Supplemental Material

A. Additional Experiments on the EPFL dataset

We test MatchFAME on the 6 EPFL datasets following the experimental setup of [13]. Each dataset includes 8 to 30 images, unlike the large number of images in the Photo Tourism datasets. Given each dataset, we generate and refine the initial keypoint matches with the same procedure introduced in [13]. We follow their convention and estimate the universe size within a significant short amount of time.

A.1. More on Cycle Consistency and Inconsistency

We referred to a cycle \(ijk\) as consistent whenever \(X_{ij}X_{jk} \leq X_{ik}\), \(X_{ik}X_{ki} \leq X_{ji}\) and \(X_{ki}X_{ij} \leq X_{kj}\). Note that \(X_{ij}X_{jk}\) is a binary matrix with ones whenever there are paths of lengths 2 between keypoints of images \(i\) and \(k\) and \(X_{kj}\) is binary matrix with ones whenever there are paths of lengths 1 (single edges) between keypoints of image \(i\) and \(k\), respectively; both are matched to a keypoint \(t_j\) in image \(j\), then they are matched to each other. Therefore, any cycle \(ijk\) with corresponding partial permutations \(X_{ij}, X_{jk}, X_{ki}\) is consistent if and only if for any \(t_i \in [m_i], t_j \in [m_j]\) and \(t_k \in [m_k]\): if two of the events \(X_{ij}(t_i,t_j) = 1, X_{jk}(t_j,t_k) = 1, X_{ki}(t_k,t_i) = 1\) hold true, then the third one holds true as well.

This equivalent reformulation of cycle consistency further clarifies the definition of \(d_{ijk}\) in (4). For fixed \(X_{ij}, X_{jk}, X_{ki} \in \mathcal{P}^{1 \times 2}\), the denominator of the fraction in (4) can be viewed as the number of combinations of three keypoints \(a, b, c\), such that at least two of the three events

\[
X_{ij}(a,b) = 1, \ X_{jk}(b,c) = 1, \text{ and } X_{ki}(c,a) = 1 \tag{10}
\]

hold. Furthermore, the numerator of the fraction in (4) can be viewed as the total number combinations of three keypoints \(a, b, c\), such that all the three events in (10) hold. Thus, the fraction in (4) indeed measures the level of cycle consistency, and consequently \(d_{ijk}\) measures the cycle inconsistency.

We remark that an inequality of two full permutation matrices must be an equality. Therefore, for permutation synchronization the above definition of cycle consistency is equivalent with \(X_{ij}X_{jk} = X_{ik}\) (or equivalently, \(X_{jk}X_{ki} = X_{ji}\) or \(X_{ki}X_{ij} = X_{kj}\) or \(X_{ki}X_{ij}X_{jk} = I\)). That is, our definition of cycle consistency is a direct extension of the one in group synchronization.

B. Clarifications

We clarify some definitions and expand on various claims mentioned in the paper.

B.1. More on Cycle Consistency and Inconsistency

We referred to a cycle \(ijk\) as consistent whenever \(X_{ij}X_{jk} \leq X_{ik}\), \(X_{ik}X_{ki} \leq X_{ji}\) and \(X_{ji}X_{ij} \leq X_{kj}\). Note that \(X_{ij}X_{jk}\) is a binary matrix with ones whenever there are paths of lengths 2 between keypoints of images \(i\) and \(k\) and \(X_{kj}\) is binary matrix with ones whenever there are paths of lengths 1 (single edges) between keypoints of image \(i\) and \(k\), respectively; both are matched to a keypoint \(t_j\) in image \(j\), then they are matched to each other. Therefore, any cycle \(ijk\) with corresponding partial permutations \(X_{ij}, X_{jk}, X_{ki}\) is consistent if and only if for any \(t_i \in [m_i], t_j \in [m_j]\) and \(t_k \in [m_k]\): if two of the events \(X_{ij}(t_i,t_j) = 1, X_{jk}(t_j,t_k) = 1, X_{ki}(t_k,t_i) = 1\) hold true, then the third one holds true as well.

This equivalent reformulation of cycle consistency further clarifies the definition of \(d_{ijk}\) in (4). For fixed \(X_{ij}, X_{jk}, X_{ki} \in \mathcal{P}^{1 \times 2}\), the denominator of the fraction in (4) can be viewed as the number of combinations of three keypoints \(a, b, c\), such that at least two of the three events

\[
X_{ij}(a,b) = 1, \ X_{jk}(b,c) = 1, \text{ and } X_{ki}(c,a) = 1 \tag{10}
\]

B.2. Cycle-verifiability Helps in Verifying Matches in Cycles

We further interpret the cycle-verifiability condition and clarify its name. We claim that if \(ijk\) is a good cycle (w.r.t. \(ij\)) ensured by Definition 1 with \(a \in I_i \) and \(c \in I_j\), then one can verify whether \(a\) and \(c\) correctly match (i.e., \(h(a) = h(c)\)) using \(b \in I_k\). Indeed, since \(b\) matches \(a\) and \(k \in G_{ij}\), \(h(a) = h(b)\). If \(b\) and \(c\) match then since \(k \in G_{ij}\), \(h(b) = h(c)\) and consequently \(h(a) = h(c)\). Assume on the other hand that \(b\) and \(c\) do not match. If \(b\) matches another point \(c'\), then since \(k \in G_{ij}\), \(h(b) = h(c') \neq h(c)\). If \(b\) does not match any point in \(U_j\) then since \(k \in G_{ij}\), \(h(b) \notin h(U_j)\) (otherwise there exists \(c' \in U_j\) such that \(h(c') = h(b)\) and since \(k \in G_{ij}\) there has to be a match between \(b\) and \(c'\)). Since \(h(c) \in h(U_j)\) and \(h(b) \notin h(U_j)\), \(h(b) \neq h(c)\).

C. Proof of Theorem 1

The proof establishes two lemmas, Lemmas 1 and 2, and then uses them to conclude Theorem 1. It is rather technical.
Table 4. Performance on the EPFL datasets. \( n \) is the number of cameras; \( \hat{m}_i \), the approximated \( m_i \), is twice the averaged \( m_i \) over \( i \in [n] \); PR refers to the precision \( |E \cap E^g|/|E| \), which is expressed in percentage (the higher the better); \#M is the ratio (expressed in percentage) between the number of refined matches and the number of initial matches; \( T \) is runtime in seconds.

<table>
<thead>
<tr>
<th>Algorithms</th>
<th>Dataset</th>
<th>( n )</th>
<th>( \hat{m}_i )</th>
<th>PR</th>
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<td>Herz-Jesu-P25</td>
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<tr>
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<td>71.8</td>
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<td>80.6</td>
<td>72 99</td>
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<td>78 11</td>
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<td>Fountain-P11</td>
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<td>81 8</td>
<td>95.4</td>
<td>93 14</td>
<td>95.7</td>
</tr>
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and not so easy to motivate. In order to provide more intuition, we added some clarifying figures.

Conventional figures: In all of these figures, we designate by green lines good keypoint matches, by red lines bad keypoint matches and by dashed orange lines missing keypoint matches. All of these occur between keypoints of two different images. On the other hand, matches between keypoints in an image and universal 3D keypoints are designated by black dotted lines (these correspond to our formal \( h \) function). We further color the universal 3D keypoints (in \( U_{ij} \)), which represent elements of \( U_{ij}^{good} \), by green. We also color the universal 3D keypoints, which represent elements of \( U_{ij}^{bad} \), in red. In Figure 5, we slightly extend the latter convention and explain it in its caption.

Terminology Review: Recall that \( n_{\Delta} = Tr(X_{ij}X_{jk}X_{ki}) \), \( m \) is the number of all 3D keypoints, \( m_{ij} \) is the number of 3D keypoints that correspond to the 2D keypoints of images \( i \) or \( j \) and among these, \( m_{ij}^{good} \) is the number of keypoints that match wrong keypoints in the other image, or match no keypoint if a ground-truth match exists. We also denote the number of the rest of points by \( m_{ij}^{bad} \) (that is, \( m_{ij}^{good} = m_{ij} - m_{ij}^{bad} \)) and recall that these keypoints match the ground-truth keypoints or do not match any keypoint, if no ground-truth matches exist.

C.1. Upper Bound for the Cycle Inconsistency of Good Edges

This section includes the proof of the following lemma:

**Lemma 1.** For any \( ij \in E_g \), \( d_{ijk} \leq m(s_{ik}^* + s_{jk}^*) \).

We remark that in the case of group synchronization, in particular, PS, one can easily show that for any \( ij \in E \), \( |d_{ijk} - s_{ik}^*| \leq s_{jk}^* + s_{jk}^* \) (see Lemma 1 of [11]). Consequently for \( ij \in E_g \), \( d_{ijk} \leq s_{ik}^* + s_{jk}^* \). However, in PPS, without the full group structure with a bi-invariant metric, it is harder to prove the weaker bound of Lemma 1. The proof below involves various discrete combinatorial arguments.

**Proof.** Assume first that \( n_{\Delta} = 0 \) and note that (4) implies \( d_{ijk} = 1 \). Since \( d_{ijk} \neq 0 \) and \( ij \in E_g \), \( s_{ik}^* \) and \( s_{jk}^* \) cannot be both zero (otherwise this and the fact that \( ij \in E_g \) imply that \( ijk \) is cycle-consistent and thus \( d_{ijk} = 0 \)). Without loss of generality, assume \( s_{jk}^* > 0 \). We note that

\[
s_{jk}^* = \frac{m_{jk}^{bad}}{m_{jk}} \geq \frac{1}{m},
\]

which implies the desired bound:

\[
d_{ijk} = 1 = m \cdot \frac{1}{m} \leq m s_{jk}^* \leq m(s_{ik}^* + s_{jk}^*).
\]

Assume next that \( n_{\Delta} > 0 \), or equivalently,

\[
n_{\Delta} \geq 1.
\]

The next arguments require additional definitions and observations. We recall that any element of \( I_i \) represents a 2D keypoint in image \( i \). This keypoint is associated with the index vector \((i,j)\), where \( j = 1, \ldots, m_i \) and we can thus view \( I_i \) as the set of \( m_i \) index vectors. For cycle \( ijk \), \((a,b,c)\) \( \in I_i \times I_j \times I_k \) is an \((i,j,k)\) tuple if there is a match between \( a \) and both \( b \) and \( c \) (the match can be either good or bad). If \((a,b,c)\) is an \((i,j,k)\) tuple and there is no match between \( b \) and \( c \), then we refer to \((a,b,c)\) as a bad \((i,j,k)\) tuple, otherwise, it is a good \((i,j,k)\) tuple. For example, in Figure 5, there are three \((i,j,k)\) tuples: \((p_{1,2,p_{1,1},p_{1,1}})\), \((p_{1,3,p_{1,3},p_{1,3}})\), \((p_{1,4,2,p_{1,2},p_{1,3}})\), \((p_{1,4,2,p_{1,2},p_{1,3}})\) are bad \((i,j,k)\) tuples and \((p_{1,3,p_{1,3},p_{1,3}})\) is a good \((i,j,k)\) tuple.

Cycle \( ijk \), we denote by \( A_{i,j,k} \) the set of \((i,j,k)\) tuples in \( I_i \times I_j \times I_k \), by \( A_{ij}^{bad} \) the set of bad \((i,j,k)\) tuples and by \( A_{ijk}^{good} \) the set of good \((i,j,k)\) tuples.

Recall that for a cycle \( ijk \), \( n_{ij} = n_{nzz}(X_{ki}X_{ij}) \), \( n_{ij} \) and \( n_k \) are analogously defined, and \( n_{\Delta} = Tr(X_{ij}X_{jk}X_{ki}) \). Also recall the notation \( p_{ij} \) for elements of \( I_i \). We note that \((a,b,c)\) is an \((i,j,k)\) tuple if and only if (iff) \( a = p_{ia} \), \( b = p_{ib}, c = p_{ic} \), and both \( X_{ki}(w,u) = 1 \) and \( X_{ij}(u,v) = 1 \) (so \( p_{ia} \) matches to both \( p_{ib} \) and \( p_{ic} \)). We further note that the latter two requirements are equivalent with \( X_{ki}X_{ij}(w,v) = 1 \). Indeed,
since $X_{ki}X_{ij}(w,v) = \sum_{u' \in [m_i]} X_{ki}(w,u')X_{ij}(u',v)$ and $X_{ki}X_{ij}$ is a partial permutation, then $X_{ki}X_{ij}(w,v) = 1$ if $X_{ki}(w,u) = X_{ij}(u,v) = 1$ for some $u \in [m_i]$. Therefore

$$|A_{i,jk}| = n_i.$$ 

By the same way we note that

$$|A_{j,ki}| = n_j \quad \text{and} \quad |A_{k,ij}| = n_k.$$

Similarly, note that $(a,b,c)$ is a good $(i,j,k)$ tuple iff $a = p_{i,j}, b = p_{j,k}, c = p_{k,i}$, $X_{ki}(w,u) = 1$, $X_{ij}(u,v) = 1$ and $X_{jk}(v,w) = 1$. The latter three requirements are equivalent with $X_{ij}X_{jk}X_{ki}(u,u) = 1$. Indeed, the following equation

$$X_{ij}X_{jk}X_{ki}(u,u) = \sum_{v' \in [m_j], u' \in [m_k]} X_{ij}(u,v')X_{jk}(v',w')X_{ki}(w,u)$$

and the fact that $X_{ij}X_{jk}X_{ki}$ is a partial permutation imply this equivalence. Therefore,

$$|A_{i,jk}^{good}| = n_{ij} \quad \text{and} \quad |A_{i,jk}^{bad}| = n_i - n_{ij}.$$

Similarly, we conclude that

$$|A_{j,ki}^{bad}| = n_{ij} - n_{ij} \quad \text{and} \quad |A_{k,ij}^{bad}| = n_k - n_{ij}.$$

Using these observations and (4), one can rewrite $d_{ijk}$ as follows

$$d_{ijk} = \frac{|A_{i,jk}^{bad}| + |A_{j,ki}^{bad}| + |A_{k,ij}^{bad}| + 3n_{ij}}{\left|A_{i,jk}^{bad}\right| + \left|A_{j,ki}^{bad}\right| + \left|A_{k,ij}^{bad}\right| + 3n_{ij}.} \quad (12)$$

Let us assume that $(a,b,c) \in A_{i,jk}^{bad}$ and show that

$$h(c) \in U_{ik}^{bad} \cup U_{jk}^{bad}, \quad (13)$$

The assumption $ij \in E_g$ implies that there is a good match between $a$ and $b$.

We claim that if there is also a good match between $a$ and $c$, then $h(b) = h(a) = h(c) \in U_{ij} \cap U_{ik} \cap U_{jk}$. Indeed, assume $a = p_{i,j}, b = p_{j,k}, c = p_{k,i}$ and $h(a) = l$, i.e., $P_l^*(u,l) = 1$. Because there exists a good match between $a$ and $b$, $X_{ij}^*(u,v) = X_{ij}(u,v) = 1$. Since $X_{ij}^*(u,v) = P_l^*P_l^{*T}(u,v) = \sum_{d \in [m]} P_l^*(u,d)P_l^*(v,d)$ and $P_l^*$ is a partial permutation, $P_l^*(v,l) = 1$ and thus $h(b) = l$. Similarly, since there exists a good match between $a$ and $c$, $P_{c,b}^*(w,l) = 1$ and $h(c) = l$. Therefore $h(a) = h(b) = h(c) = l \in U_{ij} \cap U_{ik} \cap U_{jk}$.

Since $(a,b,c) \in A_{i,jk}^{bad}$, there is no match between $b$ and $c$ and thus $h(c) \in U_{jk}^{bad}$, which implies (13).

If on the other hand, there is a bad match between $a$ and $c$, then $h(c) \in U_{ik}^{bad}$, which also implies (13).

In view of (13), the function $f(a,b,c) = h(c)$ maps $A_{i,jk}^{bad}$ to $U_{ik}^{bad} \cup U_{jk}^{bad}$. We note that this function is injective. Indeed, since $X_{ik}$ and $X_{jk}$ are partial permutations, for any

$$h(c) \in U_{ik}^{bad} \cup U_{jk}^{bad}, \quad \text{where} \quad c \in I_k, \quad \text{if} \quad a \neq b \quad \text{in} \quad I_i \quad \text{and} \quad I_j,$$

respectively, such that there are matches between $a$ and $c$, and $f(a,b,c) = h(c)$, then $a$ and $b$ are unique. Figure 5 demonstrates $f$ and its proven injectivity in a special case. In this case, $A_{i,jk}^{bad} = \{p_{i,j}, p_{j,k}, p_{i,k}\}$, $f(p_{i,j}, p_{j,k}, p_{i,k}) = h(p_{i,k}) = 5 \in U_{ik}^{bad}$ and $f(p_{i,j}, p_{j,k}, p_{i,k}) = h(p_{i,k}) = 2 \in U_{jk}^{bad}$.

By the cardinality property of an injective map,

$$|A_{i,jk}^{bad}| + |A_{j,ki}^{bad}| + |A_{k,ij}^{bad}| \leq 3(n_{ik}^{bad} + m_{jk}^{bad}). \quad (14)$$

Therefore the combination of (11) and (14) with the definition of $s_{ik}^*$ as well as $s_{j,k}^*$ (i.e., noting that $s_{ik}^* = m_{ik}^{bad}/m_{ik} = 1 - m_{ik}^{good}/m_{ik}$ and $s_{j,k}^* = m_{j,k}^{bad}/m_{j,k} = 1 - m_{j,k}^{good}/m_{j,k}$) yields

$$d_{ijk} = \frac{|A_{i,jk}^{bad}| + |A_{j,ki}^{bad}| + |A_{k,ij}^{bad}|}{|A_{i,jk}^{bad}| + |A_{j,ki}^{bad}| + |A_{k,ij}^{bad}| + 3n_{ij}} \leq \frac{3(n_{ik}^{bad} + m_{jk}^{bad})}{3(n_{ik}^{bad} + m_{jk}^{bad}) + 3} \leq \frac{3n_{ik}^{bad} + 3m_{jk}^{bad}}{3n_{ik}^{bad} + 3m_{jk}^{bad} + 3} = \frac{3s_{ik}^{*} + \frac{3}{m_{ik}^{bad}}}{3s_{ik}^{*} + \frac{3}{m_{ik}^{bad}} + 3} \leq m_{ik}^{*} + m_{j,k}^{*} s_{j,k}^{*} \leq m^{*}(s_{ik}^{*} + s_{j,k}^{*}).$$
C.2. Lower Bound for the Averaged Cycle Inconsistency Among Good Cycles

This section includes the proof of the following lemma:

Lemma 2. If \( G = (V,E) \) is \( p_v \)-cycle verifiable, then

\[
\frac{1}{3} D_v s_i j \leq \frac{1}{|G|} \sum_{k \in G_{ij}} d_{ijk} \quad \forall i j \in E.
\]  

(15)

Proof. We assume several cases.

Case I: \( i j \in E_g \). The left hand side of (15) is zero and its right hand side (RHS) is also zero since for any \( i j \in E_g \) and \( k \in G_{ij} \), \( d_{ijk} = 0 \).

Case II: \( i j \in E_b \) and \( k \in G_{ij} \). Denote

\[
N_{ijk}^{bad} = |A_{ijk}^{bad}| + |A_{jk,i}^{bad}| + |A_{ki,j}^{bad}|
\]

and note that in view of (12)

\[
\sum_{k \in G_{ij}} d_{ijk} = \sum_{k \in G_{ij}} \frac{N_{ijk}^{bad}}{N_{ijk}^{bad} + 3n_{\Delta}}.
\]  

(16)

We thus need to lower bound the RHS of (16) in order to conclude (15).

We first derive the bound \( n_{\Delta} \leq m_{ij}^{good} \). Figure 6 demonstrates the definitions below and the desired bound in a very special case. Let \( D_{ijk} \) denote the set of indices of diagonal entries of \( X_{ij} X_{jk} X_{ki} \), that equal 1. Note that \( |D_{ijk}| = n_{\Delta} \), due to the fact that \( n_{\Delta} = \text{Tr}(X_{ij} X_{jk} X_{ki}) \). Also, since \( X_{ij} X_{jk} X_{ki} \) is of size \( m_i \times m_j \), \( D_{ijk} \subseteq [m_i] \). We can thus assign for \( d \in D_{ijk} \), \( p_{i,d} = a \in I_i \). Because \( X_{ij} X_{jk} X_{ki} (d,d) = 1 \), there exists \( 1 \leq u \leq m_j \) and \( 1 \leq v \leq m_k \), such that \( X_{ij} (d,u) = X_{jk} (u,v) = X_{ki} (v,d) = 1 \). Therefore, we note that for \( b := p_{j,v} \in I_j \) and \( c := p_{k,c} \in I_k \), there exist matches between \( a \) and \( b \), \( a \) and \( c \), as well as \( b \) and \( c \), as and \( a \) and \( c \).

Since \( k \in G_{ij} \), \( jk \in E_g \) and \( ki \in E_g \) and thus \( h(b) = h(c) = h(a) \) (see the same argument in the paragraph below (13), where it is enough to just assume that either \( jk \in E_g \) or \( ki \in E_g \)), we denote the latter common value by \( l \). Therefore \( X_{ij}^l (a,l) = X_{jk}^l (b,l) = X_{ki}^l (c,l) = 1 \). By the cardinality property of an injective map, \( n_{\Delta} \leq |U_{ij}^{good}| = m_{ij}^{good} \).

Next, we prove an upper bound of \( N_{ijk}^{bad} \). We assume without loss of generality that \( (a,b,c) \in A_{ijk}^{bad} \). Since \( ik, jk \in E_g \), the matches from \( a \) to \( c \) and from \( b \) to \( c \) are correct. Therefore, the match from \( a \) to \( b \) is wrong and \( h(a) \in U_{ij}^{bad} \). Denote by \( g_i : A_{ijk}^{bad} \rightarrow U_{ij}^{bad} \) the function which maps \( (a,b,c) \in A_{ijk}^{bad} \) to \( h(a) \in U_{ij}^{bad} \). Figure 7 illustrates \( g_i \) in a special case. This function is injective since for any \( x \in U_{ij}^{bad} \), \( g_i^{-1}(x) \) contains at most one element \( (a,b,c) \in A_{ijk}^{bad} \). Indeed, if \( g_i(\mathbf{a,b,c}) = x \), then there must exist \( e \in I_i \) such that \( h(a) = x \) in \( I_i \) such that there is a match between \( a \) and \( c \), and in \( I_j \) such that there is a match between \( a \) and \( b \) (and no match between \( b \) and \( c \)). Note that there is a match between at most one keypoint in \( I_e \) and \( a \) and thus there is at most one such \( c \). Similarly, there is at most one such \( b \). Since there is at most one keypoint in \( I_j \) which corresponds to the 3D keypoint \( x \), there is at most one such \( a \). The injectivity of \( g_i \) implies \( |A_{ijk}^{bad}| \leq |U_{ij}^{bad}| = m_{ij}^{good} \). Similarly, \( |A_{ijk}^{bad}| \leq m_{ij}^{good} \) and \( |A_{ijk}^{bad}| \leq m_{ij}^{good} \). Thus, for any \( k \in G_{ij} \)

\[
0 \leq N_{ijk}^{bad} \leq 3m_{ij}^{bad}.
\]  

(17)

Next, we establish a lower bound of \( N_{ijk}^{bad} \). For this purpose, we construct an injective map \( g_{ij,k} \) from \( U_{ij}^{bad} \cap h(I_k) \) to \( A_{ijk}^{bad} \cup A_{jk,i}^{bad} \cup A_{ki,j}^{bad} \). It will allow us to lower bound \( N_{ijk}^{bad} = |A_{ijk}^{bad}| + |A_{jk,i}^{bad}| + |A_{ki,j}^{bad}| \) by the cardinality of
we conclude WLOG that \( x_{ijk} \) is verifiable w.r.t. \( h(I_j) \). Therefore any element of \( U_{ij}^{bad} \cap h(I_k) \) is either in \( h(I_j) \) or \( h(I_j) \cap h(I_k) \).

In the case where \( x \in U_{ij}^{bad} \cap h(I_k) \) and \( x \in h(I_j) \), we will show that there exist either \( (a,b,c) \in A_{ij,k}^{bad} \) or \( (c,a,b) \in A_{ij,k}^{bad} \) such that \( h(c) = x \). In the case where \( x \in U_{ij}^{bad} \cap h(I_k) \) and \( x \in h(I_j) \cap h(I_k) \), then one can similarly show that there exists either \( (b,c,a) \in A_{ij,k}^{bad} \) or \( (c,a,b) \in A_{ij,k}^{bad} \) such that \( h(c) = x \). These arguments induce a map \( g_{ij,k} \) from \( U_{ij}^{bad} \cap h(I_k) \) to \( A_{ij,k}^{bad} \cup A_{ij,k}^{bad} \) or \( A_{j,k}^{bad} \cup A_{k,j}^{bad} \), which maps \( x \) to its corresponding bad tuple. Since \( h(c) = x \), \( g_{ij,k} \) is injective. Figure 7 illustrates \( g_{ij,k} \) in a special case.

We thus assume that \( x \in U_{ij}^{bad} \cap h(I_k) \) and \( x \in h(I_j) \). Note that the latter requirement implies the existence of \( a \in I_i \) such that \( h(a) = x \). Since \( x \in h(I_k) \), there exists \( c \in I_k \) such that \( h(c) = x \) and \( a \in I_i \). Note that there cannot be a good match between \( a \) and \( c \). In the first case there exists \( b \in I_j \) such that \( h(b) \neq h(a) \) and since we showed above that \( h(a) = h(c) \), we conclude that \( h(b) \neq h(c) \). The latter observation and the fact that \( j \in E_g \) imply that there is no match between \( b \) and \( c \) and thus \( (a,b,c) \) is a bad (\( i,j,k \)) tuple, that is, \( (a,b,c) \in A_{ij,k}^{bad} \). In the second case there exists \( b \in I_j \) such that \( h(b) = h(a) \), but there is no match between \( b \) and \( a \) (the previous case considered the scenario where there exists \( b \in I_j \) such that \( a \) and \( b \) match; furthermore, if \( h(a) \neq h(b) \) for all \( b \in I_j \), then \( x \in U_{ij}^{good} \)). Since \( h(a) = h(c) = h(b) \) and \( j \in E_g \), there is a match between \( b \) and \( c \). Therefore, \( (c,a,b) \) is a bad \( (k,i,j) \) tuple, that is, \( (c,a,b) \in A_{k,j}^{bad} \). Following the above ideas, this concludes the injectivity of \( g_{ij,k} \). This injectivity implies

\[
\sum_{x \in U_{ij}^{bad}} 1_{x \in h(I_k)} = |U_{ij}^{bad} \cap h(I_k)| \leq |A_{ij,k}^{bad} \cup A_{ij,k}^{bad} \cup A_{k,j}^{bad} | \\
\leq |A_{ij,k}^{bad}| + |A_{ij,k}^{bad}| + |A_{k,j}^{bad}| = N_{ij}^{bad}. \tag{18}
\]

In order to apply (18) we lower bound a certain sum of \( 1_{x \in h(I_k)} \). Our argument assumes that \( x \in U_{ij}^{bad} \). Since \( x \in U_{ij} \), we conclude WLOG that \( x \in h(I_k) \). Therefore there exists \( a \in I_i \) such that \( h(a) = x \). By the \( p_{ij} \)-cycle verifiability condition, \( a \) is verifiable w.r.t. \( ij \) in at least \( p_{ij}|G_{ij}| \) good cycles. For any such cycle \( ijk \) that \( a \) is verifiable in, let \( b \in I_k \) match \( a \) (for convenience, we demonstrate \( x, a, b \) and \( k \) in Figure 8). Since \( ijk \in E_g \), the match between \( a \) and \( b \) is a good match and thus \( x = h(a) = h(b) \). Since \( b \in I_k \) and \( h(b) \in h(I_k) \) and thus \( x \in h(I_k) \). That is, we have proved that if \( k \in G_{ij} \) and \( a \) is verifiable in \( ijk \), then \( x \in h(I_k) \). We have at least \( p_{ij}|G_{ij}| \) such \( k \)'s and thus

\[
\sum_{k \in G_{ij}} 1_{x \in h(I_k)} \geq p_{ij}|G_{ij}|. \tag{19}
\]

We combine the above two inequalities as follows. Summing both sides of (18) over \( k \in G_{ij} \), exchanging the order of summation and applying (19) result in

\[
p_{ij}|G_{ij}|m_{ij}^{bad} = p_{ij}|G_{ij}| |U_{ij}^{bad}| \leq \sum_{k \in G_{ij}} N_{ij}^{bad}. \tag{20}
\]

Using the above bound we will bound from below

\[
\sum_{k \in G_{ij}} N_{ij}^{bad} + 3m_{ij}^{good} \geq \sum_{k \in G_{ij}} N_{ij}^{bad} \tag{21}
\]

Note that \( F(0) = 0 \),

\[
F(3m_{ij}^{bad}) = \frac{3m_{ij}^{bad}}{3m_{ij}^{bad} + 3m_{ij}^{good}} = s_{ij}^{good}
\]

and \( F(x) \) is concave. Applying the definition of \( F \), Jensen’s inequality, (20) and (21) yield

\[
\sum_{k \in G_{ij}} \frac{N_{ij}^{bad}}{N_{ij}^{bad} + 3m_{ij}^{good}} \geq \sum_{k \in G_{ij}} F(N_{ij}^{bad}) \geq \sum_{k \in G_{ij}} ((1 - \frac{N_{ij}^{bad}}{3m_{ij}^{bad}}) F(0) + \frac{N_{ij}^{bad}}{3m_{ij}^{bad}} F(3m_{ij}^{bad})) \geq \sum_{k \in G_{ij}} \frac{N_{ij}^{bad}}{3m_{ij}^{bad}} F(3m_{ij}^{bad}) \geq \frac{1}{3} p_{ij}|G_{ij}| F(3m_{ij}^{bad}) = \frac{1}{3} p_{ij}|G_{ij}| s_{ij}^{good}.
\]

The combination of this inequality with (16) concludes the proof of the lemma. \( \square \)
C.3. Conclusion of Theorem 1

We prove the main theorem by induction, using Lemmas 1 and 2. For $t = 0$, the definition of $s_{ij}^{(0)}$, Lemma 2 and the definition of $\lambda$ imply that for all $ij \in E$:

$$s_{ij}^{(0)} = \sum_{k \in N_{ij}} d_{ijk} |N_{ij}| = \sum_{k \in G_{ij}} d_{ijk} |N_{ij}| \geq \frac{p_v}{3} |G_{ij}| s_{ij} \geq \frac{p_v}{3} (1 - \lambda) s_{ij}.$$  

We further note by using again the above definitions and the fact that for all $ij \in E$, $0 \leq d_{ijk} \leq 1$ that for $ij \in E_g$,

$$s_{ij}^{(0)} = \sum_{k \in N_{ij}} d_{ijk} |N_{ij}| = \sum_{k \in B_{ij}} d_{ijk} \leq \sum_{k \in B_{ij}} 1 \leq \frac{1}{2\beta_0}.$$ 

Therefore, the theorem is proved when $t = 0$.

Next, we assume that the theorem holds for iterations $0, 1, \ldots, t$ and show that it also holds for iteration $t + 1$. Applying the definition of $s_{ij}^{(t+1)}$, the positivity of the terms in the sum, the induction assumption of $\frac{1}{2\beta_0} \geq \max_{ij \in E_b} s_{ij}^{(t)}$, Lemma 2 and the definition of $\lambda$, we obtain for any $ij \in E_b$

$$s_{ij}^{(t+1)} = \frac{\sum_{k \in N_{ij}} e^{-(s_{ik}^{(t)} + s_{jk}^{(t)})} d_{ijk} \sum_{k \in N_{ij}} e^{-(s_{ik}^{(t)} + s_{jk}^{(t)})}}{\sum_{k \in N_{ij}} e^{-\beta(s_{ik}^{(t)} + s_{jk}^{(t)})} d_{ijk}} \geq \frac{\sum_{k \in G_{ij}} e^{-(s_{ik}^{(t)} + s_{jk}^{(t)})} d_{ijk}}{\sum_{k \in N_{ij}} e^{-\beta(s_{ik}^{(t)} + s_{jk}^{(t)})}} \geq \frac{\sum_{k \in G_{ij}} e^{-1} d_{ijk}}{|N_{ij}|} \geq \frac{p_v |G_{ij}|}{3e} s_{ij}^{(t)} \geq (1 - \lambda)p_v s_{ij}^{(t)}.$$  

Note that $xe^{-ax} \leq \frac{1}{ae}$ for any $a > 0$ and $x \geq 0$. In particular, for $a = \beta_i (1 - \lambda)p_v/3e$ and $x = s_{ik} + s_{jk}$,

$$e^{-\beta_i (1 - \lambda)p_v/3e} \leq \frac{3}{\beta_i (1 - \lambda)p_v}.$$  

We note that the assumption $\lambda < 1 + \frac{6em\lambda}{(1 - \lambda)^2 p_v}$ is equivalent with $\frac{6em\lambda}{(1 - \lambda)^2 p_v} < 1$. Therefore by taking $\beta_{t+1} = r\beta_t$ with $1 < r < \frac{6em\lambda}{(1 - \lambda)^2 p_v}$, we guarantee that for any $ij \in E_g$, $s_{ij}^{(t+1)} \leq \frac{1}{2\beta_{t+1}}$, that is, $\max_{ij \in E_g} s_{ij}^{(t+1)} \leq \frac{1}{2\beta_{t+1}}$. This implication and (22) conclude the proof of the theorem.

D. Discussion of a Possible Theoretical Extension

Although our current analysis assumes no noise on the set of good edges, one can relax this assumption. Indeed, one can assume sufficiently small noise on good edges so that for all cycles $ijk$ and a sufficiently small positive constant $\delta$, $|d_{ijk} - d_{ijk}| < \delta$, where $d_{ijk}$ and $d'_{ijk}$ are respectively the cycle inconsistencies with and without noise on good edges. Using a basic perturbation analysis, similarly as in the proof of Theorem 1, with a carefully chosen set of the reweighting parameters $\beta_i$, one can prove approximate separation of good and bad edges. In particular, the maximum value of the estimated $s_{ij}$ on good edges is proportional to $\delta$. Removing the bad edges (with estimated $s_{ij}$ larger than this threshold), one can then approximately solve the PPS problem with a subsequent spectral solver. An approximate recovery theorem for the absolute partial permutations using the filtered edges can be established using spectral graph theory.