end{pmatrix} = 0. \tag{S61}$$

Combining (S60) and (S61), we have two equations in the three unknowns $\alpha_1, \alpha_2, (y_7)_2$:

$$\begin{cases} \det(\alpha_1 F_1 + \alpha_2 F_2 + F_3) = 0 \\ ((\bar{\gamma}_7)_1, (\bar{\gamma}_7)_2, 1)(\alpha_1 F_1 + \alpha_2 F_2 + F_3)(\gamma_7, 1)^\top = 0. \end{cases} \tag{S62}$$

To find the points on the 6.5-point degenerate curve, we need to also enforce rank-deficiency of the Jacobian of the system (S62) with respect to α_1, α_2. This Jacobian reads

$$J = \begin{pmatrix} \frac{\partial \det(\alpha_1 F_1 + \alpha_2 F_2 + F_3)}{\partial \alpha_1} & \frac{\partial \det(\alpha_1 F_1 + \alpha_2 F_2 + F_3)}{\partial \alpha_2} \\ \frac{\partial ((\bar{\gamma}_7)_1, (\bar{\gamma}_7)_2, 1)(\alpha_1 F_1 + \alpha_2 F_2 + F_3)(\gamma_7, 1)^\top}{\partial \alpha_1} & \frac{\partial [(\bar{\gamma}_7)_1, (\bar{\gamma}_7)_2, 1](\alpha_1 F_1 + \alpha_2 F_2 + F_3)(\gamma_7, 1)^\top}{\partial \alpha_2} \end{pmatrix}. \tag{S63}$$

To express rank-deficiency, we introduce a dummy scalar variable d_1 to represent the non-trivial null space for J. All together we have the following system of equations now:

$$\begin{cases} \det(\alpha_1 F_1 + \alpha_2 F_2 + F_3) = 0 \\ ((\bar{\gamma}_7)_1, (\bar{\gamma}_7)_2, 1)(\alpha_1 F_1 + \alpha_2 F_2 + F_3)(\gamma_7, 1)^\top = 0 \\ J(d_1, 1)^\top = 0. \end{cases}$$

To plot the 6.5-point curve, we set the parameter $(\bar{\gamma}_7)_1$ to different real values, *e.g.* to horizontally range over all the pixels in the second image. For each fixed value of $(\bar{\gamma}_7)_1$, (S64) becomes a square polynomial system in the variables $\alpha_1, \alpha_2, (\bar{\gamma}_7)_2$. The real solutions correspond to the intersection of the 6.5-point curve and the corresponding column of the image. The curve inside the image boundaries can be obtained by solving these various systems independently, see Figure S1(a). We choose to solve the polynomials using homotopy continuation [8] as implemented in the Julia package [1]. By linearly connecting the intersection points, the 6.5-point curve is rendered. In some possible applications, a full plot of the curve may not be required. For example, consider checking the distance from a given point to the curve. In that case, we can simply compute

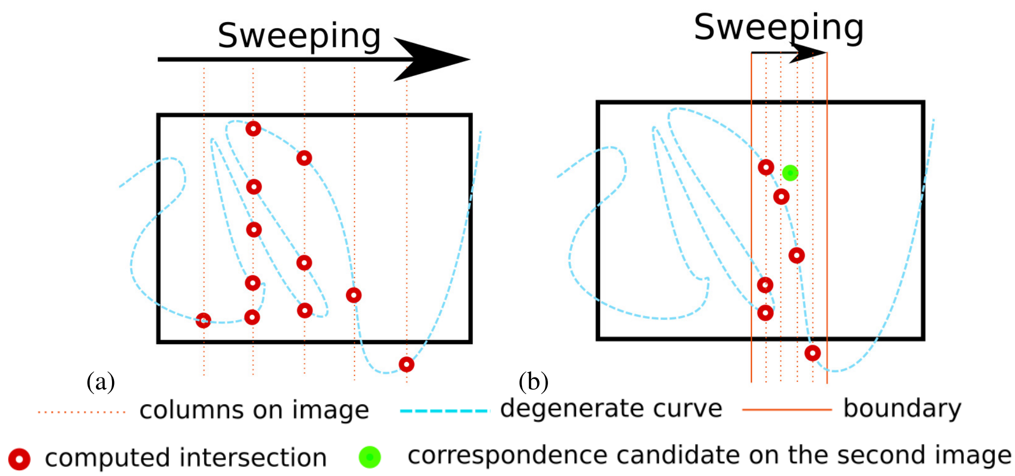

Figure S1. (a) To generate the X.5-point curves on the second image, we can sweep the image column-wise and compute the intersection with vertical lines by solving (S62). (b) Given a candidate correspondence, we can scan just a neighborhood around the candidate point.

the intersection points as $(\bar{\gamma}_7)_1$ ranges over a small interval around the correspondence candidate, see Figure S1(b). Finally, computations for different columns of the image are independent, so the described procedures are easily parallelized.

For the **calibrated case** (similarly to the uncalibrated case), with four known correspondences, we can build a linear system analogous to (S58) in the variable E. Here the null space is 5-dimensional, so we represent the essential matrix as

$$E = \alpha_1 E_1 + \alpha_2 E_2 + \alpha_3 E_3 + \alpha_4 E_4 + E_5. \tag{S64}$$

where E_i provide a basis of the null space. From [7], the essential matrix should also satisfy the following polynomial constraints:

$$\det(E) = 0 \qquad \text{and} \qquad E(E^\top E) - \frac{1}{2}\,\mathrm{trace}(EE^\top)E = 0,$$

which are in total 10 cubic equations. Similarly to uncalibrated case, to find the degenerate configurations, the Jacobian of these constraints with respect to $\alpha_1, \ldots, \alpha_4$ should be rank-deficient. The Jacobian can be built as follows:

$$J = \begin{pmatrix} \frac{\partial \det(E)}{\partial \alpha_1} & \frac{\partial \det(E)}{\partial \alpha_2} & \frac{\partial \det(E)}{\partial \alpha_3} & \frac{\partial \det(E)}{\partial \alpha_4} \\ \frac{\partial ((\bar{\gamma}_5)_1, (\bar{\gamma}_5)_2, 1) E (\gamma_5, 1)^{!\top}}{\partial \alpha_1} & \frac{\partial ((\bar{\gamma}_5)_1, (\bar{\gamma}_5)_2, 1) E (\gamma_5, 1)^\top}{\partial \alpha_2} & \frac{\partial ((\bar{\gamma}_5)_1, (\bar{\gamma}_5)_2, 1) E (\gamma_5, 1)^\top}{\partial \alpha_3} & \frac{\partial ((\bar{\gamma}_5)_1, (\bar{\gamma}_5)_2, 1) E (\gamma_5, 1)^\top}{\partial \alpha_4} \\ \frac{\partial\, \mathrm{vec}(E(E^\top E) - \frac{1}{2} trace(EE^\top)E)}{\partial \alpha_1} & \frac{\partial\, \mathrm{vec}(E(E^\top E) - \frac{1}{2} trace(EE^\top)E)}{\partial \alpha_2} & \frac{\partial\, \mathrm{vec}(E(E^\top E) - \frac{1}{2} trace(EE^\top)E)}{\partial \alpha_3} & \frac{\partial\, \mathrm{vec}(E(E^\top E) - \frac{1}{2} trace(EE^\top)E)}{\partial \alpha_4} \end{pmatrix}$$

Here $\mathrm{vec}(\cdot)$ represents the vectorization of a 3×3 matrix into a 9×1 vector, so that J is a 11×4 matrix. Then, we introduce dummy variables d_1, d_2, d_3 to express rank-deficiency of J and build the following system of equations:

$$\begin{cases} \det(E) = 0 \\ ((\bar{\gamma}_5)_1, (\bar{\gamma}_5)_2, 1) E (\gamma_5, 1)^\top = 0 \\ E(E^\top E) - \frac{1}{2} trace(EE^\top)E = 0 \\ J(d_1, d_2, d_3, 1)^\top = 0 \end{cases}$$

where $E = \alpha_1 E_1 + \alpha_2 E_2 + \alpha_3 E_3 + \alpha_4 E_4 + E_5$. Note that we have in total 22 equations and 8 unknowns. The variables are α_1, α_2, α_3, α_4, $(\bar{\gamma}_5)_1$, $(\bar{\gamma}_5)_2$, d_1, d_2 and d_3. The solutions to this system define the 4.5-point curve.

By setting $(\bar{\gamma}_5)_1$ to various different values, we can find the zero-dimensional solution sets following the same approach as in Figure S1. The real solutions for $(\bar{\gamma}_5)_2$ correspond to the intersection of the 4.5-point curve and a column of the image. The solutions to these systems are easily computed using HomotopyContinuation.jl. Note that the systems have 30 complex solutions, so that we will have at most 30 real intersection points with the various columns of the second image.

S4. Additional Experimental Results

S4.1. Extra Curve Samples

The main body showed four sample X.5-point curves for the calibrated and uncalibrated minimal problems. Figure S2 shows more synthetic curves. We have included different cases corresponding to stable and unstable problem instances.

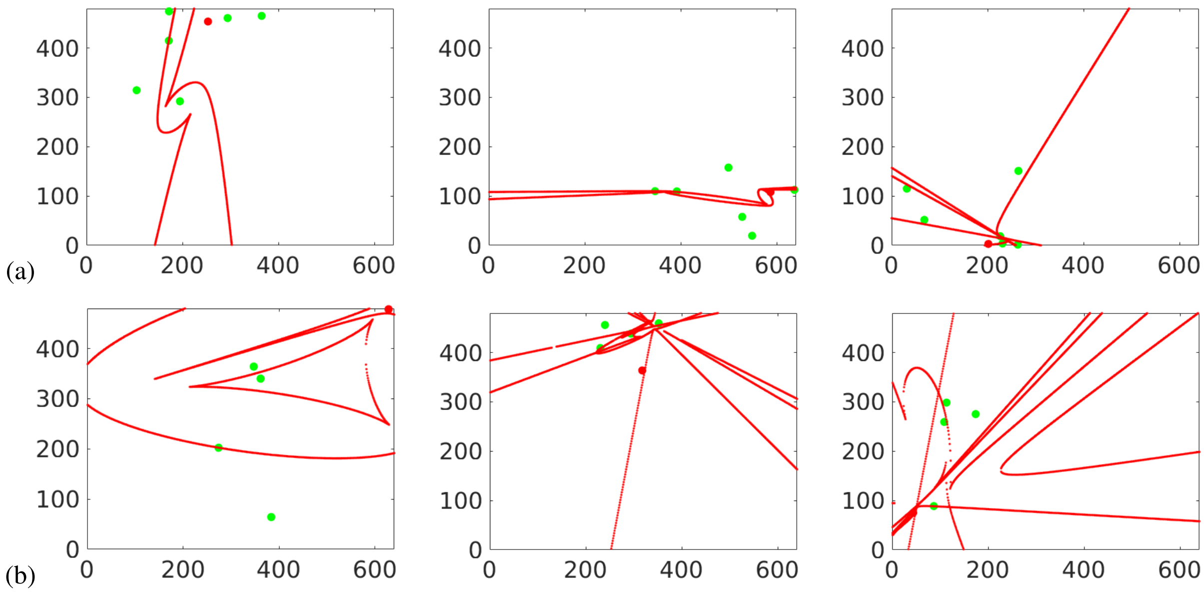

Figure S2. Sample renderings of the X.5-point curve. Points used in computing the curve are shown as green; the red point is the 5th/7th correspondence on the second image for calibrated/uncalibrated relative pose estimation; the red curve is the X.5-point curve we computed using homotopy continuation. (a) 6.5-point curves for the uncalibrated case. (b) 4.5-point curves for the calibrated case.

S4.2. Stability of Curves for Calibrated Case

Here we show sample perturbation results for the 4.5-point curve. For the synthetic dataset described in the main paper, we add $\mathcal{N}(0, 0.5)$-noise on each of the correspondences, then compute the resulting degenerate curve. In the main paper, Figure 6 shows the sample perturbation for the uncalibrated case. The corresponding cases for the calibrated case are in Figure S3. The statistics for the calibrated cases were already included in Figure 7.

S4.3. More Real Image Examples

Figure 8 in the main body showed an example using the X.5-point curve to indicate unstable configurations on real images. In this section, we display more examples on real images, see Tables S1 and S2. Note that our method is an indication of the stability of a minimal problem instance. Here we selected only all-inlier minimal problem instances whose reprojection error using the ground truth essential matrix is below a threshold (3 pixels are used). Then we computed the X.5-point curve on the second image. As mentioned in the body, the distance from the point to the curve can be used as a criteria to predict the stability. For highly unstable problem instances, the solution corresponding to the ground truth may suffer from large errors.

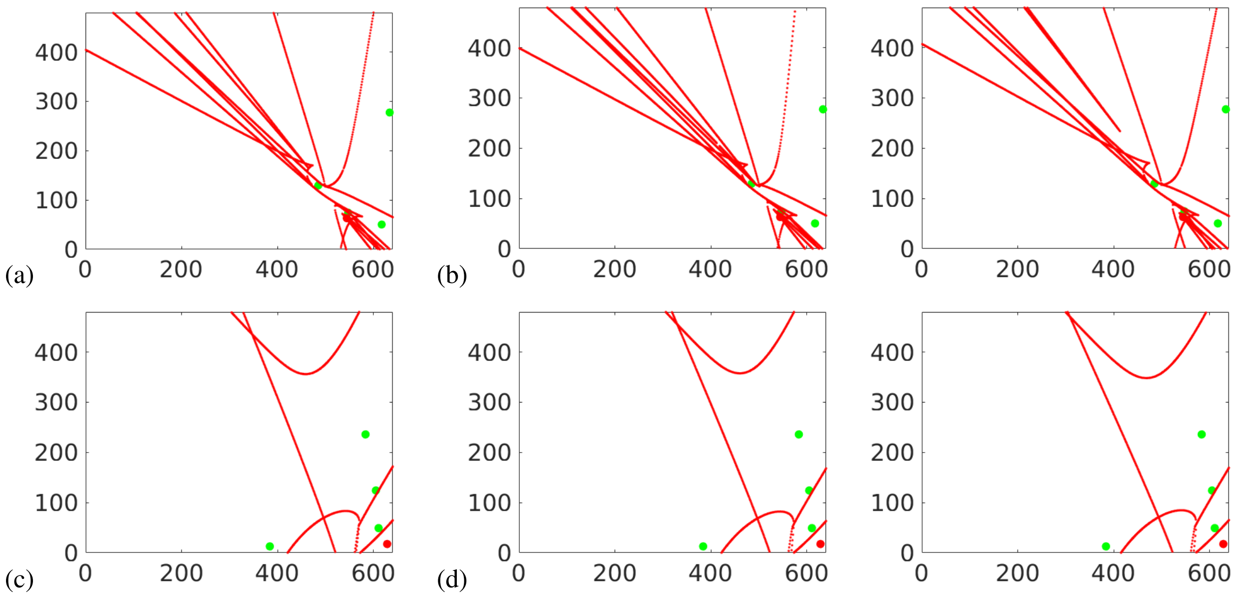

Figure S3. Illustrative result indicating the stability of the 4.5-point curve. (a) The degenerate curve for an unstable instance of E estimation. (c) The degenerate curve for a stable instance of E estimation. (b) (d) Adding different noise, the curves do not change much.

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Table S1. Further examples of the 6.5-point degenerate curve computed on real images. The estimated epipolar geometry is the estimate closest to the ground truth amongst the multiple solutions to the minimal problem.

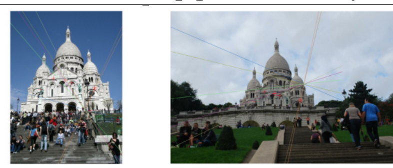
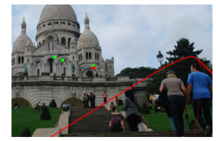
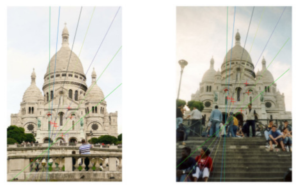
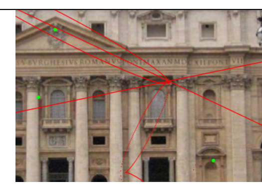
Ground Truth Epipolar Geometry	Estimated Epipolar Geometry	Degenerate Curve
		
		
		
		
		
		
		

Table S2. Further examples of the 4.5-point degenerate curve computed on real images. The estimated epipolar geometry is the estimate closest to the ground truth amongst the multiple solutions to the minimal problem.