Optimizing Elimination Templates by Greedy Parameter Search

Supplementary Material

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Here we give additional details for the main paper
E. Martyushev, J. Vrablikova, T. Pajdla. Optimizing Elimination Templates by Greedy Parameter Search. CVPR 2022.
http://github.com/martyushev/EliminationTemplates

We present some basic notions from algebraic geometry, proofs, examples of constructing elimination templates, numerical details, and additional experiments demonstrating the numerical stability.

7. Monomial orderings

A *monomial ordering* $>$ on $[X]$ is a total ordering satisfying (i) $p > 0$ for all $p \in [X]$ and (ii) if $p > q$, then $p\,s > q\,s$ for all $p, q, s \in [X]$. We are particularly interested in the following two orderings:

1. *graded reverse lex ordering* (grevlex) compares monomials first by their total degree, and breaks ties by smallest degree in x_k, x_{k-1}, etc.

2. *weighted-degree ordering* w.r.t. a weight vector $w \in \mathbb{R}^k_+$ compares monomials first by their weighted degree (the dot product of w with the exponent vector $\begin{bmatrix}\alpha_1 & \dots & \alpha_k\end{bmatrix}^\top$), and breaks ties by reverse lexicographic order as in grevlex.

8. Proof of Theorem 1

The following theorem is not a new result of this work. It is "folclore" in algebraic geometry and has been used, *e.g.*, in [9,33,36], but we could not find it formulated clearly and concisely in the literature. Thus, we present it here for the sake of completeness.

Theorem 1. *The elimination template is well defined,* i.e., *for any s-tuple of polynomials $F = (f_1, \dots, f_s)$ such that ideal $\langle F\rangle$ is zero-dimensional, there exists a set of shifts $A \cdot F$ satisfying both conditions from Definition 1.*

Proof. Let us first show that there is a set of polynomials A such that all reducible monomials $\mathcal{R}$ appear in the support of $A \cdot F$. Let $G = \{g_1, \dots, g_l\}$ be a reduced Gröbner basis of the zero-dimensional ideal $J = \langle F\rangle$ and let $\mathtt{LM}(G)$ be the set of its leading monomials, with $\langle \mathtt{LM}(G)\rangle = \langle \mathtt{LM}(J)\rangle$ [13, p. 78 Definition 5]. Let $\mathcal{B}$ be the set of standard monomials representing a linear basis B of the quotient ring $\mathbb{K}[X]/J$ for G. Then, for every reducible monomial $r \in \mathcal{R} = \{a\,b : b \in \mathcal{B}\} \setminus \mathcal{B}$ for B and any $a \in \mathbb{K}[X]$, there holds true $r \in \langle \mathtt{LM}(G)\rangle$, because r is a multiple of some $b \in \mathcal{B}$ but r is not an element of $\mathcal{B}$. The set $\mathcal{B}$ of the standard monomials is finite [13, p. 251 Theorem 6]. Thus, $\mathcal{R}$ is finite too, and we can write $\mathcal{R} = \{r_1, \dots, r_n\}$. For every $k \in \mathbb{N}$, $k \le n$, we can write $r_k = m_{k1}g_1 + \dots + m_{kl}g_l + p_k$, where $m_{ki} \in \mathbb{K}[X]$, $\deg(r_k) \ge \deg(m_{ki}g_i)$ for every $i \in \{1, \dots, l\}$, and $p_k \in \mathbb{K}[X]$ is the polynomial satisfying $\overline{r_k}^G = p_k = \overline{p_k}^G$ [13, p. 64 Theorem 3]. Moreover, for every $i \in \{1, \dots, l\}$, there exists $q_{ij} \in \mathbb{K}[X]$ such that $g_i \in G$ can be written as $g_i = \sum_j q_{ij} f_j$. We can write $r_k = \sum_i \sum_j m_{ki}q_{ij}f_j + p_k$. Let $\{m_{ki}q_{ij}\} \in A_j$ for all $i \in \{1, \dots, l\}$, $j \in \{1, \dots, s\}$, $k \in \{1, \dots, n\}$. Then, the Macaulay matrix $M(A \cdot F)$ has a non-zero element in every column corresponding to a monomial from $\mathcal{R}$.

Let us next show that the eliminated matrix $\widetilde{M}(A \cdot F)$ contains a pivot in every column corresponding to a monomial from $\mathcal{R}$. Denote by $\widetilde{P}$ the set of polynomials $\widetilde{P} = \widetilde{M}(A \cdot F) \cdot [X]_{A \cdot F}$. For a set S denote $\operatorname{hull}(S)$ the linear space over $\mathbb{K}$ spanned by the elements of S, *i.e.* $\operatorname{hull}(S) = \{\sum_j c_j s_j \,:\, s_j \in S, c_j \in \mathbb{K}\}$.

Suppose $\{m_{ki}q_{ij}\} \in A_j$ for all $i \in \{1, \dots, l\}$, $j \in \{1, \dots, s\}$, $k \in \{1, \dots, n\}$. For every k we have $r_k = \sum_i \sum_j m_{ki}q_{ij}f_j + p_k$, hence $r_k - p_k \in \operatorname{hull}(A \cdot F) = \operatorname{hull}(\widetilde{P})$. The polynomial p_k is a linear combination of elements from $\mathcal{B}$, thus the polynomial $r_k - p_k$ contains only one reducible monomial r_k and no excessive monomials. Since $\widetilde{M}(A \cdot F)$ is in the reduced row echelon form, there is a row in $\widetilde{M}(A \cdot F)$ corresponding to the polynomial $r_k - p_k \in \operatorname{hull}(\widetilde{P})$ with zero coefficients at all excessive monomials and all reducible monomials except for r_k. Hence, there is a pivot in every column of $\widetilde{M}(A \cdot F)$ corresponding to a reducible monomial. It follows that $\widetilde{M}(A \cdot F)$ must have the form (2) meaning that $M(A \cdot F)$ is the elimination template. $\square$

9. Examples

In this section, we provide several examples of constructing elimination templates and using them to compute solutions of polynomial systems.

Example 1. In the first example we demonstrate th construction of the elimination template for a set of two polynomials in $\mathbb{Q}[x, y]$. We derive the action matrix and show how to extract the solution of the system from the action matrix.

Let $J = \langle F \rangle$, where $F = \{f_1, f_2\} = \{x^2 + y^2 - 1, x^2 + xy + y^2 - 1\} \subset \mathbb{Q}[x, y]$. The Gröbner basis of J w.r.t. grevlex with $x > y$ is $G = \{xy, x^2 + y^2 - 1, y^3 - y\}$. The standard basis of $\mathbb{Q}[x, y]/J$ is $\mathcal{B} = \{y^2, y, x, 1\}$. If x is the action variable, then the action matrix is

$$T_x = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Let us construct vector V define in Eq. (5):

$$V = xv(\mathcal{B}) - T_x v(\mathcal{B}) = \begin{bmatrix} xy^2 \\ x^2 + y^2 - 1 \\ xy \\ 0 \end{bmatrix}.$$

Since $V \subset J$, see Sec. 3, there exists matrix H_0 such that $V = H_0 v(F)$. By tracing the computation of the Gröbner basis G we found

$$H_0 = \begin{bmatrix} -y & y \\ 1 & 0 \\ -1 & 1 \\ 0 & 0 \end{bmatrix}.$$

It follows that it is enough to take the set of shifts $A \cdot F = \{yf_2, yf_1, f_2, f_1\}$. We divide the set of monomials $[X]_{A \cdot F}$ into the subsets $\overline{\mathcal{B}} = \mathcal{B} \cap [X]_{A \cdot F} = \{y^2, y, 1\}$, $\mathcal{R} = \{xy^2, xy, x^2\}$ and $\mathcal{E} = \{x^2y, y^3\}$. This yields the elimination template

$$M(A \cdot F) = \begin{bmatrix} M_{\mathcal{E}} & M_{\mathcal{R}} & M_{\overline{\mathcal{B}}} \end{bmatrix}$$

$$= \begin{array}{c} \\ yf_2 \\ yf_1 \\ f_2 \\ f_1 \end{array} \begin{array}{c} \begin{array}{cccccccc} x^2y & y^3 & xy^2 & xy & x^2 & y^2 & y & 1 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 \end{bmatrix} \end{array}.$$

The reduced row echelon form of $M(A \cdot F)$ has the form

$$\widetilde{M}(A \cdot F) = \left[\begin{array}{c|c|c} \widetilde{M}_{\mathcal{E}} & 0 & * \\ \hline 0 & I & \widetilde{M}_{\mathcal{B}} \end{array}\right]$$

$$= \left[\begin{array}{cc|ccc|cccc} 1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 \end{array}\right].$$

Then the action matrix T_x is read off as $\begin{bmatrix} -\widetilde{M}_{\mathcal{B}} \\ P \end{bmatrix}$, where $P = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$ satisfies $x = Pv(\mathcal{B})$.

The eigenvalues of T_x, *i.e.* $\{0, \pm 1\}$, where the geometric multiplicity of the eigenvalue $\lambda = 0$ equals 2, *i.e.* the eigen-space associated with $\lambda = 0$ is 2-dimensional. Hence, the x-components of the roots are $0, 0, -1, 1$. The y-components can be derived from the eigenvectors resulting in the following roots: $\{(0, -1), (0, 1), (-1, 0), (1, 0)\}$.

Example 2. In this example we demonstrate the usage of non-standard bases of the quotient space. Having an action matrix related to the standard basis $\widehat{B}$, we can construct the action matrix related to a non-standard basis $\mathcal{B}$ by a change-of-basis matrix. Another option is to construct a set of shifts and divide its monomials so that the basis monomials are the ones from the non-standard basis $\mathcal{B}$. The action matrix derived from the resulting elimination template is the action matrix related to $\mathcal{B}$.

Let $J = \langle F \rangle$, where $F = \{f_1, f_2\} = \{x^3 + y^2 - 1, x - y - 1\} \subset \mathbb{Q}[x, y]$. The Gröbner basis of J w.r.t. grevlex with $x > y$ is $G = \{x - y - 1, y^3 + 4y^2 + 3y\}$. The standard basis of $\mathbb{Q}[x, y]/J$ is $\widehat{\mathcal{B}} = \{1, y, y^2\}$. If x is the action variable, then the related action matrix is

$$\widehat{T}_x = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -3 & -3 \end{bmatrix}.$$

Now let us consider the non-standard basis $\mathcal{B} = \{x^2, y, 1\}$. The respective change-of-basis matrix S, *i.e.* a matrix satisfying $\overline{v(\mathcal{B})}^G = S\, v(\widehat{\mathcal{B}})$, has the form

$$S = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Then the matrix of the action operator in the basis $\mathcal{B}$ is

$$T_x = S\,\widehat{T}_x S^{-1} = \begin{bmatrix} -1 & 2 & 2 \\ 1 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix}.$$

The vector V define in Eq. (5) has the form

$$V = xv(\mathcal{B}) - T_x v(\mathcal{B}) = \begin{bmatrix} x^3 + x^2 - 2y - 2 \\ -x^2 + xy + y + 1 \\ x - y - 1 \end{bmatrix}.$$

Since $V \subset J$, see Sec. 3, there exists matrix H_0 such that $V = H_0 v(F)$. By tracing the computation of the Gröbner basis G we found

$$H_0 = \begin{bmatrix} 1 & x+y+1 \\ 0 & -x-1 \\ 0 & 1 \end{bmatrix}.$$

It follows that it is enough to take the set of shifts $A \cdot F = \{xf_2, yf_2, f_2, f_1\}$. We divide the set of monomials $[X]_{A \cdot F}$ into the subsets $\overline{\mathcal{B}} = \mathcal{B} = \{x^2, y, 1\}$, $\mathcal{R} = \{x^3, xy, x\}$ and $\mathcal{E} = \{y^2\}$. This yields the elimination template

$$M(A \cdot F) = \begin{bmatrix} M_{\mathcal{E}} & M_{\mathcal{R}} & M_{\mathcal{B}} \end{bmatrix}$$

$$= \begin{array}{c} \\ xf_2 \\ yf_2 \\ f_2 \\ f_1 \end{array} \begin{array}{c} \begin{array}{ccccccc} y^2 & x^3 & xy & x & x^2 & y & 1 \end{array} \\ \begin{bmatrix} 0 & 0 & -1 & -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \end{array}.$$

The reduced row echelon form of $M(A \cdot F)$ is as follows

$$\widetilde{M}(A \cdot F) = \left[\begin{array}{c|c|c} \widetilde{M_{\mathcal{E}}} & 0 & * \\ \hline 0 & I & \widetilde{M_{\mathcal{B}}} \end{array}\right]$$

$$= \left[\begin{array}{c|ccc|ccc} 1 & 0 & 0 & 0 & -1 & 2 & 1 \\ \hline 0 & 1 & 0 & 0 & 1 & -2 & -2 \\ 0 & 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & -1 \end{array}\right].$$

Then the action matrix T_x is exactly $-\widetilde{M_{\mathcal{B}}}$. The eigenvalues of T_x, *i.e.* $\{-2, 0, 1\}$, give us the x-components of the roots. The y-components can be derived from the eigenvectors, which are

$$\begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Hence, we get the roots $\{(-2, -3), (0, -1), (1, 0)\}$.

Example 3. In this example we consider a set of polynomials with the same structure as in Example 2 but with different coefficients. We use the same elimination template as in the previous example and just plug in the corresponding coefficient. Then we derive the action matrix. We again use the non-standard basis $\mathcal{B} = \{x^2, y, 1\}$.

Let $F = \{f_1, f_2\} = \{x^3 - \sqrt{2}y^2 - 3, x - \sqrt{3}y + 4\} \subset \mathbb{R}[x, y]$.

We can use the same set of shifts $A \cdot F = \{xf_2, yf_2, f_2, f_1\}$ to construct the elimination template

$$M(A \cdot F) = \begin{array}{c} \begin{array}{ccccccc} y^2 & x^3 & xy & x & x^2 & y & 1 \end{array} \\ \begin{bmatrix} 0 & 0 & -\sqrt{3} & 4 & 1 & 0 & 0 \\ -\sqrt{3} & 0 & 1 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 & 0 & -\sqrt{3} & -1 \\ \sqrt{2} & 1 & 0 & 0 & 0 & 0 & -3 \end{bmatrix} \end{array}.$$

Finding the reduced row echelon form of $M(A \cdot F)$ results in the following action matrix:

$$T_x = \begin{bmatrix} \frac{\sqrt{2}}{3} & \frac{8\sqrt{6}}{3} & -\frac{16\sqrt{2}}{3} - 3 \\ -\frac{\sqrt{3}}{3} & -4 & \frac{16\sqrt{3}}{3} \\ 0 & -\sqrt{3} & 4 \end{bmatrix}.$$

Finally, from the eigenvectors of T_x we derive the roots of $F = 0$: $\{(2.955, 4.015), (-1.242 + 1.423i, 1.592 + 0.822i), (-1.242 - 1.423i, 1.592 - 0.822i)\}$.

Example 4. In this example we consider a non-radical ideal. We use the standard basis $\mathcal{B}$. It is enough to use a set of shifts such that not all the monomials from $\mathcal{B}$ are included. One can then add zero columns to the elimination template. To get the full action matrix, we need to add the permutation matrix from Eq.(4) to the matrix obtained from the elimination template. Then we derive roots of the system from the action matrix and its eigenvectors.

Let $J = \langle F \rangle$, where $F = \{f_1, f_2\} = \{x^2 - y^2, y^2 - x\} \subset \mathbb{Q}[x, y]$. The reduced Gröbner basis w.r.t. grevlex with $x > y$ is $G = \{y^2 - x, x^2 - x\}$ and the standard basis of $\mathbb{Q}[x, y]/J$ is $\mathcal{B} = \{xy, x, y, 1\}$.

If y is the action variable, then the action matrix is

$$T_y = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Let us construct vector V define in Eq. (5):

$$V = yv(\mathcal{B}) - T_y v(\mathcal{B}) = \begin{bmatrix} xy^2 - x \\ y^2 - x \\ 0 \\ 0 \end{bmatrix}.$$

Since $V \subset J$, see Sec. 3, there exists matrix H_0 such that $V = H_0 v(F)$. By tracing the computation of the Gröbner basis G we found

$$H_0 = \begin{bmatrix} 1 & x+1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

It follows that it is enough to take the set of shifts $A \cdot F = \{xf_2, f_2, f_1\}$. We divide the set of monomials $[X]_{A \cdot F}$ into the subsets $\overline{\mathcal{B}} = \mathcal{B} \cap [X]_{A \cdot F} = \{x\}$, $\mathcal{R} = \{xy^2, y^2\}$ and $\mathcal{E} = \{x^2\}$. This yields the elimination template

$$M(A \cdot F) = \begin{bmatrix} M_{\mathcal{E}} & M_{\mathcal{R}} & M_{\overline{\mathcal{B}}} \end{bmatrix}$$

$$= \begin{array}{c} \\ xf_2 \\ f_2 \\ f_1 \end{array} \begin{array}{c} \begin{array}{cccc} x^2 & xy^2 & y^2 & x \end{array} \\ \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \end{array}.$$

The reduced row echelon form of $M(A \cdot F)$ is the matrix

$$\widetilde{M}(A \cdot F) = \left[\begin{array}{c|c|c} \widetilde{M_{\mathcal{E}}} & 0 & * \\ \hline 0 & I & \widetilde{M_{\overline{\mathcal{B}}}} \end{array}\right] = \left[\begin{array}{c|cc|c} 1 & 0 & 0 & -1 \\ \hline 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right].$$

We can add zero columns corresponding to the basic monomials from $\mathcal{B} \setminus \overline{\mathcal{B}} = \{xy, y, 1\}$ to the matrix $\widetilde{M}(A \cdot F)$. This yields

$$\left[\begin{array}{c|c|c} \widetilde{M_{\mathcal{E}}} & 0 & * \\ \hline 0 & I & \widetilde{M_{\mathcal{B}}} \end{array}\right] = \left[\begin{array}{c|cc|cccc} 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \end{array}\right].$$

Then the action matrix T_y is read off as $\begin{bmatrix} -\widetilde{M_{\mathcal{B}}} \\ P \end{bmatrix}$, where $P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ satisfies $\begin{bmatrix} xy \\ x \end{bmatrix} = Pv(\mathcal{B})$.

The eigenvalues of T_y are $\{0, \pm 1\}$. The geometric multiplicity of the eigenvalue $\lambda = 0$ equals 1, whereas its algebraic multiplicity is 2 implying that T_y is non-diagonalizable. The y-components of the roots are $0, -1, 1$. The x-components can be derived from the eigenvectors resulting in the following roots: $\{(0, 0), (1, -1), (1, 1)\}$, where the root $(0, 0)$ is of multiplicity 2.

10. Proof of Proposition 1

The following proposition validates the Schur complement reduction described in Subsec. 4.2.

Proposition 1. *Let M be an elimination template represented in the following block form*

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{9}$$

where A is a square invertible matrix and its columns correspond to some excessive monomials. Then the Schur complement of A, i.e. *matrix $M/A = D - CA^{-1}B$, is an elimination template too.*

Proof. Recall that an elimination template is partitioned as $M = \begin{bmatrix} M_{\mathcal{E}} & M_{\mathcal{R}} & M_{\overline{\mathcal{B}}} \end{bmatrix}$, where $\mathcal{E}$, $\mathcal{R}$ and $\overline{\mathcal{B}}$ are the sets of excessive, reducible and basic monomials respectively. By the definition of template, the reduced row echelon form of M must have the form

$$\widetilde{M} = \begin{bmatrix} \widetilde{M_{\mathcal{E}}} & 0 & * \\ 0 & I & \widetilde{M_{\overline{\mathcal{B}}}} \\ 0 & 0 & 0 \end{bmatrix},$$

where $\begin{bmatrix} \widetilde{M_{\mathcal{E}}} \\ 0 \end{bmatrix}$ is the reduced row echelon form of matrix $M_{\mathcal{E}}$. On the other hand, according to the block form (8), we have $\widetilde{M} = \begin{bmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{bmatrix}$, where $\widetilde{A}$ is a square invertible submatrix of $\widetilde{M_{\mathcal{E}}}$. Thus, $\widetilde{A} = I$ and $\widetilde{C} = 0$. Let $\mathcal{E}_A$ be the set of excessive monomials corresponding to the columns of matrix A. Then we have $\widetilde{M_{\mathcal{E}}} = \begin{bmatrix} I & * \\ 0 & \widetilde{M_{\mathcal{E} \setminus \mathcal{E}_A}} \end{bmatrix}$. It follows that the reduced row echelon form of M/A is

$$\widetilde{M/A} = \widetilde{D} = \begin{bmatrix} \widetilde{M_{\mathcal{E} \setminus \mathcal{E}_A}} & 0 & * \\ 0 & I & \widetilde{M_{\overline{\mathcal{B}}}} \\ 0 & 0 & 0 \end{bmatrix}$$

and hence M/A is a template. □

11. Proof of Proposition 2

Here we prove a simple necessary condition for a template to be minimal.

Proposition 2. *Let M'' be an elimination template of size $s'' \times n''$ whose columns arranged w.r.t. the partition $\mathcal{E} \cup \mathcal{R} \cup \overline{\mathcal{B}}$. Then there exists a template M of size $s \times n$ so that $s \leq s''$, $n \leq n''$ and $n - s = \#\overline{\mathcal{B}}$.*

Proof. Let M'' be an elimination template of size $s'' \times n''$. First, we take a maximal subset of independent rows of M'' to get template M' of size $s \times n'$ with $s \leq s'$.

Let M' be partitioned as follows

$$M' = \begin{bmatrix} M'_{\mathcal{E}} & M'_{\mathcal{R}} & M'_{\overline{\mathcal{B}}} \end{bmatrix}.$$

As M' is an elimination template, its reduced row echelon form must be as follows

$$\widetilde{M'} = \begin{bmatrix} \widetilde{M'_{\mathcal{E}}} & 0 & * \\ 0 & I & * \end{bmatrix},$$

where I is the identity matrix of order $\#\mathcal{R}$. Removing the columns from $M'_{\mathcal{E}}$ that do not have pivots in $\widetilde{M'_{\mathcal{E}}}$ results in matrix M of size $m \times n$, where $n - s = \#\overline{\mathcal{B}}$. Clearly, matrix M is also an elimination template as its reduced row echelon form is given by

$$\widetilde{M} = \begin{bmatrix} I & 0 & * \\ 0 & I & * \end{bmatrix}.$$

Since the columns from $M'_{\mathcal{E}}$ that do not have pivots in $\widetilde{M'_{\mathcal{E}}}$ do not change the reduced row echelon form of the rest of the matrix, it follows that the right most $\#\overline{\mathcal{B}}$ columns in $\widetilde{M}$ are exactly the same as in $\widetilde{M'}$. □

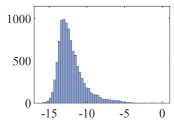
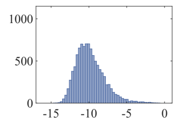
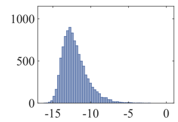
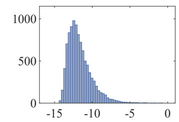
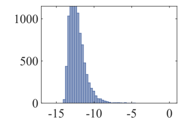
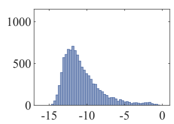
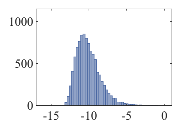
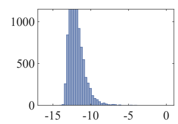
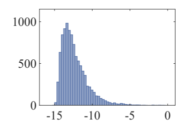
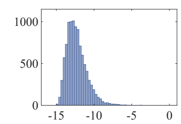
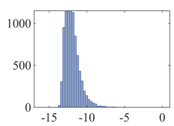
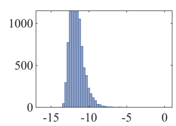
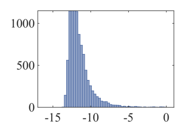
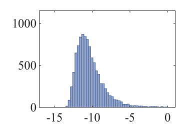
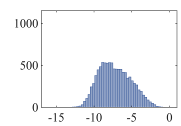
Problem #	3 (nstd)	9 (std)	10 (std)	15 (std)	16 (std)
Error distrib.					
Template size	11×26	76×100	55×74	57×73	65×85
$\#\mathcal{P}$	15	39	30	54	35
Med. error	3.30e–13	8.08e–11	4.19e–13	1.04e–12	3.41e–13
Ave. time (ms)	0.4	1.2	3.1	1.0	0.9
Problem #	17 (nstd)	20 (std)	21 (std)	22 (nstd)	23 (std)
Error distrib.					
Template size	139×155	139×163	99×119	31×47	18×28
$\#\mathcal{P}$	40	68	48	20	10
Med. error	5.52e–12	4.67e–11	7.95e–13	1.71e–13	3.06e–13
Ave. time (ms)	2.5	3.6	2.0	0.5	0.9
Problem #	28 (std)	29 (std)	31 (std)	32 (std)	33 (std)
Error distrib.					
Template size	120×140	134×162	217×248	126×162	209×277
$\#\mathcal{P}$	80	76	85	67	117
Med. error	6.11e–13	1.63e–12	1.31e–12	2.09e–11	3.77e–08
Ave. time (ms)	2.6	3.7	4.2	2.3	8.5

Table 3. Tests of numerical accuracy and runtime for some our minimal solvers from Tab. 1 and Tab. 2 of the main paper. Each histogram shows $\log_{10}$ of numerical error distribution on 10^4 trials. It is also shown the template size of each problem and the number of permissible monomials ($\#\mathcal{P}$) used for the column pivoting strategy from [11], see Sec. 12. For problems #3 and #23 the column pivoting was not applied as for those problems $\mathcal{P}$ is exactly the set of basic monomials. The runtime includes both constructing the coefficient matrix of the initial system and finding its solutions.

12. Notes on column pivoting

In Subsect. 2.4 of the main paper, we read off the action matrix from the reduced row echelon form of the elimination template. For large elimination templates, this method may be impractical for the following two reasons. First, it is slow since constructing the full reduced row echelon form is time-consuming. Second, this approach is often numerically unstable. This means that due to round-off and truncation errors the output roots, when back substituted into the initial polynomials, result in values that are far from being zeros.

Here we recall an alternative approach from [9–11] for the action matrix construction. This approach is faster than the one based on the reduced row echelon form and moreover it admits a numerically more accurate generalization.

Let M be an elimination template partitioned as $M = \begin{bmatrix} M_\mathcal{E} & M_\mathcal{R} & M_\mathcal{B} \end{bmatrix}$, where $\mathcal{E}$, $\mathcal{R}$ and $\mathcal{B}$ are the sets of excessive, reducible and basic monomials respectively. Let the set of basic monomials $\mathcal{B}$ be partitioned as $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$, where $\mathcal{B}_2 = \{a\, b \,:\, b \in \mathcal{B}\} \cap \mathcal{B}$ and $\mathcal{B}_1 = \mathcal{B} \setminus \mathcal{B}_2$.

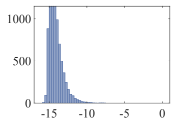
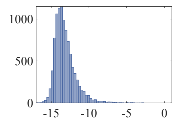
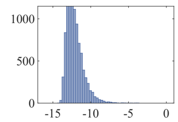
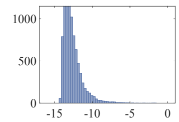
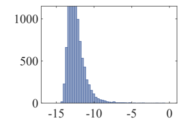
The LU decomposition of matrix $M_\mathcal{E}$ can be generally written as $M_\mathcal{E} = \begin{bmatrix} \Pi_\mathcal{E} L_\mathcal{E} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} U_\mathcal{E} \\ 0 \end{bmatrix}$, where $U_\mathcal{E}$ and $L_\mathcal{E}$ are upper- and lower-triangular matrices respectively, $\Pi_\mathcal{E}$ is a row permutation matrix. Then we define

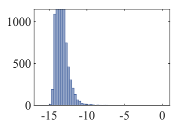
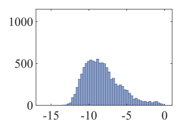
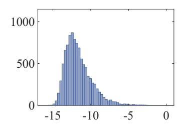
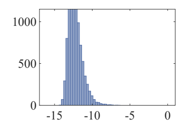
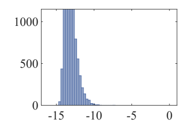
$$M' = \begin{bmatrix} (\Pi_\mathcal{E} L_\mathcal{E})^{-1} & 0 \\ 0 & I \end{bmatrix} M = \begin{bmatrix} U_\mathcal{E} & * & * \\ 0 & M'_\mathcal{R} & M'_\mathcal{B} \end{bmatrix},$$

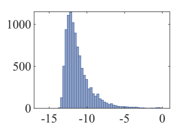
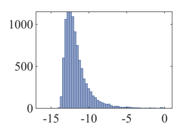
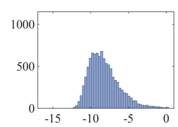
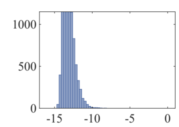
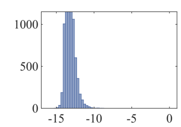
where $M'_\mathcal{R}$ is square and invertible. It follows that $M'_\mathcal{R} v(\mathcal{R}) = -M'_\mathcal{B} v(\mathcal{B})$ and hence the action matrix reads

$$T_a = \begin{bmatrix} -(M'_\mathcal{R})^{-1} M'_\mathcal{B} \\ P \end{bmatrix},$$

where P is a binary matrix, *i.e.* a matrix consisting of 0 and 1, such that $v(\mathcal{B}_2) = P v(\mathcal{B})$.

Problem #	1 (nstd)	2 (std)	4 (nstd)	5 (nstd)	6 (std)
Error distrib.					
Template size	7×15	11×20	14×40	18×36	52×68
$\#\mathcal{P}$	8	12	30	30	38
Med. error	3.67e–15	3.52e–14	4.64e–13	9.10e–14	2.46e–13
Ave. time (ms)	0.2	0.5	1.5	0.5	0.8

Problem #	7 (std)	8 (nstd)	11 (nstd)	12 (std)	13 (std)
Error distrib.					
Template size	28×40	39×95	22×41	51×70	47×55
$\#\mathcal{P}$	20	75	26	35	21
Med. error	2.87e–14	2.63e–09	1.11e–12	4.11e–13	5.99e–14
Ave. time (ms)	0.8	3.6	0.7	7.2	0.7

Problem #	14 (std)	18 (std)	19 (std)	24 (std)	25 (nstd)
Error distrib.					
Template size	87×114	87×114	118×158	47×55	16×36
$\#\mathcal{P}$	80	80	40	21	25
Med. error	2.24e–12	8.00e–13	3.01e–09	6.10e–14	4.78e–14
Ave. time (ms)	2.4	2.4	2.9	0.7	0.7

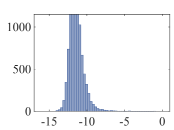
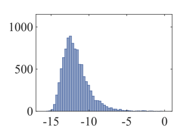
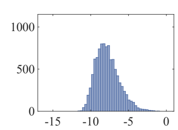
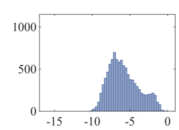
Problem #	26 (std)	27 (nstd)	30 (nstd)	34 (std)
Error distrib.				
Template size	37×81	40×46	385×433	144×284
$\#\mathcal{P}$	50	7	82	165
Med. error	2.96e–12	9.27e–13	1.09e–08	7.47e–07
Ave. time (ms)	3.1	0.6	19.8	97

Table 4. A continuation of Tab. 3 for the remaining 19 minimal solvers. Each histogram shows $\log_{10}$ of numerical error distribution on 10^4 trials. It is also shown the template size of each problem and the number of permissible monomials ($\#\mathcal{P}$) used for the column pivoting strategy from [11], see Sec. 12. For problem #1 the column pivoting was not applied as for this problem $\mathcal{P}$ is exactly the set of basic monomials. For problem #19 the column pivoting was not applied as it led to worse results. The runtime includes both constructing the coefficient matrix of the initial system and finding its solutions

As it was noted in [11], matrix $M'_{\mathcal{R}}$ is often ill conditioned and this is the main cause of numerical instabilities in solving polynomial systems. Also in [11] the authors proposed the following heuristic method of improving stability. First, the set of basic monomials $\mathcal{B}$ is replaced with the set of *permissible monomials* $\mathcal{P} = \{p \in \mathcal{X} \,:\, ap \in \mathcal{X}\}$. The partitions for $\mathcal{X}$ and M now become

$$\mathcal{X} = \mathcal{E} \cup \mathcal{R} \cup \mathcal{P} \quad \text{and} \quad M = \begin{bmatrix} M_{\mathcal{E}} & M_{\mathcal{R}} & M_{\mathcal{P}} \end{bmatrix}$$

respectively. Here $\mathcal{R} = \{ap \,:\, p \in \mathcal{P}\} \setminus \mathcal{P}$ and $\mathcal{E}$ consists of monomials which are neither in $\mathcal{R}$ nor in $\mathcal{P}$. Then the LU decomposition is applied to matrix $\begin{bmatrix} M_{\mathcal{E}} & M_{\mathcal{R}} \end{bmatrix}$:

$$M' = \begin{bmatrix} U_{\mathcal{E}} & * & * \\ 0 & U_{\mathcal{R}} & M'_{\mathcal{P}} \\ 0 & 0 & N'_{\mathcal{P}} \end{bmatrix},$$

where $U_{\mathcal{E}}$, $U_{\mathcal{R}}$ are upper-triangular matrices and $U_{\mathcal{R}}$ is square and invertible. This is the starting point for the column pivoting strategy. Let the (pivoted) QR decomposition of matrix $N'_{\mathcal{P}}$ be

$$N'_{\mathcal{P}}\Pi = Q \begin{bmatrix} U_{\mathcal{P}\setminus\mathcal{B}} & N''_{\mathcal{B}} \end{bmatrix},$$

where Π is the column permutation matrix, Q is orthogonal matrix, $U_{\mathcal{P}\setminus\mathcal{B}}$ is upper-triangular, square and invertible. Pivoting defined by the matrix Π helps to reduce the condition number of $U_{\mathcal{P}\setminus\mathcal{B}}$ and hence makes the further computation of its inverse matrix numerically more accurate. Let us define

$$M'' = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & Q^\top \end{bmatrix} M' \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \Pi \end{bmatrix} = \begin{bmatrix} U_{\mathcal{E}} & * & * & * \\ 0 & U_{\mathcal{R}} & M''_{\mathcal{P}\setminus\mathcal{B}} & M''_{\mathcal{B}} \\ 0 & 0 & U_{\mathcal{P}\setminus\mathcal{B}} & N''_{\mathcal{B}} \end{bmatrix},$$

where $M'_{\mathcal{P}}\Pi = \begin{bmatrix} M''_{\mathcal{P}\setminus\mathcal{B}} & M''_{\mathcal{B}} \end{bmatrix}$. If $\Pi^\top v(\mathcal{P}) = \begin{bmatrix} v(\mathcal{P}\setminus\mathcal{B}) \\ v(\mathcal{B}) \end{bmatrix}$, then it follows that

$$\begin{bmatrix} v(\mathcal{R}) \\ v(\mathcal{P}\setminus\mathcal{B}) \end{bmatrix} = -\begin{bmatrix} U_{\mathcal{R}} & M''_{\mathcal{P}\setminus\mathcal{B}} \\ 0 & U_{\mathcal{P}\setminus\mathcal{B}} \end{bmatrix}^{-1} \begin{bmatrix} M''_{\mathcal{B}} \\ N''_{\mathcal{B}} \end{bmatrix} v(\mathcal{B})$$
$$= -\begin{bmatrix} U_{\mathcal{R}}^{-1} M''_{\mathcal{B}} - (U_{\mathcal{R}}^{-1} M''_{\mathcal{P}\setminus\mathcal{B}})\,(U_{\mathcal{P}\setminus\mathcal{B}}^{-1} N''_{\mathcal{B}}) \\ U_{\mathcal{P}\setminus\mathcal{B}}^{-1} N''_{\mathcal{B}} \end{bmatrix} v(\mathcal{B}). \quad (10)$$

We note that the set of basic monomials $\mathcal{B}$ depends on the permutation Π, which in turn depends on the entries of template M. Therefore, in general $\mathcal{B}$ can vary depending on problem instance. Since any multiple $a\, b$ for $b \in \mathcal{B}$ belongs to $\mathcal{R} \cup \mathcal{P}$, it follows that the action matrix for the new basis $\mathcal{B}$ can be read off from (10).

The column pivoting is a universal tool that may significantly enhance numerical accuracy with a certain computational overhead. It can be always applied provided that $\#\mathcal{P} > \#\mathcal{B}$.

13. Experimental results

In this section we test the speed and numerical accuracy of our Matlab solvers for all the minimal problems from Tab. 1 and Tab. 2 of the main paper. The experiments were performed on a system with Intel Core i5 CPU @ 2.3 GHz and 8 GB of RAM. The results are presented in Tab. 3 and Tab. 4.

In case the templates for standard and non-standard bases had the same size, we chose the one with smaller numerical error. The column pivoting strategy (see Sec. 12) was applied for all solvers with $\#\mathcal{P} > \#\mathcal{B}$. However, for some problems, the set of permissible monomials was manually reduced to improve the speed/accuracy trade-off.

Finally, the numerical error is defined as follows. Let the polynomial system $F = 0$ be written in the form $M(F)U = 0$, where $M(F)$ and $U = v([X]_F)$ are the Macaulay matrix and monomial vector respectively. The matrix $M(F)$ is normalized so that each its row has unit length. Let $\dim \mathbb{K}[X]/\langle F\rangle = d$, i number all solutions to $F = 0$ including complex ones and U_i be the monomial vector U evaluated at the ith solution. We measure the numerical error of our solvers by the value

$$\left\| M(F) \begin{bmatrix} \frac{U_1}{\|U_1\|_2} & \dots & \frac{U_d}{\|U_d\|_2} \end{bmatrix} \right\|_2,$$

where $\|\cdot\|_2$ is the Frobenius norm.