Supplementary Material for Efficient Second-Order Plane Adjustment

Lipu Zhou
Meituan, Beijing, China
zhoulipu@meituan.com

1. Introduction

This document provides the following contents:

- A brief introduction to the implicit function theorem which was used to derive Lemma 1.
- The proof of Lemma 1.
- The proof of Lemma 2.
- The proof of Theorem 1.
- The proof of Lemma 3.
- The proof of Theorem 2.
- The partial derivatives of entries in $\mathbf{M}_i$.

2. Implicit Function Theorem

Here we introduce the implicit function theorem [1]. We used it to derive Lemma 1.

Implicit Function Theorem *Let* $\boldsymbol{f} : \mathbb{R}^{n+m} \to \mathbb{R}^m$ *be a continuously differentiable function, and let* $\mathbb{R}^{n+m}$ *have coordinates* $[\boldsymbol{x}, \boldsymbol{y}]$. *Fix a point* $[\boldsymbol{a}, \boldsymbol{b}] = [a_1, \cdots a_n, b_1 \cdots b_m]$ *with* $f(\boldsymbol{a}, \boldsymbol{b}) = \mathbf{0}$, *where* $\mathbf{0} \in \mathbb{R}^m$ *is the zero vector. If the Jacobian matrix of* f *with respect to* $\boldsymbol{y}$ *is invertible at* $[\boldsymbol{a}, \boldsymbol{b}]$, *then there exists an open set* $\mathbb{U} \subset \mathbb{R}^n$ *containing* $\boldsymbol{a}$ *such that there exists a unique continuously differentiable function* $\boldsymbol{g} : \mathbb{U} \to \mathbf{R}^m$ *such that* $\boldsymbol{g}(\boldsymbol{a}) = \boldsymbol{b}$, *and* $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{g}(\boldsymbol{x})) = 0$ *for all* $\boldsymbol{x} \in \mathbb{U}$. *Moreover, the Jacobian matrix of* $\boldsymbol{g}$ *in* $\mathbb{U}$ *with respect to* $\boldsymbol{x}$ *is given by the matrix product:*

$$\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}}(\boldsymbol{x}) = -\mathbf{J}_{\boldsymbol{f},\boldsymbol{y}}(\boldsymbol{x}, \boldsymbol{g}(\boldsymbol{x}))^{-1} \mathbf{J}_{\boldsymbol{f},\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{g}(\boldsymbol{x})) \tag{1}$$

where $\mathbf{J}_{\boldsymbol{f},\boldsymbol{y}}(\boldsymbol{x}, \boldsymbol{g}(\boldsymbol{x}))$ *is the Jacobian matrix of* $\boldsymbol{f}$ *with respect to* $\boldsymbol{y}$ *at* $[\boldsymbol{x}, \boldsymbol{g}(\boldsymbol{x})]$, *and* $\mathbf{J}_{\boldsymbol{f},\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{g}(\boldsymbol{x}))$ *is the Jacobian matrix of* $\boldsymbol{f}$ *with respect to* $\boldsymbol{x}$ *at* $[\boldsymbol{x}, \boldsymbol{g}(\boldsymbol{x})]$.

From the implicit function theorem, we know that we can get $\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}}(\boldsymbol{x})$ without knowing the exact form of $\boldsymbol{g}(\boldsymbol{x})$.

3. Proof of Lemma 1

$\lambda_{i,3}$ is the smallest root of

$$-\lambda_{i,3}^3 + a_i \lambda_{i,3}^2 + b_i \lambda_{i,3} + c_i = 0. \tag{2}$$

Let us define

$$\boldsymbol{\chi}_i = \begin{bmatrix} \lambda_{i,3}^2 \\ \lambda_{i,3} \\ 1 \end{bmatrix}, \ \boldsymbol{\eta}_i = \begin{bmatrix} a_i \\ b_i \\ c_i \end{bmatrix} \boldsymbol{\kappa}_i = \begin{bmatrix} -3 \\ 2a_i \\ b_i \end{bmatrix}, \boldsymbol{\delta}_{jm}^i = \frac{\partial \boldsymbol{\eta}_i}{\partial x_{jm}} \tag{3}$$

Suppose that $\boldsymbol{\pi}_i$ is observed at $\boldsymbol{x}_j$, as demonstrated in Fig. 1. Assume x_{jm} is the mth element of $\boldsymbol{x}_j$. The following lemma provides the formula of $\frac{\partial \lambda_{i,3}}{\partial x_{jm}}$.

Lemma 1. *Using the notations in (3), we have*

$$\frac{\partial \lambda_{i,3}}{\partial x_{jm}} = -\varphi_i \boldsymbol{\delta}_{jm}^i \cdot \boldsymbol{\chi}_i, \tag{4}$$

where $\cdot$ *represents the dot product and* $\varphi_i = (\boldsymbol{\kappa}_i \cdot \boldsymbol{\chi}_i)^{-1}$ *and* $\boldsymbol{\delta}_{jm}^i = \frac{\partial \boldsymbol{\eta}_i}{\partial x_{jm}}$.

Proof. a_i, b_i, c_i are functions of poses in $\mathbb{X}_i$. Here we only consider one variable x_{jm} of $\boldsymbol{x}_j$ (*i.e.*, the mth entry of $\boldsymbol{x}_j \in \mathbb{X}_i$). To compute $\frac{\partial \lambda_{i,3}}{\partial x_{jm}}$, we treat x_{jm} as the only unknown and other variables in $\mathbb{X}_i$ as constants. Thus, a_i, b_i, c_i are functions of x_{jm}. Then, we define

$$f(x_{jm}, \lambda_{i,3}) = -\lambda_{i,3}^3 + a_i \lambda_{i,3}^2 + b_i \lambda_{i,3} + c_i. \tag{5}$$

Then we have

$$\begin{aligned} \frac{\partial f}{\partial \lambda_{i,3}} &= -3\lambda_{i,3}^2 + 2a_i \lambda_{i,3} + b_i, \\ \frac{\partial f}{\partial x_{jm}} &= \lambda_{i,3}^2 \frac{\partial a_i}{\partial x_{jm}} + \lambda_{i,3} \frac{\partial b_i}{\partial x_{jm}} + \frac{\partial c_i}{\partial x_{jm}}. \end{aligned} \tag{6}$$

According to the definition of $\boldsymbol{\delta}_{jm}^i$ in (4), it has the form

$$\boldsymbol{\delta}_{jm}^i = \frac{\partial \boldsymbol{\eta}_j}{\partial x_{jm}} = \begin{bmatrix} \frac{\partial a_i}{x_{jm}} \\ \frac{\partial b_i}{x_{jm}} \\ \frac{\partial c_i}{x_{jm}} \end{bmatrix} \tag{7}$$

Substituting the definitions of φ_i, $\boldsymbol{\chi}_i$, $\boldsymbol{\kappa}_i$, and $\boldsymbol{\delta}_{jm}$ into (6), we have

$$\begin{aligned}\frac{\partial f}{\partial \lambda_{i,3}} &= \boldsymbol{\kappa}_i \cdot \boldsymbol{\chi}_i = \varphi_i^{-1},\\ \frac{\partial f}{\partial x_{jm}} &= \boldsymbol{\delta}_{jm} \cdot \boldsymbol{\chi}_i.\end{aligned} \tag{8}$$

Using the implicit function theorem, for $f(x_{jm}, \lambda_{i,3}) = 0$, we have

$$\frac{\partial \lambda_{i,3}}{\partial x_{jm}} = -\frac{\frac{\partial f}{\partial x_{jm}}}{\frac{\partial f}{\partial \lambda_{i,3}}} \tag{9}$$

Substituting (8) into (9), we finally get

$$\frac{\partial \lambda_{i,3}}{\partial x_{jm}} = -\varphi_i \boldsymbol{\delta}_{jm}^i \cdot \boldsymbol{\chi}_i. \tag{10}$$

□

4. Proof of Lemma 2

Suppose that $\boldsymbol{\pi}_i$ is observed at $\boldsymbol{x}_j$ and $\boldsymbol{x}_k$, as demonstrated in Fig. 1. Assume x_{jm} is the mth element of $\boldsymbol{x}_j$, and x_{kn} is the nth element of $\boldsymbol{x}_k$. The following lemma provides the formula of $\frac{\partial^2 \lambda_{i,3}}{\partial x_{jm} \partial x_{kn}}$.

Lemma 2. *Using notations in (3) and (4), we have*

$$\frac{\partial^2 \lambda_{i,3}}{\partial x_{jm} \partial x_{kn}} = -\varphi_i \left(\boldsymbol{\delta}_{jm}^i \cdot \frac{\partial \boldsymbol{\chi}_i}{\partial x_{kn}} + \boldsymbol{\chi}_i \cdot \frac{\partial \boldsymbol{\delta}_{jm}^i}{\partial x_{kn}} - \frac{\partial \lambda_{i,3}}{\partial x_{jm}} \frac{\partial \varphi_i^{-1}}{\partial x_{kn}} \right). \tag{11}$$

Proof. We first compute the partial derivative of $\frac{\partial \lambda_{i,3}}{\partial x_{jm}}$ in (4) with respect to x_{kn}. According to the production rule of calculus, we have

$$\frac{\partial^2 \lambda_{i,3}}{\partial x_{jm} \partial x_{km}} = -\varphi_i \boldsymbol{\delta}_{jm}^i \cdot \frac{\partial \boldsymbol{\chi}_i}{\partial x_{kn}} - \varphi_i \boldsymbol{\chi}_i \cdot \frac{\partial \boldsymbol{\delta}_{jm}^i}{\partial x_{kn}} - \boldsymbol{\delta}_{jm}^i \cdot \boldsymbol{\chi}_i \frac{\partial \varphi_i}{\partial x_{kn}}. \tag{12}$$

Let us first focus on the term $\frac{\partial \varphi_i}{\partial x_{kn}}$ in (12). As $\varphi_i = (\boldsymbol{\kappa}_i \cdot \boldsymbol{\chi}_i)^{-1}$, we have

$$\frac{\partial \varphi_i}{\partial x_{kn}} = -\left(\boldsymbol{\kappa}_i \cdot \boldsymbol{\chi}_i\right)^{-2} \frac{\partial \left(\boldsymbol{\kappa}_i \cdot \boldsymbol{\chi}_i\right)}{\partial x_{kn}} = -\varphi_i^2 \frac{\partial \varphi_i^{-1}}{\partial x_{kn}}. \tag{13}$$

Now let us consider $\boldsymbol{\delta}_{jm}^i \cdot \boldsymbol{\chi}_i \frac{\partial \varphi_i}{\partial x_{kn}}$ that is the third term in (12). Using (13), we have

$$\begin{aligned}\boldsymbol{\delta}_{jm}^i \cdot \boldsymbol{\chi}_i \frac{\partial \varphi_i}{\partial x_{kn}} &= -\boldsymbol{\delta}_{jm}^i \cdot \boldsymbol{\chi}_i \varphi_i^2 \frac{\partial \varphi_i^{-1}}{\partial x_{kn}} \\ &= -\varphi_i \underbrace{\left(\varphi_i \boldsymbol{\delta}_{jm}^i \cdot \boldsymbol{\chi}_i\right)}_{\frac{\partial \lambda_{i,3}}{\partial x_{jm}}} \frac{\partial \varphi_i^{-1}}{\partial x_{kn}} \\ &= -\varphi_i \frac{\partial \lambda_{i,3}}{\partial x_{jm}} \frac{\partial \varphi_i^{-1}}{\partial x_{kn}}\end{aligned} \tag{14}$$

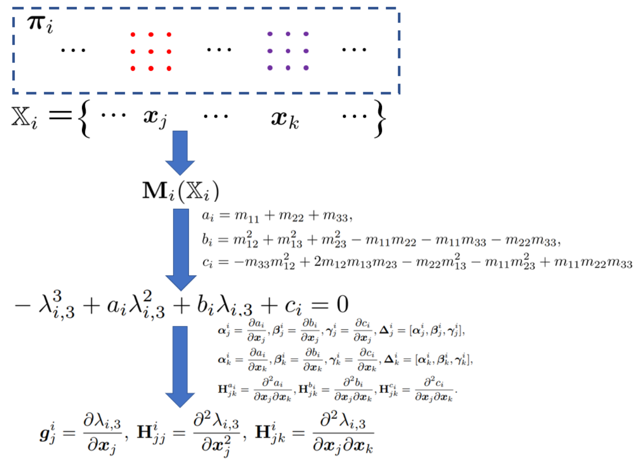

Figure 1. Summary of our algorithm. $\mathbb{X}_i$ is the set of poses which can see $\boldsymbol{\pi}_i$. Here $\boldsymbol{x}_j \in \mathbb{X}_i$ and $\boldsymbol{x}_k \in \mathbb{X}_i$. The key point of our algorithm is to get $\boldsymbol{g}_j^i = \frac{\partial \lambda_{i,3}}{\partial \boldsymbol{x}_j}$, $\mathbf{H}_{jj}^i = \frac{\partial^2 \lambda_{i,3}}{\partial \boldsymbol{x}_j^2}$, and $\mathbf{H}_{jk}^i = \frac{\partial^2 \lambda_{i,3}}{\partial \boldsymbol{x}_j \partial \boldsymbol{x}_k}$. Theorem 1 provides their formulas. From Theorem 1, we know that the partial derivatives of a_i, b_i, and c_i with respect to $\boldsymbol{x}_j$ and $\boldsymbol{x}_k$ in (16) are crucial. As a_i, b_i, and c_i are polynomials with respect to m_{ef} ($e = 1, 2, 3$ and $f = 1, 2, 3$), to get the partial derivatives of a_i, b_i, and c_i with respect to $\boldsymbol{x}_j$ and $\boldsymbol{x}_k$, we need to compute the partial derivatives of m_{ef} with respect to $\boldsymbol{x}_j$ and $\boldsymbol{x}_k$. Section 8 proves these partial derivatives of m_{ef}.

Substituting (14) into (12), we have

$$\begin{aligned}\frac{\partial^2 \lambda_{i,3}}{\partial x_{jm} \partial x_{km}} &= -\varphi_i \boldsymbol{\delta}_{jm}^i \cdot \frac{\partial \boldsymbol{\chi}_i}{\partial x_{kn}} - \varphi_i \boldsymbol{\chi}_i \cdot \frac{\partial \boldsymbol{\delta}_{jm}^i}{\partial x_{kn}} + \varphi_i \frac{\partial \lambda_{i,3}}{\partial x_{jm}} \frac{\partial \varphi_i^{-1}}{\partial x_{kn}} \\ &= -\varphi_i \left(\boldsymbol{\delta}_{jm}^i \cdot \frac{\partial \boldsymbol{\chi}_i}{\partial x_{kn}} + \boldsymbol{\chi}_i \cdot \frac{\partial \boldsymbol{\delta}_{jm}^i}{\partial x_{kn}} - \frac{\partial \lambda_{i,3}}{\partial x_{jm}} \frac{\partial \varphi_i^{-1}}{\partial x_{kn}} \right)\end{aligned} \tag{15}$$

□

5. Proof of Theorem 1

Let us define

$$\begin{aligned}&\boldsymbol{\alpha}_j^i = \frac{\partial a_i}{\partial \boldsymbol{x}_j}, \boldsymbol{\beta}_j^i = \frac{\partial b_i}{\partial \boldsymbol{x}_j}, \boldsymbol{\gamma}_j^i = \frac{\partial c_i}{\partial \boldsymbol{x}_j}, \boldsymbol{\Delta}_j^i = [\boldsymbol{\alpha}_j^i, \boldsymbol{\beta}_j^i, \boldsymbol{\gamma}_j^i],\\ &\boldsymbol{\alpha}_k^i = \frac{\partial a_i}{\partial \boldsymbol{x}_k}, \boldsymbol{\beta}_k^i = \frac{\partial b_i}{\partial \boldsymbol{x}_k}, \boldsymbol{\gamma}_k^i = \frac{\partial c_i}{\partial \boldsymbol{x}_k}, \boldsymbol{\Delta}_k^i = [\boldsymbol{\alpha}_k^i, \boldsymbol{\beta}_k^i, \boldsymbol{\gamma}_k^i],\\ &\mathbf{H}_{jk}^{a_i} = \frac{\partial^2 a_i}{\partial \boldsymbol{x}_j \partial \boldsymbol{x}_k}, \mathbf{H}_{jk}^{b_i} = \frac{\partial^2 b_i}{\partial \boldsymbol{x}_j \partial \boldsymbol{x}_k}, \mathbf{H}_{jk}^{c_i} = \frac{\partial^2 c_i}{\partial \boldsymbol{x}_j \partial \boldsymbol{x}_k}.\end{aligned} \tag{16}$$

Using the above lemmas and notations, we can derive $\boldsymbol{g}_j^i$ and $\mathbf{H}_{jk}^i$.

Theorem 1. *Using the notations in (3), (4) and (16), $\boldsymbol{g}_j^i$ and $\mathbf{H}_{jk}^i$ have the forms*

$$\begin{aligned}\boldsymbol{g}_j^i &= -\varphi_i \boldsymbol{\Delta}_j^i \boldsymbol{\chi}_i,\\ \mathbf{H}_{jk}^i &= \varphi_i \left(\mathbf{K}_{jk}^i - \lambda_{3,i}^2 \mathbf{H}_{jk}^{a_i} - \lambda_{3,i} \mathbf{H}_{jk}^{b_i} - \mathbf{H}_{jk}^{c_i} \right),\end{aligned} \tag{17}$$

where $\mathbf{K}^i_{jk} = \boldsymbol{g}^i_j \boldsymbol{u}^T - \boldsymbol{v}(\boldsymbol{g}^i_k)^T$, $\boldsymbol{u} = 2\lambda_{i,3}\boldsymbol{\alpha}^i_k + \boldsymbol{\beta}^i_k + (2a - 6\lambda_{i,3})\boldsymbol{g}^i_k$, *and* $\boldsymbol{v} = 2\lambda_{i,3}\boldsymbol{\alpha}^i_j + \boldsymbol{\beta}^i_j$, *and similar to* $\boldsymbol{g}^i_j$, $\boldsymbol{g}^i_k = -\varphi_i \boldsymbol{\Delta}^i_k \boldsymbol{\chi}_i$ *is the gradient block for* $\boldsymbol{x}_k$.

Proof. Expanding the definition of $\boldsymbol{\Delta}^i_j$ in (16), we have

$$\boldsymbol{\Delta}^i_j = \begin{bmatrix} \frac{\partial a_i}{\partial x_{j1}} & \frac{\partial b_i}{\partial x_{j1}} & \frac{\partial c_i}{\partial x_{j1}} \\ \vdots & \vdots & \vdots \\ \frac{\partial a_i}{\partial x_{jm}} & \frac{\partial b_i}{\partial x_{jm}} & \frac{\partial c_i}{\partial x_{jm}} \\ \vdots & \vdots & \vdots \\ \frac{\partial a_i}{\partial x_{j6}} & \frac{\partial b_i}{\partial x_{j6}} & \frac{\partial c_i}{\partial x_{j6}} \end{bmatrix} \in \mathbb{R}^{6\times 6} \tag{18}$$

Substituting the definition of $\boldsymbol{\delta}^i_{jm}$ in (7) into (18), we can write $\boldsymbol{\Delta}^i_j$ as

$$\boldsymbol{\Delta}^i_j = \begin{bmatrix} {\boldsymbol{\delta}^i_{j1}}^T \\ \vdots \\ {\boldsymbol{\delta}^i_{jm}}^T \\ \vdots \\ {\boldsymbol{\delta}^i_{j6}}^T \end{bmatrix} \tag{19}$$

Assume x_{jm} is the mth variable of $\boldsymbol{x}_j$. Then $\boldsymbol{g}^i_j$ can be written as

$$\boldsymbol{g}^i_j = \frac{\partial \lambda_{i,3}}{\partial \boldsymbol{x}_j} = \begin{bmatrix} \frac{\partial \lambda_{i,3}}{\partial x_{j1}} \\ \vdots \\ \frac{\partial \lambda_{i,3}}{\partial x_{jm}} \\ \vdots \\ \frac{\partial \lambda_{i,3}}{\partial x_{j6}} \end{bmatrix} \in \mathbb{R}^6. \tag{20}$$

Here $\frac{\partial \lambda_{i,3}}{\partial x_{jm}}$ is the mth element of $\boldsymbol{g}^i_j$. Substituting (19) into $\boldsymbol{g}^i_j$ in (17), we can obtain the formula of $\frac{\partial \lambda_{i,3}}{\partial x_{jm}}$ as

$$\frac{\partial \lambda_{i,3}}{\partial x_{jm}} = -\varphi_i {\boldsymbol{\delta}^i_{jm}}^T \boldsymbol{\chi}_i = -\varphi_i \boldsymbol{\delta}^i_{jm} \cdot \boldsymbol{\chi}_i. \tag{21}$$

The above formula is what we proved in Lemma 1. Using Lemma 1, we known that the formula of $\boldsymbol{g}^i_j$ in (17) is correct.

Now we consider the Hessian matrix. According to the definitions of $\boldsymbol{\delta}^i_{kn}$, $\boldsymbol{\chi}_i$, and $\boldsymbol{\delta}^i_{jm}$, we have

$$\boldsymbol{\delta}^i_{kn} = \begin{bmatrix} \frac{\partial a_i}{\partial x_{kn}} \\ \frac{\partial b_i}{\partial x_{kn}} \\ \frac{\partial c_i}{\partial x_{kn}} \end{bmatrix}, \frac{\partial \boldsymbol{\chi}_i}{\partial x_{j,m}} = \begin{bmatrix} 2\lambda^2_{i,3} \frac{\partial \lambda_{i,3}}{\partial x_{j,m}} \\ \frac{\partial \lambda_{i,3}}{\partial x_{j,m}} \\ 0 \end{bmatrix}, \frac{\partial \boldsymbol{\delta}^i_{jm}}{\partial x_{kn}} = \begin{bmatrix} \frac{\partial^2 a_i}{\partial x_{jm}\partial x_{kn}} \\ \frac{\partial^2 b_i}{\partial x_{jm}\partial x_{kn}} \\ \frac{\partial^2 c_i}{\partial x_{jm}\partial x_{kn}} \end{bmatrix} \tag{22}$$

In addition, using the definition of φ_i in (4), we obtain

$$\begin{aligned} \frac{\partial \varphi_i^{-1}}{\partial x_{km}} &= \boldsymbol{\chi}_i \cdot \frac{\partial \boldsymbol{\kappa}_i}{\partial x_{km}} + \boldsymbol{\kappa}_i \cdot \frac{\partial \boldsymbol{\chi}_i}{\partial x_{km}} \\ &= 2\lambda_{i,3} \frac{\partial a_i}{\partial x_{kn}} + \frac{\partial b_i}{\partial x_{kn}} + (2a - 6\lambda_{i,3}) \frac{\partial \lambda_{i,3}}{\partial x_{kn}} \end{aligned} \tag{23}$$

Let us denote the entry in the mth row and nth column of $\mathbf{H}^i_{jk}$ as $\mathbf{H}^i_{jk}(m,n)$. Using the formula of $\mathbf{H}^i_{jk}$ in (17) and the variables in (22) and (23), we have

$$\begin{aligned} \mathbf{H}^i_{jk}(m,n) = &- \underbrace{\varphi_i \frac{\partial \lambda_{i,3}}{\partial x_{kn}} \left(2\lambda_{i,3} \frac{\partial a_i}{\partial x_{jm}} + \frac{\partial b_i}{\partial x_{jm}} \right)}_{\varphi_i \boldsymbol{\delta}^i_{kn} \cdot \frac{\partial \boldsymbol{\chi}_i}{\partial x_{jm}}} - \\ &\underbrace{\varphi_i \left(\lambda^2_{i,3} \frac{\partial^2 a_i}{\partial x_{jm}\partial x_{kn}} + \lambda_{i,3} \frac{\partial^2 b_i}{\partial x_{jm}\partial x_{kn}} + \frac{\partial^2 c_i}{\partial x_{jm}\partial x_{kn}} \right)}_{\varphi_i \boldsymbol{\chi}_i \cdot \frac{\partial \boldsymbol{\delta}^i_{jm}}{\partial x_{kn}}} + \\ &\underbrace{\varphi_i \frac{\partial \lambda_{i,3}}{\partial x_{jm}} \left(2\lambda_{i,3} \frac{\partial a_i}{\partial x_{kn}} + \frac{\partial b_i}{\partial x_{kn}} + (2a - 6\lambda_{i,3}) \frac{\partial \lambda_{i,3}}{\partial x_{kn}} \right)}_{\varphi_i \frac{\partial \lambda_{i,3}}{\partial x_{jm}} \frac{\partial \varphi_i^{-1}}{\partial x_{km}}} \\ = &- \varphi_i \left(\boldsymbol{\delta}^i_{jm} \cdot \frac{\partial \boldsymbol{\chi}_i}{\partial x_{kn}} + \boldsymbol{\chi}_i \cdot \frac{\partial \boldsymbol{\delta}^i_{jm}}{\partial x_{kn}} - \frac{\partial \lambda_{i,3}}{\partial x_{jm}} \frac{\partial \varphi_i^{-1}}{\partial x_{km}} \right) \end{aligned} \tag{24}$$

On the other hand, we know

$$\mathbf{H}^i_{jk}(m,n) = \frac{\partial^2 \lambda_{i,3}}{\partial x_{jm}\partial x_{kn}}. \tag{25}$$

Comparing (11) and (24), we know that the formula of $\mathbf{H}^i_{jk}$ in (17) is correct. □

6. Proof of Lemma 3

$\lambda_{i,3}$ is the smallest eigenvalue of $\mathbf{M}_i(\mathbb{X}_i)$:

$$\mathbf{M}_i(\mathbb{X}_i) = \sum_{j\in obs(\boldsymbol{\pi}_i)} \mathbf{S}_{ij} - N_i \bar{\boldsymbol{p}}_i \bar{\boldsymbol{p}}_i^T, \tag{26}$$

where $\bar{\boldsymbol{p}}_i = \frac{1}{N_i} \sum_{j\in obs(\boldsymbol{\pi}_i)} \sum_{k=1}^{N_{ij}} \boldsymbol{p}^g_{ijk}$ and $\mathbf{S}_{ij} = \sum_{k=1}^{N_{ij}} \boldsymbol{p}^g_{ijk} \left(\boldsymbol{p}^g_{ijk}\right)^T$. Here the elements in $\mathbf{M}_i$, $\mathbf{S}_{ij}$ and $\bar{\boldsymbol{p}}_i$ in (26) are all functions of the poses in $\mathbb{X}_i$. As $\boldsymbol{p}^g_{ijk} = \mathbf{T}_j \tilde{p}_{ijk}$, we have

$$\begin{aligned} \mathbf{S}_{ij} &= \mathbf{T}_j \underbrace{\sum_{k=1}^{N_{ij}} \tilde{\boldsymbol{p}}_{ijk} \tilde{\boldsymbol{p}}_{ijk}^T}_{\mathbf{U}_{ij}} \mathbf{T}_j^T = \mathbf{T}_j \mathbf{U}_{ij} \mathbf{T}_j^T, \\ \bar{\boldsymbol{p}}_i &= \frac{1}{N_i} \sum_{j\in obs(\boldsymbol{\pi}_i)} \mathbf{T}_j \underbrace{\sum_{k=1}^{N_{ij}} \tilde{\boldsymbol{p}}_{ijk}}_{\tilde{\boldsymbol{p}}_{ij}} = \frac{1}{N_i} \sum_{j\in obs(\boldsymbol{\pi}_i)} \mathbf{T}_j \tilde{\boldsymbol{p}}_{ij}. \end{aligned} \tag{27}$$

Here $\mathbf{U}_{ij}$ and $\tilde{\boldsymbol{p}}_{ij}$ in (27) are constants.

Lemma 3. *In terms of* $\boldsymbol{x}_j$ *and* $\boldsymbol{x}_k$, $\bar{\boldsymbol{p}}_i$ *in (27) has the form*

$$\bar{\boldsymbol{p}}_i(\boldsymbol{x}_j, \boldsymbol{x}_k) = \mathbf{T}_j \boldsymbol{q}_{ij} + \mathbf{T}_k \boldsymbol{q}_{ik} + \boldsymbol{c}_{ijk}, \tag{28}$$

where $\boldsymbol{q}_{ij} = \frac{1}{N_i}\tilde{\boldsymbol{p}}_{ij}$*,* $\boldsymbol{q}_{ik} = \frac{1}{N_i}\tilde{\boldsymbol{p}}_{ik}$*, and* $\boldsymbol{c}_{ijk} = \frac{1}{N_i}\sum_{n\in\mathbb{O}_{jk}}\mathbf{T}_n\tilde{\boldsymbol{p}}_{in}$*. Here* $\mathbb{O}_{jk} = obs(\boldsymbol{\pi}_i) - \{j,k\}$ *represents the set of indexes of the poses that can observe* $\boldsymbol{\pi}_i$*, excluding the* $\boldsymbol{j}$*th and* $\boldsymbol{k}$*th poses.*

In terms of $\boldsymbol{x}_j$*,* $\bar{\boldsymbol{p}}_i$ *has the form*

$$\bar{\boldsymbol{p}}_i(\boldsymbol{x}_j) = \mathbf{T}_j\mathbf{q}_{ij} + \boldsymbol{c}_{ij}, \tag{29}$$

where $\boldsymbol{c}_{ij} = \mathbf{T}_k\mathbf{q}_{ik} + \boldsymbol{c}_{ijk}$*.*

Proof. For $j \in obs(\boldsymbol{\pi}_i)$ and $k \in obs(\boldsymbol{\pi}_i)$, we take $\frac{1}{N_i}\mathbf{T}_j\tilde{\boldsymbol{p}}_{ij}$ and $\frac{1}{N_i}\mathbf{T}_k\tilde{\boldsymbol{p}}_{ik}$ out the summation. Then, we can write $\bar{\boldsymbol{p}}_i$ as

$$\begin{aligned}\bar{\boldsymbol{p}}_i(\boldsymbol{x}_j,\boldsymbol{x}_k) &= \mathbf{T}_j\underbrace{\frac{1}{N_i}\tilde{\boldsymbol{p}}_{ij}}_{\boldsymbol{q}_{ij}} + \mathbf{T}_k\underbrace{\frac{1}{N_i}\tilde{\boldsymbol{p}}_{ik}}_{\boldsymbol{q}_{ik}} + \underbrace{\frac{1}{N_i}\sum_{n\in\mathbb{O}_{jk}}\mathbf{T}_n\tilde{\boldsymbol{p}}_{in}}_{\boldsymbol{c}_{ijk}}.\\ &= \mathbf{T}_j\boldsymbol{q}_{ij} + \mathbf{T}_k\boldsymbol{q}_{ik} + \boldsymbol{c}_{ijk}.\end{aligned} \tag{30}$$

For $j \in obs(\boldsymbol{\pi}_i)$, we can write (30) as

$$\begin{aligned}\bar{\boldsymbol{p}}_i(\boldsymbol{x}_j,\boldsymbol{x}_k) &= \mathbf{T}_j\boldsymbol{q}_{ij} + \underbrace{\mathbf{T}_k\boldsymbol{q}_{ik} + \boldsymbol{c}_{ijk}}_{\boldsymbol{c}_{ij}}\\ &= \mathbf{T}_j\boldsymbol{q}_{ij} + \boldsymbol{c}_{ij}\end{aligned} \tag{31}$$

□

7. Proof of Theorem 2

Theorem 2. *In terms of* $\boldsymbol{x}_j$*,* $\mathbf{M}_i$ *in (26) can be written as*

$$\mathbf{M}_i(\boldsymbol{x}_j) = \mathbf{T}_j\mathbf{Q}_j^i\mathbf{T}_j^T + \mathbf{T}_j\mathbf{K}_j^i + (\mathbf{K}_j^i)^T\mathbf{T}_j^T + \mathbf{C}_j^i, \tag{32}$$

where $\mathbf{Q}_j^i = \mathbf{U}_{ij} - N_j\boldsymbol{q}_{ij}\boldsymbol{q}_{ij}^T$ *and* $\mathbf{K}_j^i = -N_i\boldsymbol{q}_{ij}\boldsymbol{c}_{ij}^T$*. Here* $\mathbf{U}_{ij}$ *and* $\boldsymbol{q}_{ij}$ *are defined in (27) and (28), respectively.*

In terms of $\boldsymbol{x}_j$ *and* $\boldsymbol{x}_k$*,* $\mathbf{M}_i$ *can be written as*

$$\mathbf{M}_i(\boldsymbol{x}_j,\boldsymbol{x}_k) = \mathbf{T}_j\mathbf{O}_{jk}^i\mathbf{T}_k^T + \mathbf{T}_k(\mathbf{O}_{jk}^i)^T\mathbf{T}_j^T + \mathbf{C}_{jk}^i. \tag{33}$$

where $\mathbf{O}_{jk}^i = -N_i\boldsymbol{q}_{ij}\boldsymbol{q}_{ik}^T$*.*

Proof. Substituting (31) into (26) and using the formula of $\mathbf{S}_{ij} = \mathbf{T}_j\mathbf{U}\mathbf{T}_j^T$ in (27), we have

$$\begin{aligned}\mathbf{M}_i(\boldsymbol{x}_i) = &\sum_{j\in obs(\boldsymbol{\pi}_i)}\mathbf{S}_{ij} - N_i(\mathbf{T}_j\boldsymbol{q}_{ij}+\boldsymbol{c}_{ij})(\mathbf{T}_j\boldsymbol{q}_{ij}+\boldsymbol{c}_{ij})^T\\ =&\mathbf{S}_{ij} - \mathbf{T}_j\left(N_i\boldsymbol{q}_{ij}\boldsymbol{q}_{ij}^T\right)\mathbf{T}_j^T - \mathbf{T}_j\underbrace{\left(N_j\boldsymbol{q}_{ij}\boldsymbol{c}_{ij}^T\right)}_{-\mathbf{K}_j^i} -\\ &\underbrace{\left(N_j\boldsymbol{c}_{ij}\boldsymbol{p}_{ij}^T\right)}_{(-\mathbf{K}_j^i)^T}\mathbf{T}_j^T + \underbrace{\sum_{\substack{n\in obs(\boldsymbol{\pi}_i)\\ n\neq j}}\mathbf{S}_{in} - N_j\boldsymbol{c}_{ij}\boldsymbol{c}_{ij}^T}_{\mathbf{C}_j^i}\\ =&\mathbf{T}_j\mathbf{U}_{ij}\mathbf{T}_j^T - \mathbf{T}_j\left(N_i\boldsymbol{q}_{ij}\boldsymbol{q}_{ij}^T\right)\mathbf{T}_j^T + \mathbf{T}_j\mathbf{K}_j^i + \left(\mathbf{K}_j^i\right)^T\mathbf{T}_j^T + \mathbf{C}_j^i\\ =&\mathbf{T}_j\underbrace{\left(\mathbf{U}_{ij} - N_j\boldsymbol{q}_{ij}\boldsymbol{q}_{ij}^T\right)}_{\mathbf{Q}_j^i}\mathbf{T}_j^T + \mathbf{T}_j\mathbf{K}_j^i + \left(\mathbf{K}_j^i\right)^T\mathbf{T}_j^T + \mathbf{C}_j^i\\ =&\mathbf{T}_j\mathbf{Q}_j^i\mathbf{T}_j^T + \mathbf{T}_j\mathbf{K}_j^i + (\mathbf{K}_j^i)^T\mathbf{T}_j^T + \mathbf{C}_j^i\end{aligned}$$

Thus we get the formula of $\mathbf{M}_i(\boldsymbol{x}_j)$ in (32). Now let us prove (33). Let us define

$$\begin{aligned}\mathbf{E}_j^i &= \mathbf{T}_j\boldsymbol{q}_{ij}\left(\mathbf{T}_j\boldsymbol{q}_{ij} + \boldsymbol{c}_{ijk}\right)^T\\ \mathbf{E}_k^i &= \mathbf{T}_k\boldsymbol{q}_{ik}\left(\mathbf{T}_k\boldsymbol{q}_{ik} + \boldsymbol{c}_{ijk}\right)^T\end{aligned} \tag{34}$$

Substituting (30) into (26), we obtain

$$\begin{aligned}\mathbf{M}_i(\boldsymbol{x}_j,\boldsymbol{x}_k) =&\mathbf{T}_j\underbrace{\left(-N\boldsymbol{q}_{ij}\boldsymbol{q}_{ik}^T\right)}_{\mathbf{O}_j^i}\mathbf{T}_k^T + \mathbf{T}_j\underbrace{\left(-N\boldsymbol{q}_{ik}\boldsymbol{q}_{ij}^T\right)}_{(\mathbf{O}_j^i)^T}\mathbf{T}_k^T +\\ &\underbrace{\sum_{j\in obs(\boldsymbol{\pi}_i)}\mathbf{S}_{ij} - N_i\left(\mathbf{E}_j^i + \mathbf{E}_k^i + \boldsymbol{c}_{ijk}\bar{\boldsymbol{p}}_i(\boldsymbol{x}_j,\boldsymbol{x}_k)^T\right)}_{\mathbf{C}_{jk}^i}\\ =&\mathbf{T}_j\mathbf{O}_{jk}^i\mathbf{T}_k^T + \mathbf{T}_k(\mathbf{O}_{jk}^i)^T\mathbf{T}_j^T + \mathbf{C}_{jk}^i\end{aligned} \tag{35}$$

Thus we get the formula of $\mathbf{M}_i(\boldsymbol{x}_j,\boldsymbol{x}_k)$ in (33). □

8. Partial Derivatives of Entries in $\mathbf{M}_i$

As illustrated in Fig. 1, the derivatives of a_i, b_i and c_i in (16) are the crux to get $\boldsymbol{g}_j^i$, $\mathbf{H}_{jj}^i$, and $\mathbf{H}_{jk}^i$. The a_i, b_i and c_i in (2) are first-, second-, and third-order polynomials with respect to the elements in $\mathbf{M}_i$, respectively. Let us denote the eth row and fth column entry of $\mathbf{M}_i$ as m_{ef}. According to the chain rule in calculus, to compute the partial derivatives in (16), we have to calculate

$$\frac{\partial m_{ef}}{\partial \boldsymbol{x}_j},\ \frac{\partial^2 m_{ef}}{\partial \boldsymbol{x}_j^2},\ \text{and}\ \frac{\partial^2 m_{ef}}{\partial \boldsymbol{x}_j\partial \boldsymbol{x}_k}. \tag{36}$$

From our paper, we know that we only need to compute their value at $\boldsymbol{x}_0 = [0;0;0;0;0;0]$. Assume q_{ef}, k_{ef}, and o_{ef} are the eth row and fth column entries of $\mathbf{Q}_j^i$, $\mathbf{K}_j^i$, and $\mathbf{O}_{jk}^i$, respectively. Using Theorem 2, the values of $\frac{\partial m_{ef}}{\partial \boldsymbol{x}_j}$, $\frac{\partial^2 m_{ef}}{\partial \boldsymbol{x}_j\partial \boldsymbol{x}_k}$, and $\frac{\partial^2 m_{ef}}{\partial \boldsymbol{x}_j^2}$ at $\boldsymbol{x}_0$ have the forms in Table 1, Table 2, and Table 3, respectively. With $\frac{\partial m_{ef}}{\partial \boldsymbol{x}_j}$, $\frac{\partial^2 m_{ef}}{\partial \boldsymbol{x}_j\partial \boldsymbol{x}_k}$, and $\frac{\partial^2 m_{ef}}{\partial \boldsymbol{x}_j^2}$, it is easy to get the derivatives of a_i, b_i, and c_i in (16).

a_i, b_i, and c_i are polynomials. Using the relationship between the monomials can simplify the computation. For instance, as shown in Fig. 1, $m_{11}m_{22}$ is a term of b_i, and $m_{11}m_{22}m_{33}$ is a term of c_i. As $m_{11}m_{22}m_{33} = (m_{11}m_{22})\,m_{33}$. The first- and second-order partial derivatives of $m_{11}m_{22}$ can be used to compute the first- and second-order partial derivatives of $m_{11}m_{22}m_{33}$.

References

[1] Implicit Funtion Theorem. https://en.wikipedia.org/wiki/Implicit_function_theorem. 1

$$\left.\frac{\partial m_{11}}{\partial \boldsymbol{x}_j}\right|_{\boldsymbol{x}_j=\boldsymbol{x}_0} = \begin{bmatrix} 0 \\ -2k_{31}-2q_{13} \\ 2k_{21}+2q_{12} \\ -4k_{32}-4q_{23} \\ 2k_{22}-2k_{33}+2q_{22}-2q_{33} \\ 4k_{23}+4q_{23} \end{bmatrix} \qquad \left.\frac{\partial m_{12}}{\partial \boldsymbol{x}_j}\right|_{\boldsymbol{x}_j=\boldsymbol{x}_0} = \begin{bmatrix} 4k_{31}+4q_{13} \\ 2k_{32}+2q_{23} \\ 2k_{33}-2k_{11}-2q_{11}+2q_{33} \\ 0 \\ -2k_{12}-2q_{12} \\ -4k_{13}-4q_{13} \end{bmatrix}$$

$$\left.\frac{\partial m_{13}}{\partial \boldsymbol{x}_j}\right|_{\boldsymbol{x}_j=\boldsymbol{x}_0} = \begin{bmatrix} -4k_{21}-4q_{12} \\ 2k_{11}-2k_{22}+2q_{11}-2q_{22} \\ -2k_{23}-2q_{23} \\ 4k_{12}+4q_{12} \\ 2k_{13}+2q_{13} \\ 0 \end{bmatrix} \qquad \left.\frac{\partial m_{22}}{\partial \boldsymbol{x}_j}\right|_{\boldsymbol{x}_j=\boldsymbol{x}_0} = \begin{bmatrix} 2k_{41}+2q_{14} \\ k_{42}+q_{24} \\ k_{43}+q_{34} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\frac{\partial m_{23}}{\partial \boldsymbol{x}_j}\right|_{\boldsymbol{x}_j=\boldsymbol{x}_0} = \begin{bmatrix} 0 \\ k_{41}+q_{14} \\ 0 \\ 2k_{42}+2q_{24} \\ k_{43}+q_{34} \\ 0 \end{bmatrix} \qquad \left.\frac{\partial m_{33}}{\partial \boldsymbol{x}_j}\right|_{\boldsymbol{x}_j=\boldsymbol{x}_0} = \begin{bmatrix} 0 \\ 0 \\ k_{41}+q_{14} \\ 0 \\ k_{42}+q_{24} \\ 2k_{43}+2q_{34} \end{bmatrix}$$

Table 1. $\frac{\partial m_{ef}}{\partial \boldsymbol{x}_j}$ at $\boldsymbol{x}_j = \boldsymbol{x}_0$.

$$\left.\frac{\partial^2 m_{11}}{\partial \boldsymbol{x}_j \partial \boldsymbol{x}_k}\right|_{\substack{\boldsymbol{x}_j=\boldsymbol{x}_0 \\ \boldsymbol{x}_k=\boldsymbol{x}_0}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8o_{33} & -8o_{32} & 4o_{34} & 0 & 0 \\ 0 & -8o_{23} & 8o_{22} & -4o_{24} & 0 & 0 \\ 0 & 4o_{43} & -4o_{42} & 2o_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \left.\frac{\partial^2 m_{13}}{\partial \boldsymbol{x}_j \partial \boldsymbol{x}_k}\right|_{\substack{\boldsymbol{x}_j=\boldsymbol{x}_0 \\ \boldsymbol{x}_k=\boldsymbol{x}_0}} = \begin{bmatrix} 0 & 4o_{23} & -4o_{22} & 2o_{24} & 0 & 0 \\ 4o_{32} & -4o_{13}-4o_{31} & 4o_{12} & -2o_{14} & 0 & 2o_{34} \\ -4o_{22} & 4o_{21} & 0 & 0 & 0 & -2o_{24} \\ 2o_{42} & -2o_{41} & 0 & 0 & 0 & o_{44} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2o_{43} & -2o_{42} & o_{44} & 0 & 0 \end{bmatrix}$$

$$\left.\frac{\partial^2 m_{22}}{\partial \boldsymbol{x}_j \partial \boldsymbol{x}_k}\right|_{\substack{\boldsymbol{x}_j=\boldsymbol{x}_0 \\ \boldsymbol{x}_k=\boldsymbol{x}_0}} = \begin{bmatrix} 8o_{33} & 0 & -8o_{31} & 0 & -4o_{34} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -8o_{13} & 0 & 8o_{11} & 0 & 4o_{14} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -4o_{43} & 0 & 4o_{41} & 0 & 2o_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \left.\frac{\partial^2 m_{12}}{\partial \boldsymbol{x}_j \partial \boldsymbol{x}_k}\right|_{\substack{\boldsymbol{x}_j=\boldsymbol{x}_0 \\ \boldsymbol{x}_k=\boldsymbol{x}_0}} = \begin{bmatrix} 0 & -4o_{33} & 4o_{32} & -2o_{34} & 0 & 0 \\ -4o_{33} & 0 & 4o_{31} & 0 & 2o_{34} & 0 \\ 4o_{23} & 4o_{13} & -4o_{12}-4o_{21} & 2o_{14} & -2o_{24} & 0 \\ -2o_{43} & 0 & 2o_{41} & 0 & o_{44} & 0 \\ 0 & 2o_{43} & -2o_{42} & o_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left.\frac{\partial^2 m_{33}}{\partial \boldsymbol{x}_j \partial \boldsymbol{x}_k}\right|_{\substack{\boldsymbol{x}_j=\boldsymbol{x}_0 \\ \boldsymbol{x}_k=\boldsymbol{x}_0}} = \begin{bmatrix} 8o_{22} & -8o_{21} & 0 & 0 & 0 & 4o_{24} \\ -8o_{12} & 8o_{11} & 0 & 0 & 0 & -4o_{14} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 4o_{42} & -4o_{41} & 0 & 0 & 0 & 2o_{44} \end{bmatrix} \qquad \left.\frac{\partial^2 m_{23}}{\partial \boldsymbol{x}_j \partial \boldsymbol{x}_k}\right|_{\substack{\boldsymbol{x}_j=\boldsymbol{x}_0 \\ \boldsymbol{x}_k=\boldsymbol{x}_0}} = \begin{bmatrix} -4o_{23}-4o_{32} & 4o_{31} & 4o_{21} & 0 & 2o_{24} & -2o_{34} \\ 4o_{13} & 0 & -4o_{11} & 0 & -2o_{14} & 0 \\ 4o_{12} & -4o_{11} & 0 & 0 & 0 & 2o_{14} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2o_{42} & -2o_{41} & 0 & 0 & 0 & o_{44} \\ -2o_{43} & 0 & 2o_{41} & 0 & o_{44} & 0 \end{bmatrix}$$

Table 2. $\frac{\partial^2 m_{ef}}{\partial \boldsymbol{x}_j \partial \boldsymbol{x}_k}$ at $\boldsymbol{x}_j = \boldsymbol{x}_0$ and $\boldsymbol{x}_k = \boldsymbol{x}_0$.

$$\left.\frac{\partial^2 m_{11}}{\partial \boldsymbol{x}_j^2}\right|_{\boldsymbol{x}_j=\boldsymbol{x}_0} = \begin{bmatrix} 0 & 4k_{21}+4q_{12} & 4k_{31}+4q_{13} & 0 & 0 & 0 \\ 4k_{21}+4q_{12} & 8q_{33}-8q_{11}-8k_{11} & -8q_{23} & 4q_{34} & 0 & 0 \\ 4k_{31}+4q_{13} & -8q_{23} & 8q_{22}-8q_{11}-8k_{11} & -4q_{24} & 0 & 0 \\ 0 & 4q_{34} & -4q_{24} & 2q_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left.\frac{\partial^2 m_{12}}{\partial \boldsymbol{x}_j^2}\right|_{\boldsymbol{x}_j=\boldsymbol{x}_0} = \begin{bmatrix} -4k_{21}-4q_{12} & 2k_{11}+2k_{22}+2q_{11}+2q_{22}-4q_{33} & 2k_{32}+6q_{23} & -2q_{34} & 0 & 0 \\ 2k_{11}+2k_{22}+2q_{11}+2q_{22}-4q_{33} & -4k_{12}-4q_{12} & 2k_{31}+6q_{13} & 0 & 2q_{34} & 0 \\ 2k_{32}+6q_{23} & 2k_{31}+6q_{13} & -4k_{12}-4k_{21}-16q_{12} & 2q_{14} & -2q_{24} & 0 \\ -2q_{34} & 0 & 2q_{14}, 0 & q_{44}, 0 & & \\ 0 & 2q_{34} & -2q_{24} & q_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left.\frac{\partial^2 m_{13}}{\partial \boldsymbol{x}_j^2}\right|_{\boldsymbol{x}_j=\boldsymbol{x}_0} = \begin{bmatrix} -4k_{31}-4q_{13} & 2k_{23}+6q_{23} & 2k_{11}+2k_{33}+2q_{11}-4q_{22}+2q_{33} & 2q_{24} & 0 & 0 \\ 2k_{23}+6q_{23} & -4k_{13}-4k_{31}-16q_{13} & 2k_{21}+6q_{12} & -2q_{14} & 0 & 2q_{34} \\ 2k_{11}+2k_{33}+2q_{11}-4q_{22}+2q_{33} & 2k_{21}+6q_{12} & -4k_{13}-4q_{13} & 0 & 0 & -2q_{24} \\ 2q_{24} & -2q_{14} & 0 & 0 & 0 & q_{44} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2q_{34} & -2q_{24} & q_{44} & 0 & 0 \end{bmatrix}$$

$$\left.\frac{\partial^2 m_{22}}{\partial \boldsymbol{x}_j^2}\right|_{\boldsymbol{x}_j=\boldsymbol{x}_0} = \begin{bmatrix} 8q_{33}-8q_{22}-8k_{22} & 4k_{12}+4q_{12} & -8q_{13} & 0 & -4q_{34} & 0 \\ 4k_{12}+4q_{12} & 0 & 4k_{32}+4q_{23} & 0 & 0 & 0 \\ -8q_{13} & 4k_{32}+4q_{23} & 8q_{11}-8k_{22}-8q_{22} & 0 & 4q_{14} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -4q_{34} & 0 & 4q_{14} & 0 & 2q_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left.\frac{\partial^2 m_{23}}{\partial \boldsymbol{x}_j^2}\right|_{\boldsymbol{x}_j=\boldsymbol{x}_0} = \begin{bmatrix} -4k_{23}-4k_{32}-16q_{23} & 2k_{13}+6q_{13} & 2k_{12}+6q_{12} & 0 & 2q_{24} & -2q_{34} \\ 2k_{13}+6q_{13} & -4k_{32}-4q_{23} & 2k_{22}+2k_{33}-4q_{11}+2q_{22}+2q_{33} & 0 & -2q_{14} & 0 \\ 2k_{12}+6q_{12} & 2k_{22}+2k_{33}-4q_{11}+2q_{22}+2q_{33} & -4k_{23}-4q_{23} & 0 & 0 & 2q_{14}1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2q_{24} & -2q_{14} & 0 & 0 & 0 & q_{44} \\ -2q_{34} & 0 & 2q_{14} & 0 & q_{44} & 0 \end{bmatrix}$$

$$\left.\frac{\partial^2 m_{33}}{\partial \boldsymbol{x}_j^2}\right|_{\boldsymbol{x}_j=\boldsymbol{x}_0} = \begin{bmatrix} 8q_{22}-8k_{33}-8q_{33} & -8q_{12} & 4k_{13}+4q_{13} & 0 & 0 & 4q_{24} \\ -8q_{12} & 8q_{11}-8k_{33}-8q_{33} & 4k_{23}+4q_{23} & 0 & 0 & -4q_{14} \\ 4k_{13}+4q_{13} & 4k_{23}+4q_{23} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 4q_{24} & -4q_{14} & 0 & 0 & 0 & 2q_{44} \end{bmatrix}$$

Table 3. $\frac{\partial^2 m_{ef}}{\partial \boldsymbol{x}_j^2}$ at $\boldsymbol{x}_j = \boldsymbol{x}_0$.