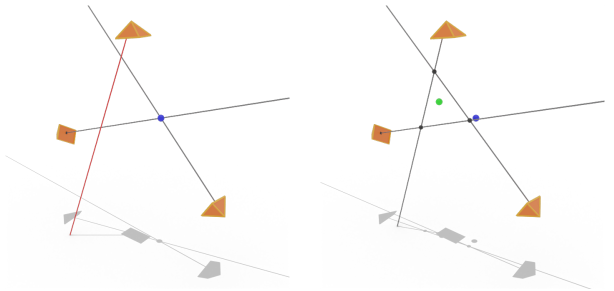

(a) Observations $\tilde{x}_i$ (b) Invalid estimated reprojections x_i

Figure 8. Blue point: ground truth. Green point: invalid point estimate obtained by rounding the solution to Eq. (RT). Black points: intersection of viewing rays.

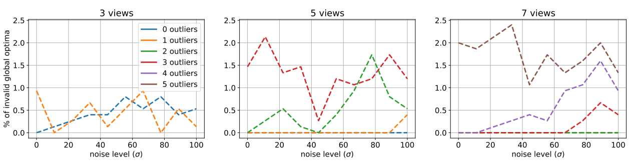

Figure 9. Percentage of invalid global optima found by Eq. (RT) in the simulated experiments. We found none for 25 and 30 views.

A. Co-planar solution to epipolar relaxation

When all camera centers are co-planar, any configuration of observations x_i of viewing rays which lie in the camera plane will satisfy the epipolar constraints, despite not necessarily corresponding to the repeojection of a single 3D point. This means solutions to Eq. (T) and Eq. (RT) might not correspond to valid 3D points, despite the relaxations being tight. Following [1], we regard any solution with such invalid global optima as non-tight in all our experiments.

An example of a configuration where an invalid global optima occurs is shown in figure Fig. 8. In the example the true 3D point lies close to the camera plane and there are two inlier observations with noise $\sigma = 0$ and one outlier whose viewing ray also lies close to the camera plane. In this case adjusting the viewing rays such that they all lie in the camera plane produces a lower cost solution than correctly labelling the third view as an outlier. In contrast, Eq. (RTF) produces the correct solution, since the 3D point is explicitly parametrized. In Fig. 9 we show the number invalid global optima found in the simulated experiments.

See [1] and [11] for a more detailed discussion on when a solution of Eq. (6) is guaranteed to generate a unique solution to Eq. (5).

B. Noise-free and outlier-free case

In this section we will show that both our relaxations are tight in the noise-free and outlier-free case, we will also prove some of the criteria needed for local stability.

We can verify whether a potential solution to Eq. (3) is globally optimal by computing the corresponding *Lagrange multipliers*, as summarized in the following fact:

Fact 1. *If $\hat{z} \in \mathbb{R}^d$ satisfies the constraints of Eq. (3) (primal feasibility) and there are Lagrange multipliers $\hat{\lambda} \in \mathbb{R}$, $\hat{\xi} \in \mathbb{R}^k$ and a corresponding multiplier matrix $S(\hat{\lambda}, \hat{\xi}) = M + \sum_{i=1}^{k} \hat{\xi}_i A_i - \hat{\lambda} E$ satisfying:*

i) Dual feasibility: $S(\hat{\lambda}, \hat{\xi}) \succcurlyeq 0$

ii) Complementarity: $S(\hat{\lambda}, \hat{\xi})\hat{z} = 0$

then the relaxation Eq. (4) is tight and $\hat{z}$ is optimal for Eq. (3).

It might seem surprising that semidefinite relaxations of geometry problems in computer vision are empirically tight to such a large extent, but [5] provides some theoretical justification for this observation. They show for instance that under a smoothness condition Eq. (4) will be a tight relaxation of Eq. (3) for problems that are close in parameter-space to solutions where the multiplier matrix has corank 1[2]. We will later show the corank 1 condition for the noise-free and outlier-free case of the triangulation problem, although we have not investigated the smoothness condition. We restate the main result in loose terms here:

Fact 2. *If we, in addition to the conditions in Fact 1, have that $S(\lambda, \mu)$ is corank 1 and ACQ (which is a smoothness condition, see [5] Definition 3.1) holds, then the relaxation Eq. (4) is locally stable, meaning that it will remain tight also for perturbed objective functions $M + \varepsilon \tilde{M}$ for small enough ε.*

The practical usefulness of Fact 2 comes from the consideration that it's often possible to show that the relaxation is tight and the stability conditions hold for noise-free measurements. This means that there is some surrounding region of noisy measurements for which the relaxation is tight as well.

B.1. Epipolar method

In the noise-free and outlier-free case we can show that the relaxation is tight with a corank 1 multiplier matrix:

Theorem 1. *The relaxation Eq. (RT) is tight with a corank 1 multiplier matrix for noise-free and outlier-free measurements $\tilde{x}_i$, $i = 1, \dots, n$.*

Proof. Partiton the lagrange multipliers as $\xi = (\varphi; \mu; \eta)$, where $\varphi_{ij} \in \mathbb{R}$ $\mu_i \in \mathbb{R}^2$ and $\eta \in \mathbb{R}$ corresponds to the constraints $(y_i; \theta_i)^T F_{ij} (y_j; \theta_j) = 0$, $\theta_i y_i = y_i$ and $\theta_i^2 = \theta_i$ respectively. Then we have:

$$S(\lambda, \varphi, \mu, \eta) = F(\varphi) + \begin{pmatrix} I & -B(\tilde{x}_i - \mu_i) & -\mu \\ * & \mathrm{diag}(\|\tilde{x}_i\|^2 + 2\eta_i) & -\frac{1}{2}c - \eta \\ * & * & \sum_{i=1}^{n} c_i - \lambda \end{pmatrix}. \tag{14}$$

[2] corank(A) = n - rank(A) for an $n \times n$ matrix A.

Where $F(\varphi) = \sum_{ij} \varphi_{ij} \bar{F}_{ij}$. Now let $\hat{\lambda} = \hat{\varphi}_{ij} = \hat{\mu}_i = 0$ and $\hat{\eta}_i = \frac{1}{2} c_i$ to get:

$$\hat{S} = S(\hat{\lambda}, \hat{\varphi}, \hat{\mu}, \hat{\eta}) = S(0, 0, 0, \frac{1}{2} c) = \begin{pmatrix} I & -B(\tilde{x}_i) & 0 \\ * & \text{diag}(\|\tilde{x}_i\|^2 + c_i) & -c \\ * & * & \sum_{i=1}^n c_i \end{pmatrix}. \quad (15)$$

This way, with $\hat{z} = (\tilde{x}; \mathbf{1}_n; 1)$ we have $\hat{S}\hat{z} = 0$. And furthermore, for arbitrary x, θ, α:

$$(x; \theta; \alpha)^T \hat{S}(x; \theta; \alpha) = = \sum_{i=0}^{n} \left(\|x_i\|^2 - 2\theta_i \tilde{x}_i + \theta_i^2 (\|\tilde{x}_i\|^2 + c_i) - 2c_i \theta_i \alpha + c_i \alpha^2 \right) = \sum_{i=0}^{n} \left(\|x_i - \theta_i \tilde{x}_i\|^2 + c_i (\alpha - \theta_i)^2 \right) \geq 0$$

so $\hat{S}$ is positive semidefinite. So the relaxation is tight by Fact 1. And since the only nonzero solution to $(x; \theta; \alpha)^T \hat{S}(x; \theta; \alpha) = 0$ up to scale is $(x; \theta; \alpha) = \hat{z}$ we have that $\hat{S}$ is corank 1. □

We emphasize that since we don't have a proof for the ACQ condition we haven't fully proved local stability. But we include the partial results in case they are useful for future works.

B.2. Fractional method

In this section we will prove two of the criteria required for local stability for the robust fractional method Eq. (RTF) for noise-free and outlier-free measurements. Local stability for the non-robust case was shown already in [4] but we will provide an alternate proof here in our notation, since it will lead into the extension to the robust case. For this we will need the stronger version of Fact 2, which we will restate here loosely (see [5] Theorem 4.5 for more details). Using the definition $A(\xi) = \sum_{i=1}^k \xi_i A_i$:

Fact 3. *If we, in addition to the conditions in Fact 1, have that:*

(i) (ACQ) ACQ holds

(ii) (smoothness) The the constraint set is smooth with respect to perturbations to the constraints

(iii) (non-branch point) The nullspace of the multiplier matrix and the tangent space of the constraint-set at the optimum don't intersect nontrivially: $\ker(\hat{S}) \cap T_{\hat{z}} = \{0\}$

(iv) (restricted slater) There exists ξ', λ' such that $A(\xi') - \lambda' E$ is positive definite on the subspace of vectors $z_\perp$ for which $\hat{S} z_\perp = 0$ and $\hat{z}^T z_\perp \neq 0$. In other words the part of the nullspace of $\hat{S}$ which is orthogonal to the solution $\hat{z}$.

The tangent space in (iii) is given by $T_{\hat{z}} = \ker(\hat{z}^T A_1; \ldots; \hat{z}^T A_k; \hat{z}^T E)$.

B.3. Non-robust version

We will show (iii-iv) for a version of Eq. (TF) with somewhat less constraints, noting that if we show (iii-iv) for the problem with less constraints we can then add in the remaining constraints back in and set the corresponding multipliers to zero to show that (iii-iv) holds for the original problem as well. Note however again that since we don't show (i-ii) the full proof is incomplete and is left for future work.

Theorem 2. *The fractional relaxation Eq. (TF) is tight and (iii-iv) hold for noise-free and outlier-free measurements $\tilde{x}_i$, $i = 1, \ldots, n$.*

Proof. We start by partitioning the Lagrange multipliers as $\xi = (\varphi; \alpha)$. Where $\varphi = (\varphi_1; \ldots; \varphi_{2n})$, and each $\varphi_i \in \mathbb{R}^4$ contains the multipliers corresponding to ith reprojection constraint multiplied by the entries of $\bar{X}$ (recall that there are two reprojection constraints per observation). Note that in the original formulation we also multiply by all the entries of $x \otimes \bar{X}$ as well, but as we will see these are not necessary for the proof to hold. And α corresponds to the Kronecker product constraints.

Since the observations $\tilde{x}$ are noise free we can denote the corresponding unique[3] 3D point in homogeneous coordinates as $\hat{X} \in \mathbb{R}^4$, normalized such that $\|\hat{X}\| = 1$. It will be convenient to introduce the reparametrization $u = \tilde{x}$ which is the same as the observation vector, except partitioned such that $u = (u_1; \ldots; u_{2n})$, $u_i \in \mathbb{R}$, *i.e.* $u_{2i+k} = \tilde{x}_{ik}$ for $i = 1, \ldots, n$, $k = 1, 2$. The primal optimum is then obtained at $\hat{z} = \bar{u} \otimes \hat{X}$, which is verified by setting $\hat{\xi} = \hat{\lambda} = 0$ to get $\hat{S}\hat{z} = (M_{\tilde{x}} \otimes I_4)(\bar{u} \otimes \hat{X}) = (M_{\tilde{x}} \bar{u}) \otimes \hat{X} = 0$.

We then note that, due to the properties of the Kronecker product[4] and that $M_{\tilde{x}}$ is positive semidefintie with corank 1, we have that $\hat{Z} = M_{\tilde{x}} \otimes I_4$ is positive semidefinite with corank 4. So the conditions of Fact 1 are satisfied and the relaxation is tight.

Since the nullspace $\ker(\hat{S})$ is 4-dimensional and contains the four orthogonal vectors $\hat{z} = \bar{u} \otimes \hat{X}$ and $\hat{z}_l = \bar{u} \otimes \hat{X}_l$ where $\hat{X}^T \hat{X}_l = 0$, $\hat{X}_l^T \hat{X}_k = 0$ for $k \neq l = 1, 2, 3$ we can parametrize $z_\perp$ from (iv) as $z_\perp = \bar{u} \otimes \hat{X}_\perp$ where $\hat{X}_\perp^T \hat{X} = 0$.

For (iii) we need to show that the vectors that span $\ker(\hat{S})$ are not in $T_{\hat{z}}$, *i.e.* for any $z \in \ker(\hat{S})$ either that $\hat{z}^T A_i z \neq 0$ for some constraint i, or that $\hat{z}^T E z \neq 0$. This

[3] assuming the observations are not degenerate, *e.g.* not all on a line.

[4] For matrices $A \in \mathbb{S}_n, B \in \mathbb{S}_m$ with eigenvalues α_i, β_j the eigenvalues of the Kronecker product $A \otimes B$ are given by the products of the eigenvalues $\alpha_i \beta_j$ for $i = 1, \ldots, n$, $j = 1, \ldots, m$.

is the case since $\hat{z}^T E \hat{z} = 1 \neq 0$ and, letting K_{ijst} be the Kronecker constraint matrix corresponding to index st of block ij, $\hat{z}^T K_{ijst} z_l = u_i u_j (\hat{X}_s \hat{X}_{lt} - \hat{X}_t \hat{X}_{ls})$ is nonzero for at least some index $ijst$ unless $u = 0$ or $\hat{X}$ and $\hat{X}_l$ are parallel, which is not the case by construction.

To show (iv), we set $\alpha' = \lambda' = 0$ and $\varphi'_i = u_i b_i - a_i$, and verify that with $z_\perp$ as above:

$$
\begin{aligned}
z_\perp^T A(\varphi', 0) z_\perp &= \sum_{i=1}^{2n} \hat{X}_\perp^T \varphi'_i (u_i b_i - a_i) \hat{X}_\perp \\
&= \sum_{i=1}^{2n} ((u_i b_i - a_i)^T \hat{X}_\perp)^2 > 0
\end{aligned}
\tag{16}
$$

where the final strict inequality follows from the fact that each term is strictly positive as $(u_i b_i - a_i)^T \hat{X} = 0$ by the original constraints and $\hat{X}_\perp$ is orthogonal to $\hat{X}$. □

We note that, while not all constraints used in Eq. (TF) are required for (iii-iv) to hold, we have found some cases where adding the additional constraints results in a tighter relaxation in the presence of noise, so we used the full set of constraints in our experiments.

B.4. Robust version

We now move on to the robust fractional method

Theorem 3. *The fractional relaxation Eq. (RTF) is tight and (iii-iv) hold for noise-free and outlier-free measurements $\tilde{x}_i$, $i = 1, \ldots, n$.*

Proof. Partition the Lagrange multipliers as $\xi = (\varphi; \mu; \eta; \alpha)$, where as in Theorem 2 φ corresponds to the reprojection constraints and α corresponds to the Kronecker constraints. We let $\mu \in \mathbb{R}^{32n}$ correspond to the constraints $\bar{X}_s \bar{X}_t (y_{ik} \theta_i - y_{ik}) = 0$ for $s, t = 1, 2, 3, 4$, $k = 1, 2$ and $i = 1, \ldots, n$. And finally we similarly have that $\eta \in \mathbb{R}^{16n} = (\eta_1; \ldots; \eta_n)$, $\eta_i \in \mathbb{R}^{16}$ corresponds to the constraints $\bar{X}_s \bar{X}_t (\theta_i^2 - \theta_i) = 0$. For each view i we collect the corresponding subset of η into a 4×4 matrix H_i defined such that $\bar{X}^T H_i \bar{X} = \sum_{s,t=1}^4 \eta_{ist} \bar{X}_s \bar{X}_t$.

To verify the global optimum we start by setting $\hat{z} = \bar{u}_\theta \otimes \hat{X}$ where $u_\theta = (\tilde{x}; 1_n)$. We then note that the constraint matrices for for the η_i-constraints can be written as a Kronecker product to get:

$$
S(0, 0, \eta, 0) = M_{\tilde{x}}^c \otimes I_4 + \sum_{i=1}^{n} T_i \otimes H_i \tag{17}
$$

where each $T_i \in \mathbb{S}_{3n+1}$ is defined such that $\bar{y}_\theta^T T_i \bar{y}_\theta = \theta_i^2 - \theta_i$ for arbitrary y_θ as in Sec. 5.2. We then set $\hat{\eta}$ such that $\hat{H}_i = c_i I_4$ and $\hat{\varphi} = \hat{\mu} = \hat{\alpha} = \hat{\lambda} = 0$ to get:

$$
\hat{S} = S(0, 0, \hat{\eta}, 0) = (M_{\tilde{x}}^c + \sum_{i=1}^{n} c_i T_i) \otimes I_4. \tag{18}
$$

Now, by the same argument as in Theorem 1 the matrix $M_{\tilde{x}}^c + \sum_{i=1}^n c_i T_i$ is positive semidefinite with corank 1, so $\hat{S}$ is positive semidefinite with corank 4. Meaning that the conditions of Fact 1 are satisfied. (iii) also follows using the same argument based on the Kronecker constraints as in Theorem 2.

Finally, for (iv) we note that $\ker(\hat{S})$ is spanned by $\hat{z}$ and $\hat{z}_l = \bar{u}_\theta \otimes \hat{X}_l$, $l = 1, 2, 3$, so by setting $\mu' = \eta' = \alpha' = \lambda' = 0$ and $\varphi'_i = u_i b_i - a_i$ restricted slater for $\hat{S}$ follows in the same way as in Eq. (16).

□