MuTri: Multi-view Tri-alignment for OCT to OCTA 3D Image Translation

Supplementary Material

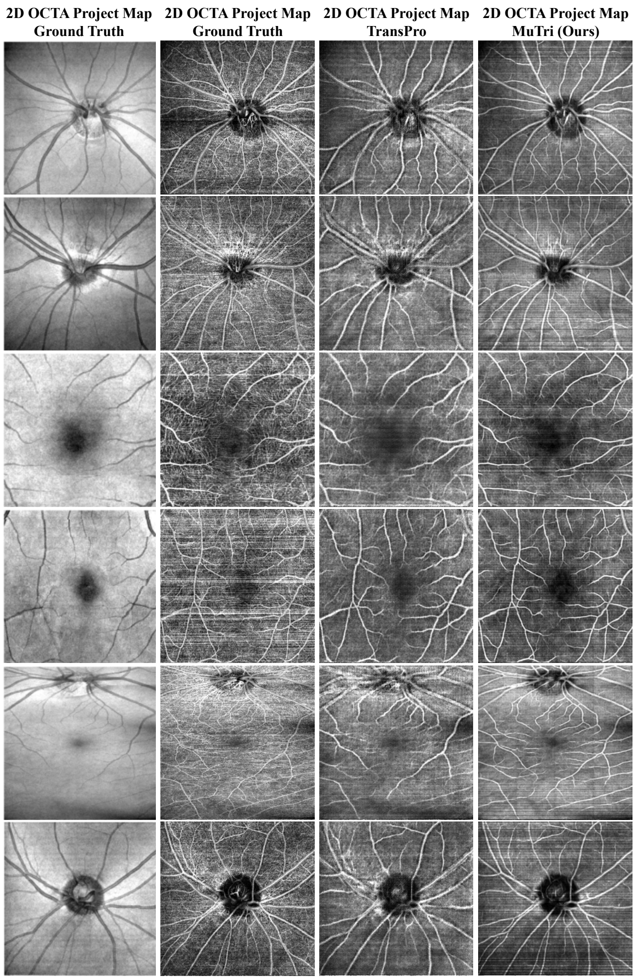

Figure 6. Visualization results of translated OCTA projection maps on OCTA2024 dataset. From left to right: 2D OCT ground truth project maps, 2D OCTA ground truth project maps, the translated 2D OCTA projection maps from TransPro, and our *MuTri*.

To understand Eq. (7), we start from the mutual information perspective to prove that minimizing $\mathcal{L}_{ConKD}$ is equal to minimizing the upper bound of the mutual information $I(\mathbf{P}, \mathbf{Q}))$ between the quantized student and teacher features from codebook $\mathbb{T}$ and $\mathbb{S}$. Specifically, given the positive pair $(\mathbf{Q}_{(\mathbf{i},\mathbf{j})}, \mathbf{P}_{(\mathbf{i},\mathbf{j})})$ and negative pairs $\{(\mathbf{Q}_{(\mathbf{i},\mathbf{j})}, \mathbf{P}_{(\mathbf{m},\mathbf{n})})\}_{m=1,n=1,\text{and}(m,n)\neq(i,j)}^{\frac{W}{S},\frac{H}{S}}$, we have the joint distribution $\mu(\mathbf{Q}, \mathbf{P})$ and the product of marginals $\mu(\mathbf{Q})\mu(\mathbf{P})$. And then, an indicator variable $\mathcal{V}$ is exploited to identify a pair $(\mathbf{Q}, \mathbf{P})$ sampled from the joint distribution $q(\mathbf{Q}, \mathbf{P}|\mathcal{V} = 1) = \mu(\mathbf{Q}, \mathbf{P})$ or product of marginals $q(\mathbf{Q}, \mathbf{P}|\mathcal{V} = 0) = \mu(\mathbf{Q})\mu(\mathbf{P})$, where $\mathcal{V} = 1$ indicates the positive pair $(\mathbf{Q}_{(\mathbf{i},\mathbf{j})}, \mathbf{P}_{(\mathbf{i},\mathbf{j})})$ and $\mathcal{V} = 0$ indicates a negative pair sampled from $\{(\mathbf{Q}_{(\mathbf{i},\mathbf{j})}, \mathbf{P}_{(\mathbf{m},\mathbf{n})})\}_{m=1,n=1,\text{and}(m,n)\neq(i,j)}^{\frac{W}{S},\frac{H}{S}}$, i.e., $(\mathbf{Q}_{(\mathbf{i},\mathbf{j})}, \mathbf{P}_{(\mathbf{i},\mathbf{j})}) \sim \mu(\mathbf{Q}, \mathbf{P})$, $\{(\mathbf{Q}_{(\mathbf{i},\mathbf{j})}, \mathbf{P}_{(\mathbf{m},\mathbf{n})})\}_{m=1,n=1,\text{and}(m,n)\neq(i,j)}^{\frac{W}{S},\frac{H}{S}} \sim \mu(\mathbf{Q})\mu(\mathbf{P})$. For our contrastive-enhanced knowledge distillation loss, we have one positive pair for every $\left(\frac{W}{S} * \frac{H}{S} - 1\right)$ negative pairs.

$$q(\mathcal{V}=1) = \frac{1}{\frac{W}{S} * \frac{H}{S}}, q(\mathcal{V}=0) = \frac{\frac{W}{S} * \frac{H}{S} - 1}{\frac{W}{S} * \frac{H}{S}}. \tag{12}$$

The class posterior of the pair $(\mathbf{Q}, \mathbf{P})$ belonging to the positive case ($\mathcal{V}$= 1) can be derived by Bayes' Theorem:

$$\begin{aligned}
&q(\mathcal{V}=1|(\mathbf{Q}, \mathbf{P})) \\
&= \frac{q\left(\mathbf{Q}, \mathbf{P} \mid \mathcal{V}=1\right) q(\mathcal{V}=1)}{q\left(\mathbf{Q}, \mathbf{P} \mid \mathcal{V}=0\right) q(\mathcal{V}=0) + q\left(\mathbf{Q}, \mathbf{P} \mid \mathcal{V}=1\right) q(\mathcal{V}=1)} \\
&= \frac{\mu\left(\mathbf{Q}, \mathbf{P}\right)}{\mu\left(\mathbf{Q}, \mathbf{P}\right) + \left(\frac{W}{S} * \frac{H}{S} - 1\right) \mu\left(\mathbf{Q}\right) \mu\left(\mathbf{P}\right)}.
\end{aligned} \tag{13}$$

Herein, we consider the log class posterior with the following derivation:

$$\begin{aligned}
&\log q\left(\mathcal{V}=1 | \left(\mathbf{Q}, \mathbf{P}\right)\right) \\
&= \log \frac{\mu\left(\mathbf{Q}, \mathbf{P}\right)}{\mu\left(\mathbf{Q}, \mathbf{P}\right) + \left(\frac{W}{S} * \frac{H}{S} - 1\right) \mu\left(\mathbf{Q}\right) \mu\left(\mathbf{P}\right)} \\
&= -\log\left(1 + \left(\frac{W}{S} * \frac{H}{S} - 1\right) \frac{\mu\left(\mathbf{Q}\right) \mu\left(\mathbf{P}\right)}{\mu\left(\mathbf{Q}, \mathbf{P}\right)}\right) \\
&\leq \log \frac{\mu\left(\mathbf{Q}, \mathbf{P}\right)}{\mu\left(\mathbf{Q}\right) \mu\left(\mathbf{P}\right)} - \log\left(\frac{W}{S} * \frac{H}{S} - 1\right)
\end{aligned} \tag{14}$$

Now, we are allowed to connect it to the mutual information by considering all cases over the above log class posterior with respect to positive and negative pairs:

$$\begin{aligned}
&\mathbb{E}_{q((\mathbf{Q},\mathbf{P})|\mathcal{V}=1)} \log q\left(\mathcal{V}=1 | \left(\mathbf{Q}, \mathbf{P}\right)\right) \\
&\leq \mathbb{E}_{\mu(\mathbf{Q},\mathbf{P})} \log \frac{\mu\left(\mathbf{Q}, \mathbf{P}\right)}{\mu\left(\mathbf{Q}\right) \mu\left(\mathbf{P}\right)} - \log\left(\frac{W}{S} * \frac{H}{S} - 1\right) \\
&= I(\mathbf{Q}, \mathbf{P}) - \log\left(\frac{W}{S} * \frac{H}{S} - 1\right)
\end{aligned} \tag{15}$$

Therefore, we have

$$I(\mathbf{P}, \mathbf{Q}) \geq \log(\frac{W}{S} \cdot \frac{H}{S} - 1) - \mathbb{E}_{q((\mathbf{Q},\mathbf{P})|\mathcal{V}=1)} \mathcal{L}_{\text{OCT}} \tag{16}$$

Herein, minimizing $\mathcal{L}_{\text{OCT}}$ will contribute to maximizing the lower bound on the mutual information between contrastive embedding pairs, to maximize the mutual information $I(\mathbf{Q}, \mathbf{P})$.