We analyze the convergence of the following optimization problem (Eq.(5) in the main paper).

$$\min_{h_1,v_1,v_2,v_3} \sum_{j \neq i} h_j^T h_i$$

s.t. $h_j^T H_{\sim i} v_3 = (q - 2d)_{c-1} v_3 \in R^{c-1}_+$,

$$\|h_i - v_1\|_2^2 + k_i^T (h_i - v_2) + \frac{\mu}{2} \|h_i - v_3\|_2^2$$

$$+ \sum_{j \neq i} k_j^T (h_j^T H_{\sim i} + v_3 - e) + \frac{\mu}{2} \|h_j^T H_{\sim i} + v_3 - e\|_2^2$$

s.t. $v_1 \in V_{box}$, $v_2 \in V_{sph}$, $v_3 \in R^{c-1}_+$,

$$\text{Eq.(1)}$$

where $e = (q - 2d)_{c-1}$, and $k_1$, $k_2$, $k_3$ are Lagrange multipliers.

We adopt the $\ell_p$-box ADMM method to solve Eq.(1), which is shown in the red lines in Algorithm 1. Next we will show that, under mild assumptions, by using this $\ell_p$-box ADMM scheme, the optimization problem Eq.(1) will converge.

We first present two assumptions that are adapted from [1], by using our notations.

**Assumption 1** Let $\mu^t$ be the value of the parameter $\mu$ at $t$-th iteration. Then the parameter $\mu$ is a finite value, i.e., $\lim_{t \to \infty} \mu^t \in (0, \infty)$.

**Assumption 2** Let $k^t = (k_1^t, k_2^t, k_3^t)$ be the values of the Lagrange multipliers $k_1$, $k_2$, $k_3$ at $t$-th iteration. Then the parameter sequence $\{k^t\}$ satisfies a) $\sum_{i=0}^{\infty} \|k^{t+1} - k^t\|_2^2 < \infty$, (also which hints that $k^{t+1} - k^t \to 0$) and b) $k^t$ is bounded for all $t$.

Note that, if we replace the range constraint $S_b(z) = \{0 \leq z \leq 1\}$ by a similar range constraint $\{-1 \leq z \leq 1\}$, we can verify that the optimization problem Eq.(1) is a special case of the problem (8) in [1]. By the Proposition 2 in [1], the problem (8) in [1] will converge under some assumptions. Similarly, we have the following theorem.

**Theorem 1** (adapted from Proposition 2 in [1]) Given Assumptions 1 and 2, then any cluster point of the whole variable sequence $\{(h_i^t, v_1^t, v_2^t, v_3^t, k_1^t, k_2^t, k_3^t)\}_0^\infty$ generated by the ADMM method will satisfy the KKT conditions of the problem Eq.(1). Moreover, $\{(h_i^t, v_1^t, v_2^t)\}_0^\infty$ will converge to the binary solutions.

The proof of Theorem 1 is nearly identical to the proof of Proposition 2 in [1] because the optimization problem Eq.(1) can be regarded as a special case of the problem (8) in [1]. The difference in range constraints $S_b(z)$ has no impact on the proof.

Then, we illustrate that, by using the ADMM method in Algorithm 1, Assumption 1 and 2 hold and the optimization problem Eq.(1) will converge.

With the update rules $\mu \leftarrow \min(\mu \mu, \max \mu)$ for the parameter $\mu$, we can see that $\mu$ is always bounded by a constant $\max \mu = 10^{10}$. Hence Assumption 1 holds.

With the update rules for $k_1$, $k_2$ and $k_3$ (Eq.(11) in the main paper) and the stopping criteria $\max(|h_i - v_1|_\infty, |h_i - v_2|_\infty, |h_i^T H_{\sim i} + v_3 - e|_\infty) \leq \epsilon$, we can verify that Assumption 2 holds.

Hence, by applying Theorem 1, the optimization prob-
lem Eq. (1) will converge.

References

[1] Baoyuan Wu and Bernard Ghanem. $\ell_p$-Box ADMM: A Versa-
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