# Adaptive Annealing for Robust Geometric Estimation
# Supplementary Material

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We first provide the least squares and robust 3D registration problems in Eqns. S1 and S2, respectively, for convenience.

$$\min_{(\mathbf{R},\mathbf{t})} \frac{1}{2}\sum_{i=1}^{N} \|\mathbf{a}_i - \mathbf{R}\mathbf{b}_i - \mathbf{t}\|^2 \tag{S1}$$

$$\min_{(\mathbf{R},\mathbf{t})} \sum_{i=1}^{N} \rho_\sigma(\|\mathbf{a}_i - \mathbf{R}\mathbf{b}_i - \mathbf{t}\|) \tag{S2}$$

## A. Weighted Umeyama Method

We restate the weighted Umeyama problem here as follows:

$$\min_{(\mathbf{R},\mathbf{t})} \quad \frac{1}{2}\sum_{i=1}^{N} \phi_i\|\mathbf{r}_i\|^2 = \frac{1}{2}\sum_{i=1}^{N} \phi_i\|\mathbf{a}_i - \mathbf{R}\mathbf{b}_i - \mathbf{t}\|^2 \tag{S3}$$

In Line 5, Algo. 1 of the main paper, we have used the weighted Umeyama method (a variant of the original Umeyama's [5]) for minimizing the weighted least squares cost, Eqn. S3. Here, we describe the method in Algo. S1.

**Algorithm S1:** Weighted Umeyama Method

**Input:** $\{\mathbf{a}_i\}$'s, $\{\mathbf{b}_i\}$'s, weights $\phi_i$'s
**Output:** Rotation $\mathbf{R}$, Translation $\mathbf{t}$

**Initialization**: No initialization for $\mathbf{R}$, $\mathbf{t}$ required.

/* $3 \times N$ matrices $\mathbf{A}_1$, $\mathbf{B}_1$ */

$\mathbf{A}_1 = \left[\sqrt{\phi_1}\mathbf{a}_1, \quad \sqrt{\phi_2}\mathbf{a}_2, \quad \ldots, \quad \sqrt{\phi_N}\mathbf{a}_N\right]$

$\mathbf{B}_1 = \left[\sqrt{\phi_1}\mathbf{b}_1, \quad \sqrt{\phi_2}\mathbf{b}_2, \quad \ldots, \quad \sqrt{\phi_N}\mathbf{b}_N\right]$

/* weight vector $\mathbf{w}$ */

$\mathbf{w} = \left[\sqrt{\phi_1}, \quad \sqrt{\phi_2}, \quad \ldots, \quad \sqrt{\phi_N}\right]$

$\mathbf{K} = \mathbf{I} - \frac{\boldsymbol{\omega}\boldsymbol{\omega}^\top}{\mathbf{w}^\top\mathbf{w}}$ // Normalization matrix

$\mathbf{\Sigma}_{ab} = \frac{\mathbf{A}_1\mathbf{K}\mathbf{B}_1^\top}{\mathbf{w}^\top\mathbf{w}}$

$[\mathbf{U}, \mathbf{D}, \mathbf{V}] = svd(\mathbf{\Sigma}_{ab})$ // $\mathbf{\Sigma}_{ab} = \mathbf{U}\mathbf{D}\mathbf{V}^\top$

$\mathbf{S} = \begin{cases} \mathbf{I} & \text{if } det(\mathbf{\Sigma}_{ab}) \geq 0 \\ diag(1, 1, \ldots, 1, -1) & \text{if } det(\mathbf{\Sigma}_{ab}) < 0 \end{cases}$

$\mathbf{R} = \mathbf{U}\mathbf{S}\mathbf{V}^\top, \mathbf{t} = \frac{1}{\mathbf{w}^\top\mathbf{w}}(\mathbf{A}_1 - \mathbf{R}\mathbf{B}_1)\mathbf{w}$

## B. Proofs of Theorems 3.1 and 3.2

**Theorem 3.1** The gradient $\mathbf{g}_{LSQ}$ and Hessian $\mathbf{H}_{LSQ}$ of the least squares cost (Eqn. S1), at the point $(\mathbf{R}, \mathbf{t})$, are given by:

$$\mathbf{g}_{LSQ} = \sum_{i=1}^{N} \mathbf{g}_{LSQ,i}; \quad \mathbf{g}_{LSQ,i} = \begin{bmatrix} -\left[\mathbf{b}_i\right]_\times \mathbf{R}^\top \mathbf{r}_i \\ -\mathbf{r}_i \end{bmatrix} \tag{S4}$$

$$\mathbf{H}_{LSQ} = \sum_{i=1}^{N} \mathbf{H}_{LSQ,i} \tag{S5}$$

$$\mathbf{H}_{LSQ,i} = \begin{bmatrix} (\mathbf{p}_i^\top \mathbf{R}\mathbf{b}_i)\mathbf{I} - \frac{1}{2}\mathbf{b}_i\mathbf{p}_i^\top\mathbf{R} - \frac{1}{2}\mathbf{R}^\top\mathbf{p}_i\mathbf{b}_i^\top & \left[\mathbf{b}_i\right]_\times \mathbf{R}^\top \\ -\mathbf{R}\left[\mathbf{b}_i\right]_\times & \mathbf{I} \end{bmatrix} \tag{S6}$$

where $\mathbf{p}_i = \mathbf{a}_i - \mathbf{t}$, $\mathbf{r}_i = \mathbf{a}_i - \mathbf{R}\mathbf{b}_i - \mathbf{t}$, and $\mathbf{I}$ is the $3 \times 3$ identity matrix.

*Proof.* Consider a term corresponding to a single observation in Eqn. S1:

$$f_i(\mathbf{R}, \mathbf{t}) = \frac{1}{2}\|\mathbf{a}_i - \mathbf{R}\mathbf{b}_i - \mathbf{t}\|^2 \tag{S7}$$

where $\mathbf{R} \in \mathbb{SO}(3)$ and $\mathbf{t} \in \mathbb{R}^3$. For ease of notation, we drop the subscript $i$ wherever applicable because we are referring only to a single observation $i$ in this proof. $\mathbb{SO}(3)$ and $\mathbb{R}^3$ are Riemannian manifolds with the natural geodesic metric on $\mathbb{SO}(3)$ (Refer to [2] for additional information on properties of $\mathbb{SO}(3)$) and the standard inner-product on $\mathbb{R}^3$ as the Riemannian metrics on the respective manifolds. Therefore, the product manifold $\mathbb{SO}(3) \times \mathbb{R}^3$ admits a canonical product Riemannian metric. This leads to the following expression of the Riemannian gradient of the least squares cost (Eqn. S1) computed at the point $(\mathbf{R}, \mathbf{t})$:

$$\mathbf{g} := \mathbf{g}_{LSQ} = \begin{bmatrix} grad_{\mathbf{R}} f(\mathbf{R}, \mathbf{t}) \\ grad_{\mathbf{t}} f(\mathbf{R}, \mathbf{t}) \end{bmatrix} \tag{S8}$$

where $grad_{\mathbf{R}} f$ is the Riemannian gradient evaluated at $\mathbf{R}$, which is defined in the following manner (refer to [1, 2]). Let $\exp_{\mathbf{R}} : \mathbb{R}^3 \to \mathbb{SO}(3)$ denote the exponential map at a point $\mathbf{R} \in \mathbb{SO}(3)$ defined by $\exp_{\mathbf{R}}([\mathbf{x}]_\times) = \mathbf{R}\exp([\mathbf{x}]_\times)$. Given a function $f : \mathbb{SO}(3) \to \mathbb{R}$, the gradient $(grad_{\mathbf{R}} f)$ and Hessian $(\mathbf{H})$ of the function $f$ at the point $\mathbf{R}$ are defined as the gradient and the Hessian of the function $f \circ \exp_{\mathbf{R}} : \mathbb{R}^3 \to \mathbb{R}$, evaluated at $\mathbf{x} = \mathbf{0}$. We denote $\tilde{f}(\mathbf{x}) := (f \circ \exp_{\mathbf{R}})(\mathbf{x})$.

$$\tilde{f}(\mathbf{x}) = f(\mathbf{R}\exp([\mathbf{x}]_\times), \mathbf{t}) = \frac{1}{2}\|\mathbf{a} - \mathbf{R}\exp([\mathbf{x}]_\times)\mathbf{b} - \mathbf{t}\|^2 \tag{S9}$$

$$\langle grad_{\mathbf{R}} f(\mathbf{R}, \mathbf{t}), \hat{\mathbf{u}} \rangle = \langle \nabla_{\mathbf{x}} \tilde{f}(\mathbf{x})|_{\mathbf{x}=\mathbf{0}}, \hat{\mathbf{u}} \rangle = D_{\hat{\mathbf{u}}} \tilde{f}(\mathbf{x})|_{\mathbf{x}=\mathbf{0}} \tag{S10}$$

where $\langle \cdot, \cdot \rangle$ is the standard Euclidean inner-product in $\mathbb{R}^3$, $D_{\mathbf{u}} \tilde{f}(\mathbf{x})$ is the directional derivative of the function $\tilde{f}(\mathbf{x})$ along the direction $\hat{\mathbf{u}}$ at the point $\mathbf{x}$ (where $\hat{\mathbf{u}} \in \mathbb{R}^3$ s.t. $\|\hat{\mathbf{u}}\| = 1$).

$$D_{\hat{\mathbf{u}}} \tilde{f}(\mathbf{x})|_{\mathbf{x}=\mathbf{0}} = \left( \frac{\mathrm{d}\tilde{f}(s\hat{\mathbf{u}})}{\mathrm{d}s} \right)_{s=0} \tag{S11}$$

Considering $q(s) = \tilde{f}(s\hat{\mathbf{u}})$ in Eqn. S11 and using it in Eqn. S10, we get

$$\langle grad_{\mathbf{R}} f(\mathbf{R}, \mathbf{t}), \hat{\mathbf{u}} \rangle = q'(s)|_{s=0} = q'(0) \tag{S12}$$

The derivative, $q'(0)$, is computed as follows:

$$
\begin{aligned}
q(s) &= \frac{1}{2}\|\mathbf{a} - \mathbf{R}\exp([s\hat{\mathbf{u}}]_\times)\mathbf{b} - \mathbf{t}\|^2 \\
q'(s) &= (\mathbf{a} - \mathbf{R}\exp(s\,[\hat{\mathbf{u}}]_\times)\mathbf{b} - \mathbf{t})^\top(-\cos(s)\mathbf{R}\,[\hat{\mathbf{u}}]_\times\,\mathbf{b} - \sin(s)\mathbf{R}\,[\hat{\mathbf{u}}]_\times^2\,\mathbf{b}_i) \\
&= \mathbf{r}_i^\top(-\cos(s)\mathbf{R}\,[\hat{\mathbf{u}}]_\times\,\mathbf{b}_i - \sin(s)\mathbf{R}\,[\hat{\mathbf{u}}]_\times^2\,\mathbf{b}_i) \\
\Rightarrow q'(0) &= -\mathbf{r}^\top\mathbf{R}\,[\hat{\mathbf{u}}]_\times\,\mathbf{b} \\
&= \mathbf{r}^\top\mathbf{R}\,[\mathbf{b}]_\times\,\hat{\mathbf{u}} \\
\Rightarrow \langle grad_{\mathbf{R}} f(\mathbf{R},\mathbf{t}), \hat{\mathbf{u}}\rangle &= \mathbf{r}^\top\mathbf{R}\,[\mathbf{b}]_\times\,\hat{\mathbf{u}} \\
\Rightarrow (grad_{\mathbf{R}} f(\mathbf{R},\mathbf{t}))^\top\,\hat{\mathbf{u}} &= \mathbf{r}^\top\mathbf{R}\,[\mathbf{b}]_\times\,\hat{\mathbf{u}} \\
\Rightarrow grad_{\mathbf{R}} f(\mathbf{R},\mathbf{t}) &= -\,[\mathbf{b}]_\times\,\mathbf{R}^\top\mathbf{r}
\end{aligned}
\tag{S13}
$$

The gradient of $f(\mathbf{R},\mathbf{t})$ with respect to $\mathbf{t} \in \mathbb{R}^3$ is given by

$$
\begin{aligned}
grad_{\mathbf{t}} f(\mathbf{R},\mathbf{t}) &= \nabla_{\mathbf{t}} f(\mathbf{R},\mathbf{t}) \\
&= \nabla_{\mathbf{t}}\left(\|\mathbf{a}_i - \mathbf{R}\mathbf{b}_i - \mathbf{t}\|^2\right) \\
&= -\mathbf{r}
\end{aligned}
\tag{S14}
$$

Therefore, the gradient can be written as (from Eqns. S13 and S14):

$$
\boxed{\mathbf{g} := \mathbf{g}_{LSQ} = \begin{bmatrix} grad_{\mathbf{R}} f(\mathbf{R},\mathbf{t}) \\ grad_{\mathbf{t}} f(\mathbf{R},\mathbf{t}) \end{bmatrix} = \begin{bmatrix} -\,[\mathbf{b}]_\times\,\mathbf{R}^\top\mathbf{r} \\ -\mathbf{r} \end{bmatrix}}
\tag{S15}
$$

The Hessian of the cost function $f(\mathbf{R},\mathbf{t})$ is given by:

$$
\mathbf{H} := \mathbf{H}_{LSQ,i} = \begin{bmatrix} \mathbf{H}_{RR} & \mathbf{H}_{Rt} \\ \mathbf{H}_{Rt}^\top & \mathbf{H}_{tt} \end{bmatrix}
\tag{S16}
$$

where $\mathbf{H}_{RR}$ is the Hessian of the function $f(\mathbf{R},\mathbf{t})$ with respect to $\mathbf{R} \in \mathbb{SO}(3)$. By definition, $\mathbf{H}_{RR}$ is equal to the Hessian of the function $\tilde{f}(\mathbf{x})$ evaluated at $\mathbf{x} = \mathbf{0}$. Therefore,

$$
\begin{aligned}
\mathbf{H}_{RR} &= \nabla^2_{\mathbf{x}}\tilde{f}(\mathbf{x}) \\
&= \mathbb{J}(\nabla_{\mathbf{x}}\tilde{f}(\mathbf{x})) \quad \text{where } \mathbb{J} \text{ is the Jacobian.}
\end{aligned}
$$

The expression for $\nabla_{\mathbf{x}}\tilde{f}(\mathbf{x})$ is given as follows:

$$
\begin{aligned}
\nabla_{\mathbf{x}}\tilde{f}(\mathbf{x}) = \Bigg\{ &-\frac{\cos(\|\mathbf{x}\|)}{\|\mathbf{x}\|^2}\mathbf{x}\mathbf{x}^\top\,[\mathbf{b}]_\times\,\mathbf{R}^\top + \frac{\sin(\|\mathbf{x}\|)}{\|\mathbf{x}\|^3}\mathbf{x}\mathbf{x}^\top\,[\mathbf{b}]_\times\,\mathbf{R}^\top - \frac{\sin(\|\mathbf{x}\|)}{\|\mathbf{x}\|}\,[\mathbf{b}]_\times\,\mathbf{R}^\top - \frac{(\mathbf{x}^\top\mathbf{b})\sin(\|\mathbf{x}\|)}{\|\mathbf{x}\|^3}\mathbf{x}\mathbf{x}^\top\mathbf{R}^\top \\
&+ \frac{2(\mathbf{x}^\top\mathbf{b})(1-\cos(\|\mathbf{x}\|))}{\|\mathbf{x}\|^4}\mathbf{x}\mathbf{x}^\top\mathbf{R}^\top - \frac{(1-\cos(\|\mathbf{x}\|))}{\|\mathbf{x}\|^2}\mathbf{b}\mathbf{x}^\top\mathbf{R}^\top \\
&- \frac{(\mathbf{x}^\top\mathbf{b})(1-\cos(\|\mathbf{x}\|))}{\|\mathbf{x}\|^2}\mathbf{R}^\top + \frac{\sin(\|\mathbf{x}\|)}{\|\mathbf{x}\|}\mathbf{x}\mathbf{b}^\top\mathbf{R}^\top \Bigg\}\,\mathbf{r}(\mathbf{x})
\end{aligned}
$$

where $\mathbf{r}(\mathbf{x}) = \left(\mathbf{a} - \mathbf{R}\exp\left([\mathbf{x}]_\times\right)\mathbf{b} - \mathbf{t}\right)$. Note that $\nabla_{\mathbf{x}}\tilde{f}(\mathbf{x})|_{\mathbf{x}=\mathbf{0}} = grad_{\mathbf{R}} f(\mathbf{R},\mathbf{t}) = -\,[\mathbf{b}]_\times\,\mathbf{R}^\top\mathbf{r}$. The Hessian of the function $\tilde{f}(\mathbf{x})$ or the Jacobian of the gradient $\nabla_{\mathbf{x}}\tilde{f}(\mathbf{x})$ with respect to $\mathbf{x}$ is given as follows. As the expressions are very lengthy, we provide the Jacobian for each of the terms in the order that they are written:

1. Term corresponding to $\left(-\frac{\cos(\|\mathbf{x}\|)}{\|\mathbf{x}\|^2}\mathbf{x}\mathbf{x}^\top\,[\mathbf{b}]_\times\,\mathbf{R}^\top\right)\mathbf{r}(\mathbf{x})$:

$$
\begin{aligned}
&-\frac{2\cos(\|\mathbf{x}\|)\sin(\|\mathbf{x}\|)}{\|\mathbf{x}\|^3}\left(\mathbf{x}^\top\mathbf{b}\right)\left(\mathbf{x}\mathbf{b}^\top\right)+\frac{\left(\mathbf{x}^\top\mathbf{b}\right)^2\sin^2(\|\mathbf{x}\|)\left(\mathbf{x}\mathbf{x}^\top\right)}{\|\mathbf{x}\|^4}-\left(\frac{\left(\mathbf{x}^\top\mathbf{b}\right)^2\cos^2(\|\mathbf{x}\|)}{\|\mathbf{x}\|^4}\mathbf{x}\mathbf{x}^\top\right)\\
&+\left(\frac{3\left(\mathbf{x}^\top\mathbf{b}\right)^2\cos(\|\mathbf{x}\|)\sin(\|\mathbf{x}\|)}{\|\mathbf{x}\|^5}\mathbf{x}\mathbf{x}^\top\right)-\left(\frac{\left(\mathbf{x}^\top\mathbf{b}\right)^2\cos(\|\mathbf{x}\|)\sin(\|\mathbf{x}\|)}{\|\mathbf{x}\|^3}\mathbf{I}\right)\\
&+\left(\frac{\cos\left(\|\mathbf{x}\|\right)^2}{\|\mathbf{x}\|^2}\|\mathbf{b}\|^2\mathbf{x}\mathbf{x}^\top-\frac{\sin\left(\|\mathbf{x}\|\right)^2}{\|\mathbf{x}\|^2}\|\mathbf{b}\|^2\mathbf{x}\mathbf{x}^\top-\frac{\cos\left(\|\mathbf{x}\|\right)\sin\left(\|\mathbf{x}\|\right)}{\|\mathbf{x}\|^3}\|\mathbf{b}\|^2\mathbf{x}\mathbf{x}^\top+\frac{\cos\left(\|\mathbf{x}\|\right)\sin\left(\|\mathbf{x}\|\right)}{\|\mathbf{x}\|}\|\mathbf{b}\|^2\mathbf{I}\right)\\
&-\left(\frac{\cos(\|\mathbf{x}\|)}{\|\mathbf{x}\|^2}\mathbf{x}^\top\left[\mathbf{b}\right]_\times\mathbf{R}^\top\mathbf{p}\mathbf{I}-\left(\frac{\sin(\|\mathbf{x}\|)}{\|\mathbf{x}\|^3}\mathbf{x}^\top\left[\mathbf{b}\right]_\times\mathbf{R}^\top\mathbf{p}\mathbf{x}\mathbf{x}^\top+2\frac{cos(\|\mathbf{x}\|)}{\|\mathbf{x}\|^4}\mathbf{x}^\top\left[\mathbf{b}\right]_\times\mathbf{R}^\top\mathbf{p}\mathbf{x}\mathbf{x}^\top\right)+\frac{\cos(\|\mathbf{x}\|)}{\|\mathbf{x}\|^2}\mathbf{x}\mathbf{p}^\top\mathbf{R}\left[\mathbf{b}\right]_\times^\top\right)
\end{aligned}
$$

2. Term corresponding to $\left(\frac{\sin(\|\mathbf{x}\|)}{\|\mathbf{x}\|^3}\mathbf{x}\mathbf{x}^\top\left[\mathbf{b}\right]_\times\mathbf{R}^\top\right)\mathbf{r}(\mathbf{x})$:

$$
\begin{aligned}
&\frac{\cos\left(\|\mathbf{x}\|\right)}{\|\mathbf{x}\|^4}\mathbf{x}^\top\left[\mathbf{b}\right]_\times\mathbf{R}^\top\mathbf{p}\mathbf{x}\mathbf{x}^\top-\frac{\left(3\sin\left(\|\mathbf{x}\|\right)\right)}{\|\mathbf{x}\|^5}\mathbf{x}^\top\left[\mathbf{b}\right]_\times\mathbf{R}^\top\mathbf{p}\mathbf{x}\mathbf{x}^\top+\frac{\sin\left(\|\mathbf{x}\|\right)}{\|\mathbf{x}\|^3}\mathbf{x}^\top\left[\mathbf{b}\right]_\times\mathbf{R}^\top\mathbf{p}\mathbf{I}+\frac{\sin\left(\|\mathbf{x}\|\right)}{\|\mathbf{x}\|^3}\mathbf{x}\left(\mathbf{p}^\top\mathbf{R}\left[\mathbf{b}\right]_\times^\top\right)\\
&+\left(\frac{2\left(\mathbf{x}^\top\mathbf{b}\right)^2\cos\left(\|\mathbf{x}\|\right)\sin\left(\|\mathbf{x}\|\right)}{\|\mathbf{x}\|^5}\mathbf{x}\mathbf{x}^\top+\frac{2\sin\left(\|\mathbf{x}\|\right)^2}{\|\mathbf{x}\|^4}(\mathbf{x}^\top\mathbf{b})\mathbf{x}\mathbf{b}^\top-\frac{4\sin\left(\|\mathbf{x}\|\right)^2\left(\mathbf{x}^\top\mathbf{b}\right)^2}{\|\mathbf{x}\|^6}\mathbf{x}\mathbf{x}^\top+\frac{\sin\left(\|\mathbf{x}\|\right)^2\left(\mathbf{x}^\top\mathbf{b}\right)^2}{\|\mathbf{x}\|^4}\mathbf{I}\right)
\end{aligned}
$$

3. Term corresponding to $\left(-\frac{\sin(\|\mathbf{x}\|)}{\|\mathbf{x}\|}\left[\mathbf{b}\right]_\times\mathbf{R}^\top\right)\mathbf{r}(\mathbf{x})$:

$$
\begin{aligned}
&-\frac{2\cos\left(\|\mathbf{x}\|\right)\sin\left(\|\mathbf{x}\|\right)}{\|\mathbf{x}\|^3}\left(\mathbf{x}^\top\mathbf{b}\right)\mathbf{b}\mathbf{x}^\top+\frac{2\sin^2\left(\|\mathbf{x}\|\right)}{\|\mathbf{x}\|^4}\left(\mathbf{x}^\top\mathbf{b}\right)\mathbf{b}\mathbf{x}^\top-\frac{\sin^2\left(\|\mathbf{x}\|\right)}{\|\mathbf{x}\|^2}\mathbf{b}\mathbf{b}^\top-\frac{\cos\left(\|\mathbf{x}\|\right)}{\|\mathbf{x}\|^2}\left[\mathbf{b}\right]_\times\mathbf{R}^\top\mathbf{p}\mathbf{x}^\top\\
&+\frac{\sin\left(\|\mathbf{x}\|\right)}{\|\mathbf{x}\|^3}\left[\mathbf{b}\right]_\times\mathbf{R}^\top\mathbf{p}\mathbf{x}^\top
\end{aligned}
$$

4. Term corresponding to $\left(-\frac{(\mathbf{x}^\top\mathbf{b})\sin(\|\mathbf{x}\|)}{\|\mathbf{x}\|^3}\mathbf{x}\mathbf{x}^\top\mathbf{R}^\top\right)\mathbf{r}(\mathbf{x})$:

$$
\begin{aligned}
&-\frac{\cos\left(\|\mathbf{x}\|\right)}{\|\mathbf{x}\|^4}\left(\mathbf{x}^\top\mathbf{b}\right)\left(\mathbf{x}^\top\mathbf{R}^\top\mathbf{p}\right)\mathbf{x}\mathbf{x}^\top+\frac{3\sin\left(\|\mathbf{x}\|\right)}{\|\mathbf{x}\|^5}(\mathbf{x}^\top\mathbf{b})(\mathbf{x}^\top\mathbf{R}^\top\mathbf{p})\mathbf{x}\mathbf{x}^\top\\
&-\frac{\sin\left(\|\mathbf{x}\|\right)}{\|\mathbf{x}\|^3}(\mathbf{x}^\top\mathbf{R}^\top\mathbf{p})\mathbf{x}\mathbf{b}^\top-\frac{\sin\left(\|\mathbf{x}\|\right)}{\|\mathbf{x}\|^3}(\mathbf{x}^\top\mathbf{b})\mathbf{x}\mathbf{p}^\top\mathbf{R}-\frac{\sin\left(\|\mathbf{x}\|\right)}{\|\mathbf{x}\|^3}(\mathbf{x}^\top\mathbf{R}^\top\mathbf{p})(\mathbf{x}^\top\mathbf{b})\mathbf{I}
\end{aligned}
$$

5. Term corresponding to $\left(\frac{2(\mathbf{x}^\top\mathbf{b})(1-\cos(\|\mathbf{x}\|))}{\|\mathbf{x}\|^4}\mathbf{x}\mathbf{x}^\top\mathbf{R}^\top\right)\mathbf{r}(\mathbf{x})$:

$$
\begin{aligned}
&\frac{2\sin\left(\|\mathbf{x}\|\right)}{\|\mathbf{x}\|^5}(\mathbf{x}^\top\mathbf{b})(\mathbf{x}^\top\mathbf{R}^\top\mathbf{p})\mathbf{x}\mathbf{x}^\top-\frac{8\left(1-\cos\left(\|\mathbf{x}\|\right)\right)}{\|\mathbf{x}\|^6}(\mathbf{x}^\top\mathbf{b})(\mathbf{x}^\top\mathbf{R}^\top\mathbf{p})\mathbf{x}\mathbf{x}^\top\\
&+\frac{2\left(1-\cos\left(\|\mathbf{x}\|\right)\right)}{\|\mathbf{x}\|^4}(\mathbf{x}^\top\mathbf{R}^\top\mathbf{p})\mathbf{x}\mathbf{b}^\top+\frac{2\left(1-\cos\left(\|\mathbf{x}\|\right)\right)}{\|\mathbf{x}\|^4}(\mathbf{x}^\top\mathbf{b})\mathbf{x}\mathbf{p}^\top\mathbf{R}+\frac{2\left(1-\cos\left(\|\mathbf{x}\|\right)\right)}{\|\mathbf{x}\|^4}(\mathbf{x}^\top\mathbf{R}^\top\mathbf{p})(\mathbf{x}^\top\mathbf{b})\mathbf{I}\\
&-\left(\frac{4\left(1-\cos\left(\|\mathbf{x}\|\right)\right)}{\|\mathbf{x}\|^4}(\mathbf{x}^\top\mathbf{b})\mathbf{x}\mathbf{b}^\top+\frac{2\left(\mathbf{x}^\top\mathbf{b}\right)^2\sin\left(\|\mathbf{x}\|\right)}{\|\mathbf{x}\|^5}\mathbf{x}\mathbf{x}^\top\right.\\
&\left.-\frac{8\left(\mathbf{x}^\top\mathbf{b}\right)^2\left(1-\cos\left(\|\mathbf{x}\|\right)\right)}{\|\mathbf{x}\|^6}\mathbf{x}\mathbf{x}^\top+\frac{2\left(\mathbf{x}^\top\mathbf{b}\right)^2\left(1-\cos\left(\|\mathbf{x}\|\right)\right)}{\|\mathbf{x}\|^4}\mathbf{I}\right)
\end{aligned}
$$

6. Term corresponding to $\left(-\frac{(1-\cos(\|\mathbf{x}\|))}{\|\mathbf{x}\|^2}\mathbf{b}\mathbf{x}^\top\mathbf{R}^\top\right)\mathbf{r}(\mathbf{x})$:

$$\begin{aligned}&\frac{\sin(\|\mathbf{x}\|)}{\|\mathbf{x}\|^3}(\mathbf{x}^\top\mathbf{b})\mathbf{b}\mathbf{x}^\top - \frac{2(1-\cos(\|\mathbf{x}\|))}{\|\mathbf{x}\|^4}(\mathbf{x}^\top\mathbf{b})\mathbf{b}\mathbf{x}^\top + \frac{(1-\cos(\|\mathbf{x}\|))}{\|\mathbf{x}\|^2}\mathbf{b}\mathbf{b}^\top\\ &-\left(\frac{\sin(\|\mathbf{x}\|)}{\|\mathbf{x}\|^3}(\mathbf{x}^\top\mathbf{R}^\top\mathbf{p})\mathbf{b}\mathbf{x}^\top - \frac{2(1-\cos(\|\mathbf{x}\|))}{\|\mathbf{x}\|^4}(\mathbf{x}^\top\mathbf{R}^\top\mathbf{p})\mathbf{b}\mathbf{x}^\top + \frac{(1-\cos(\|\mathbf{x}\|))}{\|\mathbf{x}\|^2}\mathbf{b}\mathbf{p}^\top\mathbf{R}\right)\end{aligned}$$

7. Term corresponding to $\left(-\frac{(\mathbf{x}^\top\mathbf{b})(1-\cos(\|\mathbf{x}\|))}{\|\mathbf{x}\|^2}\mathbf{R}^\top\right)\mathbf{r}(\mathbf{x})$:

$$\begin{aligned}&2\left(\mathbf{x}^\top\mathbf{b}\right)^2\frac{\sin(\|\mathbf{x}\|)(1-\cos(\|\mathbf{x}\|))}{\|\mathbf{x}\|^5}\mathbf{x}\mathbf{x}^\top + 2\frac{(1-\cos(\|\mathbf{x}\|))^2}{\|\mathbf{x}\|^4}\left(\mathbf{x}^\top\mathbf{b}\right)\mathbf{x}\mathbf{b}^\top\\ &-4\frac{(1-\cos(\|\mathbf{x}\|))^2\left(\mathbf{x}^\top\mathbf{b}\right)^2}{\|\mathbf{x}\|^6}\mathbf{x}\mathbf{x}^\top + \frac{(1-\cos(\|\mathbf{x}\|))^2\left(\mathbf{x}^\top\mathbf{b}\right)^2}{\|\mathbf{x}\|^4}\mathbf{I}\\ &-\left(\frac{\sin(\|\mathbf{x}\|)}{\|\mathbf{x}\|^3}\left(\mathbf{x}^\top\mathbf{b}\right)\mathbf{R}^\top\mathbf{p}\mathbf{x}^\top - 2\frac{(1-\cos(\|\mathbf{x}\|))}{\|\mathbf{x}\|^4}\left(\mathbf{x}^\top\mathbf{b}\right)\mathbf{R}^\top\mathbf{p}\mathbf{x}^\top + \frac{(1-\cos(\|\mathbf{x}\|))}{\|\mathbf{x}\|^2}\mathbf{R}^\top\mathbf{p}\mathbf{b}^\top\right)\\ &-\left(\frac{2\sin(\|\mathbf{x}\|)(1-\cos(\|\mathbf{x}\|))}{\|\mathbf{x}\|^3}\left(\mathbf{x}^\top\mathbf{b}\right)\mathbf{b}\mathbf{x}^\top - \frac{2(1-\cos(\|\mathbf{x}\|))^2}{\|\mathbf{x}\|^4}\left(\mathbf{x}^\top\mathbf{b}\right)\mathbf{b}\mathbf{x}^\top + \frac{(1-\cos(\|\mathbf{x}\|))^2}{\|\mathbf{x}\|^2}\mathbf{b}\mathbf{b}^\top\right)\\ &+\frac{\sin(\|\mathbf{x}\|)}{\|\mathbf{x}\|^3}(\mathbf{x}^\top\mathbf{b})\mathbf{b}\mathbf{x}^\top - \frac{2(1-\cos(\|\mathbf{x}\|))}{\|\mathbf{x}\|^4}(\mathbf{x}^\top\mathbf{b})\mathbf{b}\mathbf{x}^\top + \frac{(1-\cos(\|\mathbf{x}\|))}{\|\mathbf{x}\|^2}\mathbf{b}\mathbf{b}^\top\end{aligned}$$

8. Term corresponding to $\left(\frac{\sin(\|\mathbf{x}\|)}{\|\mathbf{x}\|}\mathbf{x}\mathbf{b}^\top\mathbf{R}^\top\right)\mathbf{r}(\mathbf{x})$:

$$\begin{aligned}&\frac{\cos(\|\mathbf{x}\|)}{\|\mathbf{x}\|^2}\left(\mathbf{b}^\top\mathbf{R}^\top\mathbf{p}\right)\mathbf{x}\mathbf{x}^\top - \frac{\sin(\|\mathbf{x}\|)}{\|\mathbf{x}\|^3}\left(\mathbf{b}^\top\mathbf{R}^\top\mathbf{p}\right)\mathbf{x}\mathbf{x}^\top + \frac{\sin(\|\mathbf{x}\|)}{\|\mathbf{x}\|}\left(\mathbf{b}^\top\mathbf{R}^\top\mathbf{p}\right)\mathbf{I}\\ &-\left(\frac{\cos^2(\|\mathbf{x}\|)}{\|\mathbf{x}\|^2}\|\mathbf{b}\|^2\mathbf{x}\mathbf{x}^\top - \frac{\sin^2(\|\mathbf{x}\|)}{\|\mathbf{x}\|^2}\|\mathbf{b}\|^2\mathbf{x}\mathbf{x}^\top - \frac{(\cos(\|\mathbf{x}\|)\sin(\|\mathbf{x}\|))}{\|\mathbf{x}\|^3}\|\mathbf{b}\|^2\mathbf{x}\mathbf{x}^\top + \frac{\cos(\|\mathbf{x}\|)\sin(\|\mathbf{x}\|)}{\|\mathbf{x}\|}\|\mathbf{b}\|^2\mathbf{I}\right)\\ &+2\frac{\cos(\|\mathbf{x}\|)\sin(\|\mathbf{x}\|)}{\|\mathbf{x}\|^3}\left(\mathbf{x}^\top\mathbf{b}\right)\mathbf{x}\mathbf{b}^\top - \left(\mathbf{x}^\top\mathbf{b}\right)^2\frac{\sin^2(\|\mathbf{x}\|)}{\|\mathbf{x}\|^4}\mathbf{x}\mathbf{x}^\top + \left(\mathbf{x}^\top\mathbf{b}\right)^2\frac{\cos^2(\|\mathbf{x}\|)}{\|\mathbf{x}\|^4}\mathbf{x}\mathbf{x}^\top\\ &-3\left(\mathbf{x}^\top\mathbf{b}\right)^2\frac{\cos(\|\mathbf{x}\|)\sin(\|\mathbf{x}\|)}{\|\mathbf{x}\|^5}\mathbf{x}\mathbf{x}^\top + \left(\mathbf{x}^\top\mathbf{b}\right)^2\frac{\cos(\|\mathbf{x}\|)\sin(\|\mathbf{x}\|)}{\|\mathbf{x}\|^3}\mathbf{I}\end{aligned}$$

Evaluating the sum of all the terms at $\mathbf{x} = \mathbf{0}$ gives:

$$\mathbf{H}_{RR} = (\mathbf{p}^\top\mathbf{R}\mathbf{b})\mathbf{I} - \frac{1}{2}\mathbf{b}\mathbf{p}^\top\mathbf{R} - \frac{1}{2}\mathbf{R}^\top\mathbf{p}\mathbf{b}^\top \tag{S17}$$

Evaluating the Jacobian of $\nabla_{\mathbf{x}}\tilde{f}(\mathbf{x})|_{\mathbf{x}=0}$ with respect to $\mathbf{t}$ gives $\mathbf{H}_{Rt}$ which is as follows:

$$\begin{aligned}\mathbf{H}_{Rt} &= \mathbb{J}_{\mathbf{t}}(-[\mathbf{b}]_\times\mathbf{R}^\top\mathbf{r})\\ &= \mathbb{J}_{\mathbf{t}}(-[\mathbf{b}]_\times\mathbf{R}^\top(\mathbf{a}-\mathbf{t}))\\ &= [\mathbf{b}]_\times\mathbf{R}^\top\end{aligned} \tag{S18}$$

In a similar manner, we can derive $\mathbf{H}_{tR} = \mathbf{H}_{Rt}^\top = -\mathbf{R}[\mathbf{b}]_\times$. The Hessian term $\mathbf{H}_{tt}$ can be obtained by taking the Jacobian of $grad_{\mathbf{t}}f(\mathbf{R},\mathbf{t})$ with respect to $\mathbf{t}$ and is given by:

$$\begin{aligned}\mathbf{H}_{tt} &= \mathbb{J}_{\mathbf{t}}(grad_{\mathbf{t}}f(\mathbf{R},\mathbf{t}))\\ &= \mathbb{J}_{\mathbf{t}}(-\mathbf{r})\\ &= \mathbb{J}_{\mathbf{t}}(-(\mathbf{a}-\mathbf{t}))\\ &= \mathbf{I}\end{aligned} \tag{S19}$$

Therefore, the Hessian matrix of the least squares cost (Eqn. S1) is given by (from Eqns. S17, S18 and S19) :

$$\boxed{\mathbf{H}_{LSQ} = \begin{bmatrix} (\mathbf{p}^\top\mathbf{R}\mathbf{b})\mathbf{I} - \frac{1}{2}\mathbf{b}\mathbf{p}^\top\mathbf{R} - \frac{1}{2}\mathbf{R}^\top\mathbf{p}\mathbf{b}^\top & [\mathbf{b}]_\times \mathbf{R}^\top \\ -\mathbf{R}\,[\mathbf{b}]_\times & \mathbf{I} \end{bmatrix}} \tag{S20}$$

□

**Theorem 3.2** The Hessian $\mathbf{H}$ of the robust Umeyama cost (Eqn. S2), at the point $(\mathbf{R}, \mathbf{t})$, is given by:

$$\mathbf{H} = \sum_{i=1}^{N} \left( -l_i \frac{\mathbf{g}_{LSQ,i}\mathbf{g}_{LSQ,i}^\top}{\|\mathbf{r}_i\|^2} + m_i \mathbf{H}_{LSQ,i} \right), \tag{S21}$$

$$\text{where } l_i = \frac{\rho'(\|\mathbf{r}_i\|)}{\|\mathbf{r}_i\|} - \rho''(\|\mathbf{r}_i\|), m_i = \frac{\rho'(\|\mathbf{r}_i\|)}{\|\mathbf{r}_i\|} \tag{S22}$$

*Proof.* The robust cost (Eqn. S2) is given by:

$$g(\mathbf{R}, \mathbf{t}) = \sum_{i=1}^{N} \rho_\sigma(\|\mathbf{a}_i - \mathbf{R}\mathbf{b}_i - \mathbf{t}\|) \tag{S23}$$

Taking the gradient, we get

$$\nabla g(\mathbf{R}, \mathbf{t}) = \sum_i \rho'\left(\|\mathbf{r}_i\|\right) \nabla\left(\|\mathbf{r}_i\|\right)$$

Now, we compute the Hessian as:

$$\mathbf{H} = \sum_i \nabla\left(\|\mathbf{r}_i\|\right) \cdot \nabla^T\left(\|\mathbf{r}_i\|\right) \cdot \rho''\left(\mathbf{r}_i\right) + \sum_i \rho'\left(\|\mathbf{r}_i\|\right) \cdot \nabla^2\left(\|\mathbf{r}_i\|\right) \tag{S24}$$

We denote $f_i = f_i(\mathbf{R}, \mathbf{t}) = \frac{1}{2}\|\mathbf{r}_i\|^2$, from Eqn. S7. Then,

$$\|\mathbf{r}_i\| \cdot \nabla\left(\|\mathbf{r}_i\|\right) = \nabla\left(\frac{1}{2}\|\mathbf{r}_i\|^2\right) \Rightarrow \nabla\left(\|\mathbf{r}_i\|\right) = \frac{\nabla f_i}{\|\mathbf{r}_i\|} \tag{S25}$$

Then,

$$\begin{aligned}
\nabla^2\left(\|\mathbf{r}_i\|\right) &= \nabla\left[\nabla\left(\frac{1}{2}\|\mathbf{r}_i\|^2\right) \cdot \frac{1}{\|\mathbf{r}_i\|}\right] \\
&= \frac{1}{\|\mathbf{r}_i\|^2}\left(\|\mathbf{r}_i\| \cdot \nabla^2\left(\frac{1}{2}\|\mathbf{r}_i\|^2\right) - \nabla\left(\frac{1}{2}\|\mathbf{r}_i\|^2\right) \cdot \nabla^T\left(\|\mathbf{r}_i\|\right)\right) \\
&= \frac{1}{\|\mathbf{r}_i\|^2}\left(\|\mathbf{r}_i\| \cdot \nabla^2\left(\frac{1}{2}\|\mathbf{r}_i\|^2\right) - \nabla\left(\frac{1}{2}\|\mathbf{r}_i\|^2\right) \cdot \nabla^T\left(\frac{1}{2}\|\mathbf{r}_i\|^2\right) \cdot \frac{1}{\|\mathbf{r}_i\|}\right) \text{[Using Eqn. S25]} \\
&= \frac{1}{\|\mathbf{r}_i\|} \cdot \nabla^2\left(\frac{1}{2}\|\mathbf{r}_i\|^2\right) - \frac{1}{\|\mathbf{r}_i\|^3}\nabla\left(\frac{1}{2}\|\mathbf{r}_i\|^2\right) \cdot \nabla\left(\frac{1}{2}\|\mathbf{r}_i\|^2\right)^T \\
&= \frac{1}{\|\mathbf{r}_i\|} \cdot \nabla^2 f_i - \frac{1}{\|\mathbf{r}_i\|^3}\nabla f_i \cdot \nabla f_i^T
\end{aligned}$$

Using this in Eqn. S24, we get,

$$\begin{aligned}
\mathbf{H} &= \sum_i \nabla\left(\|\mathbf{r}_i\|\right) \cdot \nabla^T\left(\|\mathbf{r}_i\|\right) \cdot \rho''\left(\|\mathbf{r}_i\|\right) + \sum_i \rho'\left(\|\mathbf{r}_i\|\right) \cdot \nabla^2\left(\|\mathbf{r}_i\|\right) \\
&= \sum_i \frac{\nabla f_i \cdot \nabla f_i^T}{\|\mathbf{r}_i\|^2} \cdot \rho''\left(\|\mathbf{r}_i\|\right) + \sum_i \rho'\left(\|\mathbf{r}_i\|\right)\left[\frac{1}{\|\mathbf{r}_i\|} \cdot \nabla^2 f_i - \frac{1}{\|\mathbf{r}_i\|^3}\nabla f_i \cdot \nabla f_i^T\right] \\
&= \sum_i \left\{\left(\frac{\rho''\left(\|\mathbf{r}_i\|\right)}{\|\mathbf{r}_i\|^2} - \frac{\rho'\left(\|\mathbf{r}_i\|\right)}{\|\mathbf{r}_i\|^3}\right)\nabla f_i \cdot \nabla f_i^T + \frac{\rho'\left(\|\mathbf{r}_i\|\right)}{\|\mathbf{r}_i\|} \cdot \nabla^2 f_i\right\} \\
&= \sum_i \left(-l_i \frac{\mathbf{g}_{LSQ,i}\mathbf{g}_{LSQ,i}^\top}{\|\mathbf{r}_i\|^2} + m_i \mathbf{H}_{LSQ,i}\right) \text{[as } \mathbf{g}_{LSQ,i} = \nabla f_i \text{ and } \mathbf{H}_{LSQ,i} = \nabla^2 f_i\text{]}
\end{aligned}$$

| Dataset | Time Taken (ms) | | | | |
|---|---|---|---|---|---|
| | FGR | GORE | SE3Reg | TEASER++ | GNCp (Ours) |
| armadillo | 4.66 | **0.26** | 1.56 | 0.41 | 6.99 |
| bunny | 4.63 | **0.25** | 1.53 | 0.45 | 6.83 |
| buddha | 4.66 | **0.27** | 1.53 | 0.47 | 7.04 |
| dragon | 4.95 | **0.33** | 1.59 | 0.46 | 7.86 |

Table S1. Computational time (mean taken over all instances) on *Type-1* synthetic datasets for $p_{outl} = 50\%$.

| Dataset | Time Taken (ms) | | | |
|---|---|---|---|---|
| | FGR | SE3Reg | TEASER++ | GNCp (Ours) |
| armadillo | 450.34 | **97.25** | 2240.19 | 160.65 |
| bunny | 455.77 | **97.63** | 2222.02 | 160.6 |
| buddha | 441.33 | **97.25** | 2308.69 | 155.18 |
| dragon | 450.98 | **99.49** | 2233.7 | 158.63 |

Table S2. Computational time (mean taken over all instances) on *Type-2* synthetic datasets for $p_{outl} = 50\%$.

| Dataset | Factor of increased time taken | | | |
|---|---|---|---|---|
| | FGR | SE3Reg | TEASER++ | GNCp (Ours) |
| armadillo | $\times$97 | $\times$62 | $\times$5464 | $\times$**23** |
| bunny | $\times$98 | $\times$64 | $\times$4938 | $\times$**24** |
| buddha | $\times$95 | $\times$64 | $\times$4912 | $\times$**22** |
| dragon | $\times$91 | $\times$63 | $\times$4857 | $\times$**20** |

Table S3. Factor of increased computation time (mean taken over all instances) from *Type-1* to *Type-2* datasets for $p_{outl} = 50\%$.

Therefore, the Hessian of the robust Umeyama cost (Eqn. S2) is given by:

$$\boxed{\mathbf{H} = \sum_i \left( -l_i \frac{\mathbf{g}_{LSQ,i}\mathbf{g}_{LSQ,i}^\top}{\|\mathbf{r}_i\|^2} + m_i \mathbf{H}_{LSQ,i} \right)}$$

□

## C. Computation Time on Synthetic Datasets

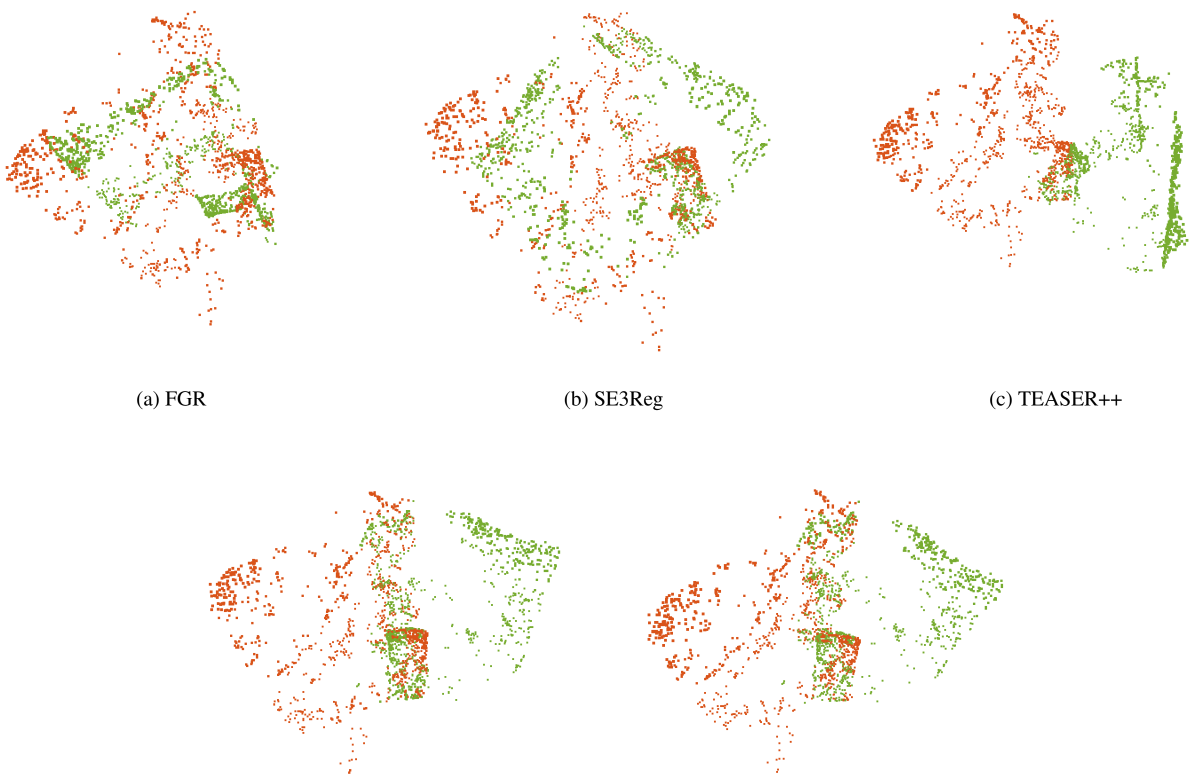
In Sec. 4.1 of the main paper, we presented the results on synthetic datasets. Here, we show the computation time for the two synthetic datasets. It can be seen, from Table S1, that for *Type-1*, which is easier to solve due to small input size ($N = 100$ matches) and low noise ($\eta = 0.01$), RANSAC based method *i.e.* GORE takes the least time. However, for a relatively hard dataset *i.e.* *Type-2*, ($N = 10000$ matches, $\eta = 0.1$), the computation time for GORE is very high (of the order of hours), but M-estimation based methods are much faster. In Table S2, it can be seen that our method (GNCp) is slightly worse than SE3Reg, with TEASER++ taking the highest computation time. In Table S3, we compare the increase in the computation time factor from *Type-1* to *Type-2*. It can be seen that our method has the least increase in the time factor compared to other methods when the problem gets harder to solve.

## D. Visual Registration Results

In Fig. 4 of the main paper, we presented visual results for the alignment with different methods on point clouds with low overlap. In Fig. S1, we show the zoomed visual results on all the methods for the same instance. It can be seen that our method registers the point clouds correctly compared to other methods.

## E. Results on 3DLoMatch Dataset

The 3DLoMatch [3] dataset has a very small inlier fraction ($\ll 50\%$) in many instances as shown in past works [3, 4]. Our method focuses on minimization of a robust cost which does not involve additional preprocessing/pruning to reduce the



(a) FGR

(b) SE3Reg

(c) TEASER++

(d) GNCp (Ours)

(e) Ground Truth

Figure S1. Two point clouds (red and green) with a low overlap in the MIT Lab sequence of 3D Match dataset. Our method registers the point clouds correctly compared to the ground truth reference.

weightage of outliers. As stated in the main paper, minimization of robust cost is reasonable to perform only when the global minimum of the robust cost is close to the ground truth. However, we observe that, only in $\approx 60\%$ of all the 1781 instances in 3DLoMatch, the global minimum is close to the ground truth (*i.e.*, the rotation and translation deviation is less than $15°$ and $0.3$ metres respectively). The mean rotation error (MRE) and the mean translation error (MTE) for the successful instances are $3.07°$ and $10.31$ metres, respectively.

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