Robust Monocular 3D Human Motion with Lasso-Based Differential Kinematics
@@@ Supplementary Material @@@

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In this supplementary document, we provide technical details on the proof of theorem 1.

1. Computation of the Observation Model

1.1. Model of the body motion

The SMPL's skeleton is considered as a tree of kinematic chains where the root is at the center hip [1]. The parent joints for a given landmark point $\mathbf{q}$ is the set of all joints contained in the serial kinematic chain linking the hip to the joint where the landmark is linked. Every SMPL's joint is 3-rotational degrees of freedom. If the total number of ADoFs is n, then a skeleton motion can be uniquely represented by a real n-vector of angular motion $\mathbf{\Omega} = [\omega_\mathbf{1}, \ldots, \omega_\mathbf{n}]$. Let us assume $\mathcal{F} = \{1, \ldots, n\}$ as the set of global indices of vector $\mathbf{\Omega}$. Let us consider the 3-rotational joints index set $\mathcal{J} = \{1, \ldots, \gamma = 24\}$ of SMPL. For every joint $i \in \mathcal{J}$, its motion is composed of a subset of differential ADoFs $\mathbf{\Omega}_i = [\omega_{3i-2}, \omega_{3i-1}, \omega_{3i}]^\top$. The vector of total articulated motion can be re-written as a concatenation of the group of differential joints' motion $\mathbf{\Omega} = [\mathbf{\Omega}_1, \ldots, \mathbf{\Omega}_{\gamma=24}]$. The twist $\Phi_j \in se(3)$, $j \in \mathcal{J}$ constructed from the unit axis of rotation and translation allows us to obtain the rotation transform thanks to the exponential $e^{\Phi_j} \in SE(3)$ [2]. Where $se(3)$ is the Lie Algebra of the group of rigid transforms $SE(3)$. Let us consider a 3D skeleton's point with constant location $\mathbf{q}_i^0$, $1 \leq i \leq l$, in the reference frame attached to joint $j \in \mathcal{J}$. l being the total number of tracked body's point. The location $\mathbf{q}_i$ of this point in the camera frame is given as

$$\mathbf{q}_i = \mathbf{T}_i \, \mathbf{q}_i^0, \tag{1}$$

where $\mathbf{T}_i \in SE(3)$ is the rigid transform from the camera to the coordinate frame that is attached to $j \in \mathcal{J}$ and is given by

$$\mathbf{T}_i = e^{\Phi_{j_1}} \, \ldots e^{\Phi_{j_m}} e^{\Phi_j}, \; i \in [l] \,. \tag{2}$$

Where $[j_1, \ldots, j_m] \subset \mathcal{J}$ is the list of parents joints of j. j_1 being always the camera to hip joint. The differential 3D displacement of point $\mathbf{q}_i$ due to pure differential 3-rotational joint motion $\mathbf{\Omega}_j$, in the camera frame, is given as

$$\dot{\mathbf{q}}_i = \mathbf{J}_j \left(\mathbf{q}_i\right) \, \mathbf{\Omega}_j, \tag{3}$$

where

$$\mathbf{J}_j(\mathbf{q}_i) = \mathrm{Rot}\left(\frac{\partial \mathbf{T}_i}{\partial \mathbf{\Omega}_j} {\mathbf{T}_i}^{-1}\right) \mathbf{q}_i + \mathrm{Trans}\left(\frac{\partial \mathbf{T}_i}{\partial \mathbf{\Omega}_j} {\mathbf{T}_i}^{-1}\right), \tag{4}$$

is a 3×3 Jacobian matrix. Rot and Trans operators extract respectively rotation and translation parts of an $SE(3)$ matrix. If we consider the effect of all differential 3-rotational joint motions of the global skeleton on a given landmark point, then

$$\dot{\mathbf{q}}_i = [\mathbf{J}_1(\mathbf{q}_i) \, \ldots \, \mathbf{J}_{\gamma=24}(\mathbf{q}_i)] \; \mathbf{\Omega}. \tag{5}$$

Where $\mathbf{J}_j = \mathbf{0}_{3\times 3}$ if joint j is not a parent of the joint to which $\mathbf{q}_i$ is attached. In SMPL model, the translation is considered to be given in the camera frame and directly added to model's vertices. When combined with the rotation motion, the entire differential translation and articulated motion allows us to rewrite equation Eq. 5 as

$$\dot{\mathbf{q}}_i = [\mathbf{J}_1(\mathbf{q}_i) \, \ldots \, \mathbf{J}_{\gamma=24}(\mathbf{q}_i)] \; \mathbf{\Omega} + \mathbf{x}. \tag{6}$$

Where $\mathbf{x} \in \mathbf{R}^3$ is motion in translation between camera and body in camera frame. In this work, we do not distinguish between separate camera or body translation and we consider the link between both entities as non-separable.

1.2. Model of the 2D landmark motion

Let us consider $\mathbf{v}(\mathbf{q}) = \left(\frac{x}{z}, \frac{y}{z}\right)^{\mathrm{T}}$ as the perspective projection of a 3D SMPL body's point $\mathbf{q} = (x, y, z)^{\mathrm{T}}$. If this 3D point undergoes a differential motion, then the 2D projection moves following the linearized formula

$$\begin{aligned} \dot{\mathbf{v}}(\mathbf{q}) &= \begin{bmatrix} \frac{1}{z} & 0 & -\frac{x}{z^2} \\ 0 & \frac{1}{z} & -\frac{y}{z^2} \end{bmatrix} \dot{\mathbf{q}} \qquad (7) \\ &= \mathbf{F}(\mathbf{q}) \, \dot{\mathbf{q}}. \qquad (8) \end{aligned}$$

If we consider a set of l of tracked body's points and integrating equation Eq. 5 in the above equation, then we obtain

$$\begin{bmatrix}\dot{\mathbf{v}}(\mathbf{q}_1)\\ \vdots \\ \dot{\mathbf{v}}(\mathbf{q}_l)\end{bmatrix} = \mathbf{P}\,[\boldsymbol{\Gamma} \quad \mathbf{L}]\,\left[\mathbf{x}^{\mathrm{T}}\,\boldsymbol{\Omega}^{\mathrm{T}}\right]^{\mathrm{T}}. \tag{9}$$

Where $\boldsymbol{\Gamma}$ is a $3l$-identity matrix and $\mathbf{P}$ is a $2l \times 3l$ matrix such that

$$\mathbf{P} = \begin{bmatrix}\mathbf{F}(\mathbf{q}_1) & & \\ & \ddots & \\ & & \mathbf{F}(\mathbf{q}_l)\end{bmatrix} \tag{10}$$

and $\mathbf{L}$ is $3l \times 3\gamma$ matrix such that

$$\mathbf{L} = \begin{bmatrix}\mathbf{J}_1(\mathbf{q}_1) & \dots & \mathbf{J}_n(\mathbf{q}_1)\\ \vdots & \ddots & \vdots \\ \mathbf{J}_1(\mathbf{q}_l) & \dots & \mathbf{J}_n(\mathbf{q}_l)\end{bmatrix}. \tag{11}$$

With the paper notation, we adopt the following contracted expression

$$\mathbf{y} = \mathbf{P}\,\boldsymbol{\Gamma}\,\mathbf{x} + \mathbf{P}\,\mathbf{L}\,\boldsymbol{\Omega}, \tag{12}$$

where $\mathbf{y}$ is the vector of all observed 2D differential displacements in the noise free case. In the presence noise, equation 4 in the main paper follows.

2. Proof of theorem 1

Proof 1 *Let us consider $(\mathbf{x}^\star, \boldsymbol{\Omega}^\star)$ as the global minimum of problem Eq. 6. The Karush-Kuhn-Tucker (KKT) necessary and sufficient conditions of problem Eq. 6 can be derived as follows*

KKT-1:

$$\left(\mathbf{P}\mathbf{L}_{\mathcal{V}^\star}\right)^{\mathrm{T}}\left(\mathbf{z} - \mathbf{P}\,\boldsymbol{\Gamma}\mathbf{x}^\star - \mathbf{P}\,\mathbf{L}_{\mathcal{V}^\star}\,\boldsymbol{\Omega}^\star_{\mathcal{V}^\star}\right) = \alpha\boldsymbol{\Sigma}\left(\boldsymbol{\Omega}^\star_{\mathcal{V}^\star}\right), \tag{13}$$

KKT-2:

$$\left|\left[\mathbf{P}\mathbf{L}\right]_k^{\mathrm{T}}\left(\mathbf{z} - \mathbf{P}\,\boldsymbol{\Gamma}\mathbf{x}^\star - \mathbf{P}\,\mathbf{L}_{\mathcal{V}^\star}\,\boldsymbol{\Omega}^\star_{\mathcal{V}^\star}\right)\right| \le \alpha, \tag{14}$$
$$\textit{for all columns } \left[\mathbf{P}\,\mathbf{L}\right]_k \textit{ not in } \mathbf{L}_{\mathcal{V}^\star}.$$

KKT-3:

$$\left(\mathbf{P}\,\boldsymbol{\Gamma}\right)^{\mathrm{T}}\left(\mathbf{z} - \mathbf{P}\,\boldsymbol{\Gamma}\mathbf{x}^\star - \mathbf{P}\,\mathbf{L}_{\mathcal{V}^\star}\,\boldsymbol{\Omega}^\star_{\mathcal{V}^\star}\right) = 0. \tag{15}$$

*From equation **??** the non-zero concatenated elements of $\boldsymbol{\Omega}^\star$ satisfy $\boldsymbol{\Omega}^\star_{\mathcal{V}^\star} = \left(\mathbf{P}\mathbf{L}_{\mathcal{V}^\star}\right)^{+}\mathbf{z}^\star$. Where $\mathbf{z}^\star = \left(\mathbf{z} - \mathbf{P}\,\boldsymbol{\Gamma}\mathbf{x}^\star\right)$. Condition 2) in theorem 1 can thus be rewritten as*

$$\boldsymbol{\Sigma}\left(\left(\mathbf{P}\mathbf{L}_{\mathcal{V}^\star}\right)^{+}\mathbf{z}^\star\right) = \tag{16}$$
$$\boldsymbol{\Sigma}\left(\left(\mathbf{P}\mathbf{L}_{\mathcal{V}^\star}\right)^{+}\mathbf{z}^\star - \alpha\left(\mathbf{P}\mathbf{L}_{\mathcal{V}^\star}{}^{\mathrm{T}}\mathbf{P}\mathbf{L}_{\mathcal{V}^\star}\right)^{-1}\boldsymbol{\Sigma}\left(\left(\mathbf{P}\mathbf{L}_{\mathcal{V}^\star}\right)^{+}\mathbf{z}^\star\right)\right)$$

If we take any differential kinematic vector $\mathbf{O}$ such that $\mathbf{O}_{\mathcal{V}^\star}$ satisfies

$$\boldsymbol{\Sigma}\left(\mathbf{O}_{\mathcal{V}^\star}\right) = \boldsymbol{\Sigma}\left(\boldsymbol{\Omega}^\star_{\mathcal{V}^\star}\right) \textit{ and} \tag{17}$$
$$\mathbf{O}_{\mathcal{V}^\star} = \left(\mathbf{P}\mathbf{L}_{\mathcal{V}^\star}\right)^{+}\mathbf{z}^\star - \alpha\left(\mathbf{P}\mathbf{L}_{\mathcal{V}^\star}{}^{\mathrm{T}}\mathbf{P}\mathbf{L}_{\mathcal{V}^\star}\right)^{-1}\boldsymbol{\Sigma}\left(\mathbf{O}_{\mathcal{V}^\star}\right). \tag{18}$$

Then it also satisfies the KKT condition of equation 13 as can be seen below

$$\textbf{\textit{KKT-1:}}\left(\mathbf{P}\mathbf{L}_{\mathcal{V}^\star}\right)^{\mathrm{T}}\left(\mathbf{z} - \mathbf{P}\,\boldsymbol{\Gamma}\mathbf{x}^\star - \mathbf{P}\,\mathbf{L}_{\mathcal{V}^\star}\,\mathbf{O}_{\mathcal{V}^\star}\right) = \tag{19}$$
$$\left(\mathbf{P}\mathbf{L}_{\mathcal{V}^\star}\right)^{\mathrm{T}}\Big(\mathbf{z} - \mathbf{P}\,\boldsymbol{\Gamma}\mathbf{x}^\star - \mathbf{P}\,\mathbf{L}_{\mathcal{V}^\star}\,\Big(\left(\mathbf{P}\mathbf{L}_{\mathcal{V}^\star}\right)^{+}\mathbf{z}^\star \tag{20}$$
$$-\alpha\left(\left(\mathbf{P}\mathbf{L}_{\mathcal{V}^\star}\right)^{\mathrm{T}}\mathbf{P}\mathbf{L}_{\mathcal{V}^\star}\right)^{-1}\boldsymbol{\Sigma}\left(\mathbf{O}_{\mathcal{V}^\star}\right)\Big)\Big) =$$
$$\underbrace{\left(\mathbf{P}\mathbf{L}_{\mathcal{V}^\star}\right)^{\mathrm{T}}\left(\mathbf{z} - \mathbf{P}\,\boldsymbol{\Gamma}\mathbf{x}^\star\right) - \left(\mathbf{P}\mathbf{L}_{\mathcal{V}^\star}\right)^{\mathrm{T}}\mathbf{z}^\star}_{=0} \tag{21}$$
$$+\alpha\,\boldsymbol{\Sigma}\left(\mathbf{O}_{\mathcal{V}^\star}\right) =$$
$$\alpha\,\boldsymbol{\Sigma}\left(\mathbf{O}_{\mathcal{V}^\star}\right), \tag{22}$$

*then **KKT-1:** is filled by $\mathbf{O}_{\mathcal{V}^\star}$.*

$$\textbf{\textit{KKT-2:}}\textit{ For all columns } \left[\mathbf{P}\,\mathbf{L}\right]_k \textit{ not in } \mathbf{L}_{\mathcal{V}^\star}, \tag{23}$$
$$\left|\left[\mathbf{P}\mathbf{L}\right]_k^{\mathrm{T}}\left(\mathbf{z} - \mathbf{P}\,\boldsymbol{\Gamma}\mathbf{x}^\star - \mathbf{P}\,\mathbf{L}_{\mathcal{V}^\star}\,\mathbf{O}_{\mathcal{V}^\star}\right)\right| =$$
$$\Big|\left[\mathbf{P}\mathbf{L}\right]_k^{\mathrm{T}}\Big(\mathbf{z} - \mathbf{P}\,\boldsymbol{\Gamma}\mathbf{x}^\star - \mathbf{P}\,\mathbf{L}_{\mathcal{V}^\star}\,\Big(\left(\mathbf{P}\mathbf{L}_{\mathcal{V}^\star}\right)^{+}\mathbf{z}^\star \tag{24}$$
$$-\alpha\left(\mathbf{P}\mathbf{L}_{\mathcal{V}^\star}{}^{\mathrm{T}}\mathbf{P}\mathbf{L}_{\mathcal{V}^\star}\right)^{-1}\boldsymbol{\Sigma}\left(\mathbf{O}_{\mathcal{V}^\star}\right)\Big)\Big)\Big| \tag{25}$$

*As $\mathbf{O}_{\mathcal{V}^\star}$ is the solution of **??**, then it satisfies*

$$\mathbf{P}\,\mathbf{L}_{\mathcal{V}^\star}\,\left(\mathbf{P}\mathbf{L}_{\mathcal{V}^\star}\right)^{+}\mathbf{z}^\star = \mathbf{P}\,\mathbf{L}_{\mathcal{V}^\star}\,\left(\mathbf{P}\mathbf{L}_{\mathcal{V}^\star}\right)^{+}\left(\mathbf{z} - \mathbf{P}\,\boldsymbol{\Gamma}\mathbf{x}^\star\right) \tag{26}$$
$$= \mathbf{P}\,\mathbf{L}_{\mathcal{V}^\star}\,\left(\mathbf{P}\mathbf{L}_{\mathcal{V}^\star}\right)^{+}\mathbf{P}\,\mathbf{L}_{\mathcal{V}^\star}\,\boldsymbol{\Omega}^\star \tag{27}$$
$$= \mathbf{P}\,\mathbf{L}_{\mathcal{V}^\star}\,\boldsymbol{\Omega}^\star, \textit{ property of pseudo-inverse} \tag{28}$$
$$= \mathbf{z}^\star. \tag{29}$$

It follows that equation 25 implies

$$25 \Rightarrow \textbf{\textit{KKT-2:}}\alpha\left|\left[\mathbf{P}\mathbf{L}\right]_k^{\mathrm{T}}\left(\mathbf{P}\mathbf{L}_{\mathcal{V}^\star}\right)^{+\mathrm{T}}\boldsymbol{\Sigma}\left(\mathbf{O}_{\mathcal{V}^\star}\right)\right| \tag{30}$$
$$< \alpha,\ \textit{given condition 1) of theorem 1.} \tag{31}$$

*Thus **KKT-2** is also filled by $\mathbf{O}_{\mathcal{V}^\star}$. With the same reasoning and the result obtained from equations 26-29, it can be proven that*

$$\left(\mathbf{P}\,\boldsymbol{\Gamma}\right)^{\mathrm{T}}\left(\mathbf{P}\mathbf{L}_{\mathcal{V}^\star}\right)^{+\mathrm{T}}\boldsymbol{\Sigma}\left(\mathbf{O}_{\mathcal{V}^\star}\right) = 0, \tag{32}$$

*which satisfies **KKT-3** if condition 3) of theorem 1 is filled.*

References

[1] Matthew Loper, Naureen Mahmood, Javier Romero, Gerard Pons-Moll, and Michael J. Black. SMPL: a skinned multi-person linear model. *ACM Transactions on Graphics*, 34(6):248:1–248:16, Oct. 2015.

[2] Richard M. Murray, Zexiang Li, S. Shankar Sastry, and S. Shankara Sastry. *A Mathematical Introduction to Robotic Manipulation*. CRC Press, Mar. 1994. Google-Books-ID: D_PqGKRo7oIC.