Order-One Rolling Shutter Cameras

Supplementary Material

This is the Supplementary Material for the paper *"M. A. Hahn, K. Kohn, O. Marigliano, T. Pajdla. Order-One Rolling Shutter Cameras. CVPR 2025."*

We provide additional notations, definitions, concepts, technical lemmas, and proofs of all results in the main paper. The code we used for computations is included as a pair of ancillary files. See `readme.txt` for how to use the code.

9. Derivations and Proofs

We start by explaining some of the basic mathematical objects we use throughout the article. Our starting point is the projective three-space $\mathbb{P}^3$, whose points are determined by their homogeneous coordinates $(x_1 : x_2 : x_3 : x_0)$.

Dual space. The *dual space* $(\mathbb{P}^3)^*$ is the set of planes in $\mathbb{P}^3$. It is isomorphic to $\mathbb{P}^3$ by identifying a plane $\Sigma = \{x \in \mathbb{P}^3 \mid \sum_i c_i x_i = 0\}$ with the coefficients $\Sigma^\vee = (c_1 : c_2 : c_3 : c_0)$ of its defining linear equation. If $L \subseteq \mathbb{P}^3$ is a line, then the set of all planes that contain L form a line in $(\mathbb{P}^3)^*$ which is called the *dual line* of L and denoted by $L^\vee$.

Grassmannian, pencil of lines. The set of lines in $\mathbb{P}^3$ is called the Grassmannian $\mathrm{Gr}(1, \mathbb{P}^3)$. Each of its elements can be represented by a matrix

$$\begin{pmatrix} c_1 & c_2 & c_3 & c_0 \\ d_1 & d_2 & d_3 & d_0 \end{pmatrix}$$

whose rows contain the coefficients of the two linear equations that define the line. Two such matrices represent the same element of $\mathrm{Gr}(1, \mathbb{P}^3)$ if there is a 2×2 invertible matrix that takes one to the other.

Elements of the Grassmannian are uniquely determined by their *dual Plücker coordinates* $(p_{12} : p_{13} : p_{10} : p_{23} : p_{20} : p_{30})$, which are computed as the six 2×2 minors $p_{ij} = c_i d_j - c_j d_i$ of the above matrix. This exhibits the Grassmannian as a subset of $\mathbb{P}^5$. It is in fact an *algebraic variety* since it is the solution set of the polynomial equation $p_{12}p_{30} - p_{13}p_{20} + p_{10}p_{23} = 0$ over $\mathbb{P}^5$.

Analogously, one can uniquely represent a line in $\mathbb{P}^3$ via its *primal Plücker coordinates*, which are the 2×2 minors of a 2×4 matrix whose rows span the line.

A *pencil* of lines is a one-dimensional family of lines passing through a common point and contained in a common plane. If the common point is at infinity, the lines of the pencil are parallel to each other. A pencil of lines in $\mathbb{P}^3$ can be seen as a line (i.e., a curve of degree one) inside $\mathrm{Gr}(1, \mathbb{P}^3)$.

Algebraic varieties, Zariski closure, degree. An *algebraic variety* is the set of solutions of a set of polynomial equations over some field. In our paper, we mostly consider varieties V inside some projective space $\mathbb{P}^n$ over the real or the complex numbers. These take the form $V(E)$ for a set E of polynomials in several variables. When working in projective space, we must require these polynomials to be homogeneous.

For a subset X of $\mathbb{P}^n$, we define its Zariski closure as the smallest variety containing X.

Let V be a variety of dimension d in $\mathbb{P}^n$. We define the degree of X as the number of complex intersection points of V with a *generic* linear space of dimension $n - d$. Here generic means that the linear space corresponds to a point outside of a Zariski–closed subset of the parameter space of linear spaces. This parameter space is the Grassmannian $\mathrm{Gr}(n-d, \mathbb{P}^n)$, a generalisation of the object defined above.

Dual varieties. Several of our proofs use projective duality of algebraic varieties. Analogously to the dual three-space, for any $n \in \mathbb{N}$, the *dual space* $(\mathbb{P}^n)^*$ is defined as the set of hyperplanes in $\mathbb{P}^n$. The elements of $(\mathbb{P}^n)^*$ are likewise represented as $(n+1)$-tuples of projective coordinates. For any subvariety V of $\mathbb{P}^n$, its *dual variety* $V^\vee$ is the Zariski closure inside $(\mathbb{P}^n)^*$ of the set of hyperplanes that are tangent at some smooth point of V.

Over the complex numbers, we have $(V^\vee)^\vee = V$. This is called *projective duality*. More concretely, we have the following over $\mathbb{C}$: For a smooth point p of V and a smooth point $H^\vee$ of the dual variety $V^\vee$, the hyperplane H is tangent to V at the point p if and only if the hyperplane $p^\vee$ is tangent to $V^\vee$ at the point $H^\vee$.

9.1. Derivation of Λ map

The camera ray through an image point results from intersecting backprojected planes of the image lines $\rho(v : t)$ and $\nu(u : s)$: the map

$$\Lambda : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathrm{Gr}(1, \mathbb{P}^3)^*$$

sends a point $((u:s),(v:t))$ to

$$\begin{aligned}
&[-t,0,v]\,R(v:t)\,[I_3 \mid -C(v:t)] \\
&\wedge [0,-s,u]\,R(v:t)\,[I_3 \mid -C(v:t)] \\
&= \begin{bmatrix} I_3 \\ -C(v:t)^\top \end{bmatrix} R(v:t)^\top \begin{bmatrix} 0 & -st & ut \\ st & 0 & -sv \\ -ut & sv & 0 \end{bmatrix} \\
&\cdot R(v:t)[I_3 \mid -C(v:t)] \\
&= \begin{bmatrix} I_3 \\ -C(v:t)^\top \end{bmatrix} R(v:t)^\top \left[\begin{bmatrix} sv \\ ut \\ st \end{bmatrix}\right]_\times \\
&\cdot R(v:t)\,[I_3 \mid -C(v:t)] \\
&= \begin{bmatrix} I_3 \\ -C(v:t)^\top \end{bmatrix} \left[R(v:t)^\top \begin{bmatrix} sv \\ ut \\ st \end{bmatrix}\right]_\times \\
&\cdot [I_3 \mid -C(v:t)].
\end{aligned} \tag{23}$$

9.2. Classifying RS$_1$ cameras

In this section, we prove Theorem 4. We assume throughout this section that the map Λ is rational.

Lemma 24. *If the map Λ is rational, so are the maps C and $\Sigma^\vee$.*

Proof. Consider $(v:t) \in \mathbb{P}^1$ general. Then, the plane $\Sigma(v:t)^\vee$ is the equation of the Zariski closure of $\bigcup_{(u:s)\in\mathbb{P}^1} \Lambda((v:t),(u:s))$, which is the span of $\Lambda((v:t),(1:0))$ and $\Lambda((v:t),(0:1))$. This shows that $\Sigma^\vee$ is a rational map. Similarly, we see that C is rational by observing that $C(v:t)$ is the intersection of $\Lambda((v:t),(1:0))$ and $\Lambda((v:t),(0:1))$. $\square$

We start with showing that our unions of pencils of lines are indeed congruences, i.e., surfaces the Grassmannian $\mathrm{Gr}(1,\mathbb{P}^3)$.

Lemma 25. *Let $X \subseteq \mathbb{P}^3 \times (\mathbb{P}^3)^*$ be an irreducible curve. For $(C,\Sigma^\vee) \in X$, consider the pencil $\mathcal{L}(C,\Sigma) := \{\xi \in \mathrm{Gr}(1,\mathbb{P}^3) \mid C \in \xi \subseteq \Sigma\}$. Then the set $\mathcal{L}_X := \bigcup_{(C,\Sigma^\vee)\in X} \mathcal{L}(C,\Sigma)$ has dimension two.*

Proof. Let $\mathcal{V} \subseteq X \times \mathcal{L}_X$ be the incidence variety of elements $(C,\Sigma^\vee,\xi)$ such that $\xi \in \mathcal{L}(C,\Sigma)$. Since all fibers over X of $\mathcal{V}$ are one-dimensional, the variety $\mathcal{V}$ is two-dimensional. It remains to show that the generic fiber of $\mathcal{V}$ over $\mathcal{L}_X$ is zero-dimensional. Pick a general $\xi \in \mathcal{L}_X$ and let $\mathcal{C}$ be the image of X in $\mathbb{P}^3$. Then the set $\{C \in \mathcal{C} \mid \xi \in \mathcal{L}(C,\Sigma) \text{ for some } \Sigma\} = \xi \cap \mathcal{C}$ is finite because ξ is general. Dually, if $\mathcal{S}$ denotes the image of X in $(\mathbb{P}^3)^*$, then the set $\{\Sigma^\vee \in \mathcal{S} \mid \xi \in \mathcal{L}(C,\Sigma) \text{ for some } C\} = \xi^\vee \cap \mathcal{S}$ is also finite. Thus, the fiber of $\mathcal{L}_X$ over ξ is finite. $\square$

A rolling shutter camera that does not move but rotates has (almost) always the order-one congruence $\overline{\mathcal{L}}$ that consists of all lines passing through the fixed center. The only exception is when the rotation cancels out the movement of the rolling line such that all rolling planes are the same plane, say Σ. In that case, the camera rays do not form a congruence, but only the pencil of lines that are contained in Σ and pass through the camera center. Such a camera can only take pictures of the points in the fixed plane Σ.

Thus, to classify all rolling shutter cameras with an order-one congruence, we may assume that the camera is actually moving (and possibly rotating). In the following, we use the rolling image lines $r \in \mathcal{R}$ synonymously with our projective parameters $(v:t) \in \mathbb{P}^1$, via the identification ρ in (1).

Theorem 26. *Consider a rolling shutter camera whose center moves along a rational curve $\mathcal{C} \subseteq \mathbb{P}^3$. The associated congruence $\overline{\mathcal{L}}$ has order one if and only if the intersection of all rolling planes $\Sigma(r)$ is a line K that satisfies one of the following two conditions:*
1) K intersects the curve $\mathcal{C}$ in $\deg\mathcal{C}-1$ many points (counted with multiplicty), or
2) $K = \mathcal{C}$ and $\Sigma(r_1) = \Sigma(r_2)$ implies $C(r_1) = C(r_2)$ for all $r_1, r_2 \in \mathcal{R}$.

In order to prove this theorem, we make use of Kummer's classication of order-one congruences according to their focal loci [32]. A *focal point* of an order-one congruence $\overline{\mathcal{L}}$ is a space point that lies on infinitely many lines on $\overline{\mathcal{L}}$. The *focal locus* of the congruence is the Zariski closure of its set of focal points. The following version of Kummer's classification is due to De Poi [45].

Theorem 27. *A congruence $\mathcal{L} \subset \mathrm{Gr}(1,\mathbb{P}^3)$ has order one if and only if it is one of the following:*

i. $\mathcal{L}$ is the set of all lines passing through a fixed point $C \in \mathbb{P}^3$. Its only focal point is C.
ii. $\mathcal{L}$ consists of all secant (and tangent) lines of a twisted cubic curve $\mathcal{C} \subset \mathbb{P}^3$. Its focal locus is $\mathcal{C}$.
iii. $\mathcal{L}$ consists of all lines that meet both a rational curve $\mathcal{C} \subset \mathbb{P}^3$ and a line $K \subset \mathbb{P}^3$ that intersects the curve $\mathcal{C}$ in $\deg\mathcal{C}-1$ many points (counted with multiplicity). Its focal locus is $\mathcal{C} \cup K$.
iv. There is a line $K \subset \mathbb{P}^3$ and a dominant morphism $\sigma : K^\vee \to K$ such that $\mathcal{L} = \bigcup_{\Sigma^\vee \in K^\vee} \{\xi \in \mathrm{Gr}(1,\mathbb{P}^3) \mid \sigma(\Sigma^\vee) \in \xi \subset \Sigma\}$. Its focal locus is K.

Order-one congruences of type i are associated with rolling shutter cameras that stand still (but possibly rotate).

Lemma 28. *The congruence $\overline{\mathcal{L}}$ associated with a rolling shutter camera cannot be of type ii.*

Proof. The congruence is a union of pencils; in fact, $\mathcal{L} = \bigcup_{r\in\mathcal{R}} \mathcal{L}(r)$, where $\mathcal{L}(r) = \{\xi \in \mathrm{Gr}(1,\mathbb{P}^3) \mid C(r) \in \xi \subset \Sigma(r)\}$. In particular, its focal locus contains the curve $\mathcal{C}$ along which the camera center moves. If the congruence $\overline{\mathcal{L}}$

were of type ii, the curve $\mathcal{C}$ would be a twisted cubic. However, any rolling plane $\Sigma(r)$ intersects the curve $\mathcal{C}$ in the camera center $C(r)$ and at most two other points, meaning that the generic line in $\mathcal{L}(r)$ is not a secant of $\mathcal{C}$. $\square$

Remark 29. The congruence associated with a rolling shutter camera has order zero if and only if all rolling planes are the same plane. This is because the set of points seen by the camera is the union of all rolling planes, which is a connected surface that contains a plane. When the congruence has order zero and all rolling planes are the same, the congruence consists of all lines in that plane and the camera moves along a rational curve in that plane.

Proof of Theorem 26. We start by observing that the set of camera rays $\mathcal{L}$ of a moving rolling shutter camera is always two-dimensional by Lemma 25. Moreover, the focal locus of the congruence $\overline{\mathcal{L}}$ contains the curve $\mathcal{C}$ that describes the movement of the camera center. In particular, $\mathcal{L}$ cannot be of type i. Hence, by Theorem 27 and Lemma 28, the congruence $\overline{\mathcal{L}}$ has order one if and only if it is of type iii or iv.

We start by assuming that the congruence is of type iii. By considering the focal locus, we find that the rational curve from iii coincides with the camera movement curve $\mathcal{C}$. Since $\mathcal{L}$ is the union of the pencils $\mathcal{L}(r) = \{\xi \in \mathrm{Gr}(1, \mathbb{P}^3) \mid C(r) \in \xi \subset \Sigma(r)\}$ and every of its lines has to meet a fixed line K, every rolling plane $\Sigma(r)$ has to contain K. The rolling planes cannot all be the same, because otherwise the congruence $\overline{\mathcal{L}}$ would consist of all lines in that plane, which is a congruence of order zero. Therefore, the intersection of the rolling planes is exactly the line K, and we are in type 1) of Theorem 26. Conversely, given a rolling shutter camera of type 1), the generic rolling plane $\Sigma(r)$ is the span of the center $C(r)$ with the line K, and thus $\overline{\mathcal{L}}$ is of type iii.

If the congruence $\overline{\mathcal{L}}$ is of type iv, the camera moves along a line $\mathcal{C} = K$. This line is the focal locus of $\overline{\mathcal{L}}$ and thus coincides with the line given in iv. We claim that every rolling plane $\Sigma(r)$ contains that line K. To see that, we assume for contradiction that one of the rolling planes $\Sigma(r)$ intersects K in a single point, namely $C(r)$. Now, consider an arbitrary plane Σ containing K (i.e., $\Sigma^\vee \in K^\vee$). The intersection $\Sigma \cap \Sigma(r)$ is a line ξ passing through the center $C(r)$. Thus, the line ξ is on the pencil $\mathcal{L}(r) \subset \mathcal{L}$. Since Σ is the unique plane that contains both ξ and K, the expression of $\mathcal{L}$ in type iv of Theorem 27 shows that $\sigma(\Sigma^\vee) \in \xi$, which implies that $\sigma(\Sigma^\vee) = C(r)$. In other words, the map $\sigma : K^\vee \to K$ is constant with image $C(r)$, which contradicts its dominance. Hence, we have shown that the rolling planes intersect in the line K (as before, they cannot all be equal, since otherwise the congruence would have order zero). Moreover, we have for any rolling plane $\Sigma(r)$ that $\sigma(\Sigma(r)^\vee) = C(r)$, and we are in type 2) of Theorem 26. Conversely, for a rolling shutter camera of type 2), we have that $K^\vee$ consists of all rolling planes and the map $\sigma : K^\vee \to K, \Sigma(r)^\vee \mapsto C(r)$ is dominant since the camera is moving. Thus, $\mathcal{L} = \bigcup_{r \in \mathcal{R}} \mathcal{L}(r)$ is of type iv. $\square$

To classify all RS_1 cameras, it remains to analyze when the congruence parametrization map Λ is birational.

Lemma 30. *The map Λ is birational if and only if, for a generic $\xi \in \mathcal{L}$, there is a unique $r \in \mathcal{R}$ such that $\xi \in \mathcal{L}(r)$.*

Proof. Λ is birational if and only if, for a generic $\xi \in \mathcal{L}$, there are unique parameters $((v : t), (u : s))$ whose image under Λ is $\mathcal{L}$. Recall that the morphism ρ in (1) identifies the parameters $(v : t) \in \mathbb{P}^1$ with $r \in \mathcal{R}$. Hence, the birationality of Λ implies for $\xi \in \mathcal{L}$ the existence of a unique $r \in \mathcal{R}$ with $\xi \in \mathcal{L}(r)$. Conversely, if a generic $\xi \in \mathcal{L}$ uniquely determines the pencil $\mathcal{L}(r)$ that contains it, then ξ is uniquely picked in that pencil via the parameters $(u : s)$, i.e., Λ is birational. $\square$

Proposition 31. *Consider a rolling shutter camera whose associated congruence $\overline{\mathcal{L}}$ has order zero. The map Λ is birational if and only if the camera moves on a line $\mathcal{C}$ and the center map $C : \mathcal{R} \dashrightarrow \mathcal{C}$ is birational.*

Proof. As in Remark 29, all rolling planes are the same plane, say Σ. The congruence is the family of all lines contained in Σ and the camera moves along a rational plane curve $\mathcal{C} \subset \Sigma$. By Lemma 30, the map Λ is birational if and only if, for a generic line $\xi \subset \Sigma$, there is a unique $r \in \mathcal{R}$ such that $C(r) \in \xi$. The latter means that the generic line ξ intersects the plane curve $\mathcal{C}$ in a single point (i.e., $\deg \mathcal{C} = 1$) and that C is a birational map. $\square$

Proposition 32. *Consider a rolling shutter camera that does not move (but possibly rotates). The map Λ is birational if and only if the intersection of all rolling planes is a line K and the rolling planes map Σ is birational.*

Proof. The congruence is the family of lines passing through the fixed camera center $C \in \mathbb{P}^3$. Hence, this situation is projectively dual to the setting in Proposition 31. In particular, when considering the rolling planes $\Sigma(r)$ as points $\Sigma(r)^\vee$ in $(\mathbb{P}^3)^*$, they form a curve in the plane $C^\vee \subset (\mathbb{P}^3)^*$. As in the proof of Proposition 31, the birationality of Λ means that that plane curve is a line (namely $K^\vee$) with a rational parametrization via the rolling planes map $\Sigma^\vee$. $\square$

Theorem 33. *Consider a moving rolling shutter camera with a congruence $\overline{\mathcal{L}}$ of positive order. The map Λ is birational if and only if the map $\mathcal{R} \dashrightarrow \mathbb{P}^3 \times (\mathbb{P}^3)^*, r \mapsto (C(r), \Sigma(r)^\vee)$ that parametrizes the pencils $\mathcal{L}(r)$ is birational. In particular, the birationality of $C : \mathcal{R} \dashrightarrow \mathcal{C}$ implies that Λ is birational.*

Proof. The pairs $(C(r), \Sigma(r)^\vee)$ trace a curve in $\mathbb{P}^3 \times (\mathbb{P}^3)^*$, which we denote by $\mathcal{D}$.

Let Λ be birational and let X be the set of pairs $(C(r), \Sigma(r)^\vee) \in \mathcal{D}$ such that there is some $r' \neq r$ with $(C(r), \Sigma(r)^\vee) = (C(r'), \Sigma(r')^\vee)$. The set X has either dimension zero or it has dimension one. In the former case, we are done since a generic element of $\mathcal{D}$ will have a unique pre-image r. In the latter case, the union $\bigcup_{(C(r),\Sigma(r)^\vee)\in X} \mathcal{L}(r) \subseteq \mathcal{L}$ is dense by Lemma 25, contradicting Lemma 30.

Now, we assume that the map $\mathcal{R} \dashrightarrow \mathcal{D}$ is birational. Then, by Lemma 30, it is enough to show that, for a generic $\xi \in \mathcal{L}$, there is a unique pair $(C, \Sigma^\vee) \in \mathcal{D}$ such that $C \in \xi \subset \Sigma$. To prove this, we consider a generic line $\xi \in \mathcal{L}$ on the congruence and assume for contradiction that there are two distinct pairs $(C_1, \Sigma_1^\vee), (C_2, \Sigma_2^\vee) \in \mathcal{D}$ such that $C_i \in \xi \subset \Sigma_i$ for $i = 1, 2$. We distinguish two cases.

First, if $C_1 \neq C_2$, the generic line ξ is a secant line of the curve $\mathcal{C}$ traced by the camera centers. Hence, the congruence $\overline{\mathcal{L}}$ is the family of secant lines of $\mathcal{C}$. In particular, all the lines ξ' in the pencil $C_1 \in \xi' \subset \Sigma_1$ have to be secants of $\mathcal{C}$, which is only possible if $\mathcal{C}$ is a plane curve contained in Σ_1. Since the same applies to the pencil given by (C_2, Σ_2), we see in particular that $\Sigma_1 = \Sigma_2$ (otherwise, the curve $\mathcal{C}$ would be the unique line in their intersection, but then its secants would not form a line congruence). However, the congruence $\overline{\mathcal{L}}$ is now simply the family of lines contained in the plane Σ_1, which has order zero; a contradiction to the assumptions in Theorem 33.

Second, if $C_1 = C_2$, then we have necessarily that $\Sigma_1 \neq \Sigma_2$. This case is projectively dual to the first case. In particular, the rolling planes $\Sigma(r)$ trace a curve in the plane $C_1^\vee \subset (\mathbb{P}^3)^*$ and the dual congruence $\overline{\mathcal{L}}^\vee \subset \mathrm{Gr}(1, (\mathbb{P}^3)^*)$ is the family of secant lines of that plane curve. Hence, $\overline{\mathcal{L}}^\vee$ consists of all lines in the plane $C_1^\vee$, and $\overline{\mathcal{L}}$ consists of all lines passing through the center C_1. However, as observed at the beginning of the proof of Theorem 26, the latter is not possible for a moving camera. $\square$

We obtain the following corollary for order one cameras.

Corollary 34. *Consider a moving rolling shutter camera whose associated congruence $\overline{\mathcal{L}}$ has order one. The map Λ is birational if and only if the rolling planes map $\Sigma^\vee$ is birational.*

Proof. Clearly, if the map $\Sigma^\vee$ is birational, then the map $r \mapsto (C(r), \Sigma(r)^\vee)$ is birational, and we can apply Theorem 33 to see that Λ is birational. For the converse direction, the same statement implies that it is enough to show that the birationality of $r \mapsto (C(r), \Sigma(r)^\vee)$ implies that $\Sigma^\vee$ is birational. To prove this, we consider the two cases of order-one congruences described in Theorem 26 separatly.

In case 2), any rolling plane $\Sigma(r)$ determines the corresponding camera center $C(r)$, and then the birationality of $r \mapsto (C(r), \Sigma(r)^\vee)$ ensures that there is a unique parameter r.

In case 1), a generic rolling plane $\Sigma(r)$ intersects the curve $\mathcal{C}$ is $\deg \mathcal{C}$ many points (counted with multiplicity). All except one of those points lie on the line K. The remaining point is the camera center $C(r)$, which is thus uniquely determined by $\Sigma(r)$. As above, the birationality of $r \mapsto (C(r), \Sigma(r)^\vee)$ ensures that the parameter r is unique. As a side note, a generic camera center $C(r)$ determines uniquely the corresponding rolling plane: $\Sigma(r) = C(r) \vee K$. Therefore, the birationality of the rolling planes map $\Sigma^\vee$ is equivalent to the birationality of the center map C. This also proves Remark 5. $\square$

Proof of Theorem 4. If the camera does not move, then $\mathcal{C}$ is a point. The associated congruence consists of all lines passing through that point and has order one. Hence, the camera has order one if and only if the map Λ is birational. By Proposition 32, the latter is equivalent to the conditions in Theorem 4. The point $\mathcal{C}$ has to lie on the line K since each rolling plane has to contain both $\mathcal{C}$ and K and there is a one-dimensional family of such planes.

If the camera moves, then $\mathcal{C}$ is a rational curve. The camera has order one if and only if the conditions in Theorem 26 and Corollary 34 are satisfied. Note that the birationality of the map $\Sigma^\vee$ implies that every rolling plane $\Sigma(r)$ uniquely determines the parameter r and thus also the corresponding camera center $C(r)$. Hence, the birationality of $\Sigma^\vee$ ensures that the cases 1) and 2) in Theorem 26 are equivalent to the types I and II in Theorem 4. $\square$

9.3. Camera Spaces

In this section, we prove Propositions 6 and 8.

Lemma 35. *The dimension of $\mathcal{H}_d$ is $3d+5$. For every line, conic, or nondegenerate rational curve $\mathcal{C}$ of degree at most five, there is a line K such that $(\mathcal{C}, K) \in \mathcal{H}_{\deg \mathcal{C}}$. For $d \geq 6$, the locus of rational degree-d curves $\mathcal{C}$ that admit a line K with $(\mathcal{C}, K) \in \mathcal{H}_d$ is a low-dimensional subset of the locus of all rational degree-d curves.*

Proof of Lemma 35. For curves $\mathcal{C}$ of degree at most two, there are clearly many such lines K. For curves $\mathcal{C}$ of degree at least three, the existence of such a line K requires the curve $\mathcal{C}$ to be *nondegenerate*, i.e., not contained in any plane (otherwise, every secant line would automatically intersect the curve in $\deg \mathcal{C}$ many points). Every nondegenerate cubic curve has a two-dimensional family of secant lines K. It is a classical fact that every nondegenerate rational quartic curve has a one-dimensional family of trisecant lines K (see e.g. [36]) and that every nondegenerate rational quintic curve has a quadrisecant line K (in fact, typically a unique one; see [14, Remark 2.16]). Together with the fact that the locus of all rational degree-d space curves has dimension $4d$, this discussion proves the first two assertions of Lemma 35 for $d \leq 5$.

For $d > 5$, the locus $\mathcal{H}'_d$ of rational curves in $\mathbb{P}^3$ of degree d that intersect some line at $d-1$ or d many points has dimension $3d+5 < 4d$ [14, Lem. 2.13]. Moreover, for the general curve in $\mathcal{H}'_d$ (with $d \geq 5$), there is a *unique* line meeting the curve in $d-1$ points (and not d points) [14, Thm. 2.15 & Rem. 2.16]. $\square$

Lemma 35 implies Remark 7, i.e., that almost any rational curve of degree $d \leq 5$ can be the center locus of a RS_1 camera of type I, while only special curves are allowed when $d \geq 6$. Next, we determine which birational maps $\Sigma^\vee : \mathbb{P}^1 \dashrightarrow K^\vee$ are allowed. The following lemma applies to all three types of RS_1 cameras, and uses the notation introduced in the paragraph before Proposition 6.

Lemma 36. *Let $C : \mathbb{P}^1 \dashrightarrow \mathbb{P}^3$ be a rational map defined over $\mathbb{R}$ and let $\mathcal{C}$ be the complex Zariski closure of its image. Let $K \subseteq \mathbb{P}^3$ be a line and let $\Sigma^\vee : \mathbb{P}^1 \dashrightarrow K^\vee$ be a birational map, both defined over $\mathbb{R}$, such that, for all $(v:t) \in \mathbb{P}^1$, the plane $\Sigma(v:t)$ contains the point $C(v:t)$. Assume that $\mathcal{C} \not\subseteq H^\infty$ and that $\mathcal{C}$ and K are related as in Type I, II, or III of Theorem 4.*

Then, the following are equivalent:

- *(a) There exists an RS_1 camera of type I, II, or III with distinguished line K, center-movement map C, and rolling planes map $\Sigma^\vee$.*
- *(b) K^∞ is a point and on its dual line $(K^\infty)^\vee \subseteq (H^\infty)^*$, there are points $A, B \in (K^\infty)^\vee$ such that $A \cdot B = 0$, $A \cdot A = B \cdot B$, and the map $\Sigma^\vee_\infty : \mathbb{P}^1 \dashrightarrow (H^\infty)^*$ is $\Sigma^\vee_\infty(v:t) = Av + Bt$.*

Proof. $(a) \Rightarrow (b)$: We begin by assuming that C, K, and $\Sigma^\vee$ are part of an RS_1 camera. In particular, the birational map $\Sigma^\vee$ is of the form (2). Thus, $\Sigma^\vee_\infty(x) = (1:0:-x) \cdot R(x)$ for $x \in \mathbb{R}$. The rotation matrix $R(x)$ preserves norms. So, after fixing a scaling for the rational map $\Sigma^\vee_\infty$, the former equation implies

$$\frac{\Sigma^\vee_\infty(x)}{\|\Sigma^\vee_\infty(x)\|} = \frac{1}{\sqrt{1+x^2}}(1,0,-x) \cdot R(x). \tag{24}$$

For general $x \in \mathbb{R}$, the projection matrix $P(x)$ maps the whole plane $\Sigma(x)$ onto the rolling line $\rho(x:1)$. In particular, exactly one camera ray $\xi(x) \subseteq \Sigma(x)$ passing through $C(x)$ will be mapped by $P(x)$ to $(0:1:0)^\top$. Since $(0:1:0) = \varphi((x:1),(1:0))$, the homogenization of the map Λ becomes well-defined at $((x:1),(1:0))$ and we see that $\xi(x) = \Lambda((x:1),(1:0))$. In particular, ξ is a rational map.

Thus, $\omega(x) := \xi_\infty(x)$ also defines a rational map. Since $P(x)\omega(x) = (0:1:0)^\top$, after fixing a scaling for the rational map ω, we have

$$R(x) \cdot \frac{\tilde{\omega}(x)^\top}{\|\tilde{\omega}(x)\|} = (0,1,0)^\top, \quad \text{where } \tilde{\omega}(x) := \left([I_3 \mid -C(x)]\,\omega(x)\right)^\top. \tag{25}$$

Since $\omega(x)$ is of the form $(\omega_1 : \omega_2 : \omega_3 : 0)^\top$, the point $\tilde{\omega}(x)$ is simply $(\omega_1 : \omega_2 : \omega_3) \in \mathbb{P}^2$. Equations (24) and (25) uniquely determine the rotation matrix $R(x)$. Indeed,

$$R(x) = \begin{bmatrix} \frac{1}{\sqrt{1+x^2}} & 0 & \frac{x}{\sqrt{1+x^2}} \\ 0 & 1 & 0 \\ -\frac{x}{\sqrt{1+x^2}} & 0 & \frac{1}{\sqrt{1+x^2}} \end{bmatrix} \cdot \begin{bmatrix} \frac{\Sigma^\vee_\infty(x)}{\|\Sigma^\vee_\infty(x)\|} \\ \frac{\tilde{\omega}(x)}{\|\tilde{\omega}(x)\|} \\ \frac{\Sigma^\vee_\infty(x) \times \tilde{\omega}(x)}{\|\Sigma^\vee_\infty(x)\| \cdot \|\tilde{\omega}(x)\|} \end{bmatrix}. \tag{26}$$

For $x, y \in \mathbb{R}$, we may now compute $\Lambda_\infty(x,y)$, the intersection of the camera ray parametrized by (x,y) with the plane H^∞. The point $\Lambda_\infty(x,y)$ is the intersection of the lines represented by

$$(1:0:-x) \cdot R(x) = \Sigma^\vee_\infty(x) \text{ and } (0:1:-y) \cdot R(x)$$

in$(H^\infty)^*$. We compute

$$\begin{aligned} \Lambda_\infty(x,y) &= \Sigma^\vee_\infty(x) \times \Big((0:1:-y) \cdot R(x)\Big) \\ &= \Sigma^\vee_\infty(x) \times \Big(\frac{\tilde{\omega}(x)}{\|\tilde{\omega}(x)\|} - y \\ &\cdot \Big(-\frac{x}{\sqrt{1+x^2}} \cdot \frac{\Sigma^\vee_\infty(x)}{\|\Sigma^\vee_\infty(x)\|} + \frac{1}{\sqrt{1+x^2}} \cdot \frac{\Sigma^\vee_\infty(x) \times \tilde{\omega}(x)}{\|\Sigma^\vee_\infty(x)\| \cdot \|\tilde{\omega}(x)\|}\Big)\Big) \\ &= \frac{\Sigma^\vee_\infty(x) \times \tilde{\omega}(x)}{\|\tilde{\omega}(x)\|} + \frac{y}{\sqrt{1+x^2}} \cdot \frac{\|\Sigma^\vee_\infty(x)\|}{\|\tilde{\omega}(x)\|} \cdot \tilde{\omega}(x). \end{aligned} \tag{27}$$

The latter expression can be scaled to a rational function since Λ is a rational map. Since the first term of $\|\tilde{\omega}(x)\| \cdot \Lambda_\infty(x,y)$ (which is $\Sigma^\vee_\infty(x) \times \tilde{\omega}(x)$) is already rational, the function $\|\tilde{\omega}(x)\| \cdot \Lambda_\infty(x,y)$ can only be scaled to a rational function if its second term $y \cdot \|\Sigma^\vee_\infty(x)\| \cdot \tilde{\omega}(x)/\sqrt{1+x^2}$ is rational as well. This means that $\|\Sigma^\vee_\infty(x)\|/\sqrt{1+x^2}$ is rational, which is equivalent to $\sqrt{1+x^2} \cdot \|\Sigma^\vee_\infty(x)\|$ being rational. For the affine linear map $\Sigma^\vee_\infty$ this means that $(1+x^2) \cdot (\Sigma^\vee_\infty(x) \cdot \Sigma^\vee_\infty(x)) = Q(x)^2$ for some quadratic polynomial Q. Since $K \subseteq \Sigma(x)$ for all x, we can write $\Sigma^\vee_\infty(x) = Ax + B$ for some $A, B \in (K^\infty)^\vee$, and so a direct computation (e.g., with `Macaulay2`) reveals that the existence of Q is equivalent to the conditions $A \cdot B = 0$ and $A \cdot A = B \cdot B$.

Since the entries of A and B are real numbers, these two conditions imply that A and B are not scalar multiples of each other. Thus, the map $\Sigma^\vee_\infty$ is not constant, which implies that K is not contained in H^∞ (otherwise we would have $\Sigma^\vee_\infty(x) = K^\vee$ for all x).

$(b) \Rightarrow (a)$: We are given the data of K, C, and $\Sigma^\vee$, and need to find an RS_1 camera that conforms to these. First, the type of our camera (I, II, or III) is readable from K and $\mathcal{C}$. Second, the rotation map R must be determined. Third, the map Λ defined in (5) has to be rational.

We define a map $\omega : \mathbb{P}^1 \dashrightarrow H^\infty$ such that the line $\xi(x)$ spanned by $\omega(x)$ and $C(x)$ will be sent to $(0:1:0)$ by the camera. Fix any line ℓ at infinity and set $\omega(x) := \Sigma(x) \wedge \ell$. Equation (26) now gives the definition of a rotation map R that conforms to the given data.

Finally, since $\sqrt{1+x^2}\|\Sigma_\infty^\vee(x)\| = Q(x)$, we see from (27) that Λ_∞ is a rational map (after multiplying by $\|\tilde{\omega}(\cdot)\|$). Therefore, also $\Lambda(x,y) = C(x) \vee \Lambda_\infty(x,y)$ is rational. $\square$

Lemma 36 gives an existence statement for RS_1 cameras of all types given the data C, K, $\Sigma^\vee$. That data determines uniquely the associated congruence $\overline{\mathcal{L}}$, but the camera rotation map $R : \mathbb{R} \to \mathrm{SO}(3)$ and the congruence parametrization Λ are not yet fixed. In fact, there are many ways of defining a camera rotation map $R : \mathbb{R} \to \mathrm{SO}(3)$ that fits the data. For $x \in \mathbb{R}$, the orientation $R(x)$ has three degrees of freedom. The first two are accounted for, up to orientation, by the rolling plane $\Sigma(x)$. The third may be fixed, up to orientation, by the choice of which line $\xi(x) \subseteq \Sigma(x)$ passing through $C(x)$ is the camera ray that the projection matrix $P(x)$ maps to $(0:1:0)$, the intersection point of all rolling lines. Such a choice is expressed as a rational map $\xi : \mathbb{P}^1 \dashrightarrow \mathrm{Gr}(1,\mathbb{P}^3)$.

Lemma 37. *Let $(C, K, \Sigma^\vee)$ satisfy the conditions from Lemma 36. The RS_1 cameras involving C, K, and $\Sigma^\vee$ are in 4-to-1 correspondence with rational maps $\xi : \mathbb{P}^1 \dashrightarrow \mathrm{Gr}(1,\mathbb{P}^3)$ satisfying $C(v:t) \in \xi(v:t) \subset \Sigma(v:t)$ for all $(v:t) \in \mathbb{P}^1$. The correspondence comes from the two choices of orientation, that on ξ and that on Σ. Changing the orientation on ξ does not change the picture-taking map Φ. Changing the orientation on Σ does change Φ, from $(\Phi_1 : \Phi_2 : \Phi_0)$ to $(\Phi_1 : -\Phi_2 : \Phi_0)$.*

Proof. Given $(C, K, \Sigma^\vee)$, we are interested in describing all possible orientation maps $R : \mathbb{R} \to \mathrm{SO}(3)$ that would conform to an RS_1 camera. As in the proof of Lemma 36, for every such camera, there is a rational map ξ that assigns to every $x \in \mathbb{R}$ the camera ray that the projection matrix $P(x)$ maps to $(0:1:0)^\top$ (see paragraph under (24)). Conversely, any such map defines an orientation map $R : \mathbb{R} \to \mathrm{SO}(3)$ via (24) and (25) (with $\omega := \xi_\infty$), *after* fixing a scaling for the rational maps $\Sigma_\infty^\vee$ and ω. That means that $R(x)$ is actually only determined up to the signs of $\Sigma_\infty^\vee(x)$ and $\omega(x)$. Changing the sign of $\Sigma_\infty^\vee(x)$ would mean to use the unit normal vector of the rolling plane $\Sigma(x)$ that points in the opposite direction, i.e., a rotation around the ray $\xi(x)$ by $180°$. Changing the sign of $\omega(x)$ amounts to a rotation around the normal vector of the plane $\Sigma(x)$ by $180°$.

Finally, we have to analyze how these sign changes affect the picture-taking map Φ. Recall from (7) that Φ is the composition of several maps. The map Γ is not affected by the sign changes. Hence, it suffices to consider how $\Lambda \circ \varphi^{-1}$ is affected. In an affine chart, this map is spelled out in (27). On the one hand, changing the sign of $\Sigma_\infty^\vee(x)$, changes $\Lambda_\infty(x,y)$ in (27) to $-\Lambda_\infty(x,-y)$. On the other hand, changing the sign of $\omega(x)$, changes $\Lambda_\infty(x,y)$ in (27) simply to $-\Lambda_\infty(x,y)$. Combining this with the definition of the map φ from (4) concludes the proof. $\square$

We are now ready to prove Proposition 6.

Proof of Proposition 6. With the lemmas established above, we can describe the parameter space of RS_1 cameras of type I, up to the above described choice of orientation. The first parameter is a line K. It is not allowed to lie at infinity due to Lemma 36. Second, we can choose any curve $\mathcal{C}$ of degree d, not at infinity, such that $(\mathcal{C}, K) \in \mathcal{H}_d$. The third parameter is a map $\Sigma_\infty^\vee$ as in Lemma 36. The rolling planes map $\Sigma^\vee$ can be read off from the the map $\Sigma_\infty^\vee$ since each rolling plane $\Sigma(v:t)$ is the span of the line $\Sigma_\infty(v:t)$ with K. By Remark 5, the map Σ determines uniquely the parametrization C of the curve $\mathcal{C}$. Finally, we need to specify the map ξ. In Type I, for general $(v:t) \in \mathbb{P}^1$, the line $\xi(v:t)$ intersects the line K at a point other than $C(v:t)$, giving rise to a rational map $\lambda : \mathbb{P}^1 \dashrightarrow K$. Conversely, every such rational map λ gives a camera-ray map $\xi := \lambda \vee C$. Thus, the orientation map R is specified, up to the $4:1$ relationship above, by λ. To summarize, the space of all type-I cameras is $\mathcal{P}_{I,d,\delta}$ in Proposition 6. There is a one-dimensional family of maps $\Sigma_\infty^\vee$ of the form described in Lemma 36, since choosing A already determines B, up to sign. Hence, the camera space has dimension $\dim \mathcal{P}_{I,d,\delta} = (3d+5)+1+(2\delta+1) = 3d+2\delta+7$. $\square$

Similarly, the parameters for an RS_1 camera of type II or III are the line K (not at infinity), the center-movement map $C : \mathbb{P}^1 \dashrightarrow K$ of degree $d \geq 0$, the map $\Sigma_\infty^\vee$ as in Lemma 36 which determines the rolling planes map $\Sigma^\vee$, and the map $\xi : \mathbb{P}^1 \dashrightarrow \mathrm{Gr}(1,\mathbb{P}^3)$ satisfying $C(v:t) \in \xi(v:t)$ and $\xi(v:t) \subseteq \Sigma(v:t)$ for all $(v:t) \in \mathbb{P}^1$. Once the data $(K, C, \Sigma^\vee)$ is fixed, the following lemma shows that the map ξ is determined by a pair of homogeneous polynomials (h,p) of degrees $(\delta, \delta+d+1)$ for some $\delta \geq 0$.

We work in the following setting: we rotate and translate K until it becomes the z-axis. Then, the center-movement map $C : \mathbb{P}^1 \dashrightarrow \mathbb{P}^3$ takes the form $(0:0:C_3:C_0)$, where C_3, C_0 are homogeneous polynomials in two variables of degree $d \geq 0$. The rolling planes map $\Sigma^\vee : \mathbb{P}^1 \dashrightarrow (\mathbb{P}^3)^*$ takes the form $(\Sigma_1 : \Sigma_2 : 0 : 0)$, where Σ_1, Σ_2 are linear homogeneous polynomials in two variables. We represent the map $\xi : \mathbb{P}^1 \dashrightarrow \mathrm{Gr}(1, \mathbb{P}^3)$ by Plücker coordinates $(p_{12} : p_{13} : p_{10} : p_{23} : p_{20} : p_{30})$, where the p_{ij} are homogeneous polynomials in two variables of arbitrary degree.

Lemma 38. *Assume that K is the z-axis and consider maps $C : \mathbb{P}^1 \dashrightarrow K$ and $\Sigma^\vee : \mathbb{P}^1 \dashrightarrow K^\vee$. A map $\xi : \mathbb{P}^1 \dashrightarrow \mathrm{Gr}(1, \mathbb{P}^3)$ satisfies $C(v : t) \in \xi(v : t)$ and $\xi(v : t) \subseteq \Sigma(v : t)$ for all $(v : t) \in \mathbb{P}^1$ if and only its Plücker coordinates are*

$$\begin{aligned}&(p_{12} : p_{13} : p_{10} : p_{23} : p_{20} : p_{30})\\ =\ &(0 : -h\Sigma_2 C_3 : -h\Sigma_2 C_0 : h\Sigma_1 C_3 : h\Sigma_1 C_0 : p_{30}),\end{aligned}$$

where h is a homogeneous polynomial in two variables.

Proof. The degree-d map C is of the form $(0 : 0 : C_3 : C_0)$. We can assume that C_3 and C_0 share no common factor, as otherwise we could make d smaller. Since the birational map $\Sigma = (\Sigma_1 : \Sigma_2 : 0 : 0)$ is not constant, we note the same for Σ_1 and Σ_2.

The condition $C(v : t) \in \xi(v : t)$ means that $\xi(v : t)$ is the span of $C(v : t)$ and some other point

$$(a_1(v : t) : a_2(v : t) : a_3(v : t) : a_0(v : t)).$$

Hence, its Plücker coordinates are

$$\begin{aligned} p_{12} &= 0 & p_{30} &= a_3 C_0 - a_0 C_3\\ p_{13} &= a_1 C_3 & p_{20} &= a_2 C_0\\ p_{10} &= a_1 C_0 & p_{23} &= a_2 C_3.\end{aligned}$$

Note that fixing the point $(C_3 : C_0)$ restricts the coordinates $p_{13}, p_{10}, p_{23}, p_{20}$, but that p_{30} is arbitrary. Dually, considering the line $\xi(v : t)$ as the intersection of two planes $\Sigma(v : t)$ and $(b_1(v : t) : b_2(v : t) : b_3(v : t) : b_0(v : t)) \in (\mathbb{P}^3)^*$, we obtain

$$\begin{aligned} p_{12} &= 0 & p_{30} &= b_2\Sigma_1 - b_1\Sigma_2\\ p_{13} &= -b_0\Sigma_2 & p_{20} &= -b_3\Sigma_1\\ p_{10} &= b_3\Sigma_2 & p_{23} &= b_0\Sigma_1.\end{aligned}$$

Now we show that $\Sigma_1 \mid a_2$. Let Σ_1^k be the highest power of Σ_1 that divides C_3. Then $\Sigma_1^k \mid p_{13}$, thus $\Sigma_1^k \mid b_0$, so $\Sigma_1^{k+1} \mid p_{23}$, hence $\Sigma_1 \mid a_2$. By a similar argument we find that $\Sigma_2 \mid a_1$. Write $a_2 = h\Sigma_1$ and $a_1 = h'\Sigma_2$. Then $h\Sigma_1\Sigma_2 C_0 = a_2\Sigma_2 C_0 = -b_3\Sigma_1\Sigma_2 = -a_1\Sigma_1 C_0 = -h'\Sigma_1\Sigma_2 C_0$. It follows that $h' = -h$ and that the Plücker coordinates of ξ have the required form. $\square$

Proof of Proposition 8. We are now ready to describe the parameter space of RS_1 cameras of type II. The proof is analogous to that of Proposition 6. Working in the same set-up as Lemma 38 we see that once $(K, C, \Sigma^\vee)$ are fixed, the map ξ is determined by the bivariate polynomials h and p_{30} from the Lemma. Let $\delta \geq 0$ be the degree of h. Then p_{30} is necessarily of degree $\delta + d + 1$. All in all, the parameter space of RS_1 cameras of types II and III is $\mathcal{P}_{II,d,\delta}$ as claimed.

Note that RS_1 cameras of type III are precisely the special case when $d = 0$ and the image of the constant map C is not allowed to be at infinity. The dimension of the camera space is $\dim \mathcal{P}_{II,d,\delta} = 4 + (2d+1) + 1 + (\delta+1) + (\delta+d+2) - 1 = 3d + 2\delta + 8$. $\square$

9.4. Constant Rotation

This section proves Propositions 9 and 10, and Remark 11.

Lemma 39. *Let $\mathcal{C} \subset \mathbb{P}^3$ be an irreducible curve of degree d. If there is a line K that intersects $\mathcal{C}$ in d points (counted with multiplicity), then $\mathcal{C}$ and K are contained in a common plane.*

Proof. Take any point on $\mathcal{C}$ that does not lie on K. Then, that point and K span a plane Σ that contains at least $d+1$ points of $\mathcal{C}$. Since $d = \deg \mathcal{C}$, the plane Σ has to contain the whole curve $\mathcal{C}$. $\square$

Lemma 40. *Let $\mathcal{C} \subset \mathbb{P}^3$ be a curve and let $K \in \mathrm{Gr}(1, \mathbb{P}^3)$. There are only finitely many planes in $\mathbb{P}^3$ that contain K and are tangent to $\mathcal{C}$ at one of the points in $\mathcal{C} \setminus K$.*

Proof. We may assume without loss of generality that $\mathcal{C}$ is irreducible. Moreover, we assume that $\mathcal{C} \neq K$, as otherwise the assertion is clear. We pick a generic point on the line K and denote by $\pi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ the projection from that point. Then, $\pi(\mathcal{C})$ is a plane curve, and every plane Σ containing K that is tangent to $\mathcal{C} \setminus K$ projects to a line $\pi(\Sigma)$ that contains the point $\pi(K)$ and is tangent to $\pi(\mathcal{C})$ at another point. In the dual projective plane $(\mathbb{P}^2)^\vee$, those lines $\pi(\Sigma)$ correspond to intersection points of $\pi(\mathcal{C})^\vee$ with the line $\pi(K)^\vee$. There are only finitely many such intersection points, since otherwise $\pi(\mathcal{C})^\vee = \pi(K)^\vee$, which would imply the contradiction that the curve $\pi(\mathcal{C})$ equals the point $\pi(K)$. $\square$

Proof of Proposition 9. Consider a rolling shutter camera with constant rotation. Without loss of generality, we assume that $R(v : t) = I_3$. Then, the rolling plane $\Sigma^\vee(v : t)$ defined as an element of $(\mathbb{P}^3)^*$ by

$$(-tC_0(v : t)\ :\ 0\ :\ vC_0(v : t)\ :\ tC_1(v : t) - vC_3(v : t))$$

is spanned by $(0 : 1 : 0 : 0)$, $(v : 0 : t : 0)$, and $C(v : t)$. In particular, all rolling planes meet in a common point. By Theorem 4 it is enough to show the following: If the intersection of the rolling planes is a line K, then the other conditions in Theorem 4 are automatically satisfied.

We start by proving that the map Σ is birational in that case. We observe that $\Sigma : (v : t) \mapsto K \vee (v : 0 : t : 0)$. Thus, we obtain its inverse Σ^{-1} by intersecting each rolling plane $\Sigma(v:t)$ with the plane $(0:0:0:1)^\vee$ at infinity. (In fact, the latter is the line $(1:0:0:0)\vee(v:0:t:0)$.)

Now it remains to show that the Zariski closure $\mathcal{C}$ of the set of camera centers is one of the three types in Theorem 4. If $\mathcal{C}$ is a point, it has to lie in each rolling plane and thus on the line K, which is type III. Otherwise, $\mathcal{C}$ is an irreducible, rational curve of degree d. If $\#(K\cap\mathcal{C})>d$, then $\mathcal{C}=K$ and we are in type II. If $\#(K\cap\mathcal{C})=d$, then $\mathcal{C}$ and K lie in a common plane, say Σ', by Lemma 39. But since $\Sigma:(v:t)\mapsto K\vee C(v:t)$, each rolling plane has to be equal to that plane Σ', which contradicts that the intersection of the rolling planes is a line. Hence, the case $\#(K\cap\mathcal{C})=d$ cannot happen.

We still have to consider the case $\#(K\cap\mathcal{C})<d$. If $d=1$, this means that $\mathcal{C}$ is a line that does not meet K, which is type I in Theorem 4. Thus, in the following, we may assume that $d\geq 2$. Our goal is to show $\#(K\cap\mathcal{C})=d-1$, and so we assume for contradiction that $\#(K\cap\mathcal{C})\leq d-2$. Then, a general rolling plane $\Sigma'\in K^\vee$ meets the curve $\mathcal{C}$ in at least two points outside of K. Those points are distinct by Lemma 40. Hence, there are distinct $(v:t),(v':t')\in\mathbb{P}^1$ such that $C(v:t),C(v':t')\in(\Sigma'\cap\mathcal{C})\setminus K$. This implies that $\Sigma'=K\vee C(v:t)=\Sigma(v:t)$ and $\Sigma'=K\vee C(v':t')=\Sigma(v':t')$, which contradicts the birationality of the map Σ. □

Proof of Proposition 10 and Remark 11. Consider a rolling shutter camera that moves with constant speed along a line $\mathcal{C}$ and does not rotate. If the camera is of order one, then we see directly from the spaces $\mathcal{P}^{\mathrm{cs}}_{I,1}$ and $\mathcal{P}^{\mathrm{cs}}_{II,1}$ in Section 4.3 that the line $\mathcal{C}$ is parallel to the projection plane Π. For the converse direction, we rotate and translate such that we can assume that the constant rotation is $R=I_3$ and that the movement starts at the origin, i.e., the birational map $C:\mathbb{P}^1\dashrightarrow\mathcal{C}$ satisfies $C(0:1)=(0:0:0:1)$. Then, the assumption that $\mathcal{C}$ is parallel to the projection plane Π means that $C(1:0)=\mathcal{C}^\infty=(a:b:0:0)$. Thus, $C(v:t)=(av:bv:0:t)$. By (2), the rolling planes are $\Sigma^\vee(v:t)=(-t:0:v:av)$. The intersection of all rolling planes is the line K spanned by the points $(0:1:0:0)$ and $(0:0:-a:1)$. Therefore, the camera has order one by Proposition 9. If $a=0$, the lines $\mathcal{C}$ and K coincide and the RS_1 camera is of type II. Otherwise, the lines $\mathcal{C}$ and K are skew and the camera has type I. This proves Proposition 10.

For Remark 11, we assume again that the constant rotation is $R=I_3$ and that the movement starts at the origin, but this time $C(v:t)=(av:bv:cv:t)$ for $c\neq 0$. By (2), the rolling planes are $\Sigma^\vee(v:t)=(-t^2:0:vt:cv^2-avt)$, so they trace a conic in $(\mathbb{P}^3)^*$. Hence, a general point $X\in\mathbb{P}^3$ is contained in two rolling planes $\Sigma(v_1:t_1)$ and $\Sigma(v_2:t_2)$ with $(v_1:t_1)\neq(v_2:t_2)$. Therefore, the point X is observed twice on the image, namely as $P(v_1:t_1)X$ and $P(v_2:t_2)X$. The associated congruence has order two (since the general point X is contained in the two congruence lines $X\vee C(v_1:t_1)$ and $X\vee C(v_2:t_2)$), while the congruence parametrization Λ is birational by Theorem 33. □

9.5. Image of lines

In this section, we prove the statements in Section 5.

Proof of Proposition 13. Let D be the degree of the image curve $\Phi(L)$. Since L is general, every rolling plane meets L in exactly one point. Hence, a generic rolling line r on the image plane meets the curve $\Phi(L)$ in a unique point outside of $(0:1:0)$. The intersection multiplicity at that unique point is one. (Otherwise the point $r^\vee$ would be on the dual curve $\Phi(L)^\vee$. The genericity of r would imply that $\Phi(L)^\vee$ is the line $(0:1:0)^\vee$, but then $\Phi(L)$ would not be a curve; a contradiction).

We can compute the degree D of the curve $\Phi(L)$ by intersecting wih any line different from $\Phi(L)$ when counting intersection multiplicities. In particular, intersecting with the generic rolling line r shows that the curve has multiplicity $D-1$ at the point $(0:1:0)$. □

Proof of Theorem 12. After rotating and translating, we may assume that the line K is the y-axis, i.e., $K=(0:1:0:0)\vee(0:0:0:1)$. Then, the birational map $\Sigma^\vee:\mathbb{P}^1\dashrightarrow K^\vee$ takes the form $\Sigma\vee=(\Sigma_1:0:\Sigma_3:0)$, where Σ_1 and Σ_3 are linear with $\gcd(\Sigma_1,\Sigma_3)=1$. We consider the center-movement map $C=(C_1:C_2:C_3:C_0)$ of degree d. In type I, the coordinate functions C_1 and C_3 have $d-1$ common roots (counted with multiplicity), i.e., $C_i=\tilde{C}\cdot\ell_i$ for $i=1,3$, where $\deg\tilde{C}=d-1$ and $\deg\ell_i=1$. Since $C(v:t)\in\Sigma(v:t)$ for all $(v:t)\in\mathbb{P}^1$, we have moreover that $(\ell_1:\ell_3)=(-\Sigma_3:\Sigma_1)$. All in all, the center-movement map is of the form

$$C=(-\tilde{C}\cdot\Sigma_3:C_2:\tilde{C}\cdot\Sigma_1:C_0). \tag{28}$$

In type I, we have $\tilde{C}\neq 0$, while the case $\tilde{C}=0$ corresponds to types II and III.

Now, let us consider a general point $X=(X_1:X_2:X_3:X_0)\in\mathbb{P}^3$. Then, $X\vee K=(X_3:0:-X_1:0)^\vee$ is one of the rolling planes and there is precisely one $(v_X:t_X)\in\mathbb{P}^1$ such that $\Sigma(v_X:t_X)=X\vee K$, i.e.,

$$(\Sigma_1(v_X:t_X):\Sigma_3(v_X:t_X))=(X_3:-X_1). \tag{29}$$

From this, we see that $(v_X:t_X)$ depends linearly on $(X_1:X_3)$, and not on X_2 or X_0. In the following, for any map $f(v:t)$ defined on $\mathbb{P}^1$, we write $f_X(X_1:X_3):=f(v_X:t_X)$. For instance, $\Sigma_X=X\vee K$ and (28) becomes

$$C_X=(\tilde{C}_X\cdot X_1:C_{2,X}:\tilde{C}_X\cdot X_3:C_{0,X}). \tag{30}$$

The line on the congruence passing through the point X is $\Gamma(X) = C_X \vee X$. To compute the image of X under the map Φ in (7), we need to find the unique $(u:s) \in \mathbb{P}^1$ such that $\Lambda((v_X : t_X), (u:s)) = \Gamma(X)$. To do that, it will be enough to consider the points at infinity, i.e., to solve $\Lambda_\infty((v_X:t_X),(u:s)) = \Gamma_\infty(X)$. From (30), we compute

$$\begin{aligned} \Gamma_\infty(X) &= (C_X \vee X)^\infty \\ &= (X_1 \cdot \alpha(X) : \beta(X) : X_3 \cdot \alpha(X)), \\ \text{where } \alpha(X) &:= C_{0,X} - X_0 \tilde{C}_X, \\ \beta(X) &:= X_2 C_{0,X} - X_0 C_{2,X}. \end{aligned} \tag{31}$$

Case 1: Non-constant rotation. We begin by computing Λ_∞ in the case of non-constant rotations. We make use of Lemma 37 and parametrize the RS_1 cameras involving $C, K, \Sigma^\vee$ using maps $\xi : \mathbb{P}^1 \dashrightarrow \mathrm{Gr}(1, \mathbb{P}^3)$ such that $C(v:t) \in \xi(v:t) \subset \Sigma(v:t)$ for all $(v:t) \in \mathbb{P}^1$. In particular, we have $\xi_\infty(v:t) \in \Sigma_\infty(v:t)$ for all $(v:t) \in \mathbb{P}^1$, which means that the map ξ_∞ is of the form $\xi_\infty = (-\tilde{\xi}\cdot\Sigma_3 : \xi_2 : \tilde{\xi}\cdot\Sigma_1)$. We will compute Λ_∞ by homogenizing (27). For that, we recall from Lemma 36 that $\Sigma_\infty^\vee(v:t) = Av + Bt$ with $A \cdot B = 0$ and $\|A\| = \|B\|$. After scaling, we may assume that the latter norm is 1. Then, $\|\Sigma_\infty^\vee(v:t)\| = \sqrt{v^2+t^2}$ and we obtain from (27) that

$$\begin{aligned} &\Lambda_\infty((v_X:t_X),(u:s)) = (\Sigma^\vee_{\infty,X} \times \xi_{\infty,X}) \cdot s \\ &+ \xi_{\infty,X} \cdot t_X \cdot u \\ &= (X_1 \cdot (a(X)s + b(X)u) : c(X)s + d(X)u : \\ &X_3 \cdot (a(X)s + b(X)u)), \end{aligned} \tag{32}$$

where $a(X) := \xi_{2,X}, b(X) := t_X \tilde{\xi}_X, c(X) := -\tilde{\xi}_X(X_1^2 + X_3^2)$, and $d(X) := t_X \xi_{2,X}$. Hence, comparing (31) and (32), we see that $\Lambda_\infty((v_X:t_X),(u:s)) = \Gamma_\infty(X)$ is equivalent to $(\alpha:\beta) = (s:u)\left[\begin{smallmatrix} a & c \\ b & d\end{smallmatrix}\right]$. The determinant of the latter 2×2 matrix is $t_X \cdot (\xi_{2,X}^2 + \tilde{\xi}_X^2(X_1^2+X_3^2))$, which is not the zero polynomial since $(X_1^2 + X_3^2)$ is not a square. Thus, for general X, the 2×2 matrix is invertible and the unique solution $(u:s) \in \mathbb{P}^1$ is given by

$$\begin{aligned} u_X &= a(X)\cdot\beta(X) - c(X)\cdot\alpha(X) \\ &= \xi_{2,X} \cdot (X_2 C_{0,X} - X_0 C_{2,X}) \\ &\quad + \tilde{\xi}_X(X_1^2+X_3^2)\cdot(C_{0,X} - X_0\tilde{C}_X), \\ s_X &= d(X)\cdot\alpha(X) - b(X)\cdot\beta(X) \\ &= t_X \Big(\xi_{2,X}\cdot(C_{0,X} - X_0 \tilde{C}_X) \\ &\quad -\tilde{\xi}_X \cdot (X_2 C_{0,X} - X_0 C_{2,X})\Big). \end{aligned}$$

In particular, we see that t_X divides s_X and therefore $\Phi(X) = (\frac{s_X}{t_X} v_X : u_X : s_X)$. To determine the degree of the map Φ, it is now sufficient to show that u_X is irreducible and to compute its degree. For that, we rewrite u_X as follows:

$$u_X = f(X) - X_0 \cdot f_0(X) + X_2 \cdot f_2(X),$$

where $f(X) := \tilde{\xi}_X(X_1^2+X_3^2)\cdot C_{0,X}$, $f_0(X) := \xi_{2,X}C_{2,X} + \tilde{\xi}_X(X_1^2+X_3^2)\cdot\tilde{C}_X$, and $f_2(X) := \xi_{2,X}C_{0,X}$. Note that f, f_0 and f_2 only depend on $(X_1 : X_3)$, and not on X_0 or X_2. Hence, the only way that u_X can be reducible is when f, f_0 and f_2 have a common factor. We now consider RS_1 cameras of type I and II separately to show that this is not possible.

Case 1a: Type I. In type I, the map ξ is given via a map $\lambda : \mathbb{P}^1 \dashrightarrow K$ such that $\xi(v:t) = C(v:t) \vee \lambda(v:t)$ for general $(v:t) \in \mathbb{P}^1$. Writing $\lambda = (0:\lambda_2:0:\lambda_0)$, this means that

$$\xi_\infty = (-\lambda_0\tilde{C}\cdot\Sigma_3 : \lambda_0 C_2 - \lambda_2 C_0 : \lambda_0\tilde{C}\cdot\Sigma_1),$$

i.e., $\tilde{\xi} = \lambda_0 \cdot \tilde{C}$ and $\xi_2 = \lambda_0 C_2 - \lambda_2 C_0$. Then, $f_0(X) = \lambda_{0,X}(C_{2,X}^2 + \tilde{C}_X^2(X_1^2+X_3^2)) - \lambda_{2,X}C_{0,X}C_{2,X}$. Hence, for every choice of C_0 and for sufficiently general $\lambda_0, \lambda_2, \tilde{C}, C_2$, we have that $\gcd(C_{0,X}, f_0) = 1$ and $\gcd(\xi_{2,X}, f) = 1$. Thus, f, f_0 and f_2 cannot have a common factor for a general RS_1 camera in $\mathcal{P}_{I,d,\delta}$. Therefore, u_X is generally irreducible and we conclude that $\deg\Phi = \deg u_X = 2d+\delta+1$, where $d = \deg C$ and $\delta = \deg\lambda$. The irreducibility argument worked for *every* choice of C_0, in particular in the case $d = 1$ and $C_0 = t$, which corresponds to the camera center moving on a line with constant speed. Hence, we obtain $\deg\Phi = \delta+3$ for a general RS_1 camera in $\mathcal{P}^{\mathrm{cs}}_{I,1,\delta}$.

Case 1b: Type II and III. In this case, $\tilde{C} = 0$. By Lemma 38, the Plücker coordinates of the map ξ are $(-h\Sigma_3 C_2 : 0 : -h\Sigma_3 C_0 : -h\Sigma_1 C_2 : p : h\Sigma_1 C_0)$, for some polynomials h and p. Hence, $\tilde{\xi} = hC_0$ and $\xi_2 = p$. Thus, we obtain that $f(X) = h_X C_{0,X}^2(X_1^2+X_3^2)$ and $f_0(X) = p_X C_{2,X}$, and so $\gcd(f, f_0) = 1$ and u_X is irreducible for every choice of C_0 and for sufficiently general h, p, C_2. Therefore, for a general RS_1 camera in $\mathcal{P}_{II,d,\delta}$, we find that $\deg\Phi = \deg u_X = 2d+\delta+2$, where $d = \deg C$ and $\delta = \deg h$. Also, a general RS_1 camera in $\mathcal{P}^{\mathrm{cs}}_{II,1,\delta}$ (where $C_0 = t$) satisfies $\deg\Phi = \delta+4$.

Case 2: Constant rotation. The second row of the constant rotation matrix R has to be $K^\infty = (0:1:0)$. Hence, we may assume that $R = I_3$. Then, $\Sigma^\vee = (-t:0:v:0)$ and we see from (29) that $(v_X : t_X) = (X_1 : X_3)$. We compute $\Lambda_\infty((v_X:t_X),(u:s))$ as the intersection of the lines that are dual to the points $\Sigma^\vee_{\infty,X} = (-X_3 : 0 : X_1)$ and $((0:-s:u)P(v_X:t_X))^\infty = (0:-s:u)$:

$$\Lambda_\infty((v_X:t_X),(u:s)) = (sX_1 : uX_3 : sX_3).$$

Comparing this with (31), we see that the unique solution $(u:s) \in \mathbb{P}^1$ to $\Lambda_\infty((v_X:t_X),(u:s)) = \Gamma_\infty(X)$ is

$(u_X : s_X) = (\beta(X) : X_3 \cdot \alpha(X))$. Therefore, $\Phi(X) = (X_1 \cdot \alpha(X) : \beta(X) : X_3 \cdot \alpha(X))$.

We will now show that $\beta(X) = X_2 C_{0,X} - X_0 C_{2,X}$ is generally irreducible. For every choice of C_0 and sufficiently general C_2, we have $\gcd(C_{0,X}, C_{2,X}) = 1$. The latter means that $\beta(X)$ is irreducible since $C_{0,X}$ and $C_{2,X}$ only depend on $(X_1 : X_3)$. This argument does not involve $\tilde{C}$. Hence, a general RS_1 camera in either $\mathcal{P}_{I,d}$ or $\mathcal{P}_{II,d}$ satisfies $\deg \Phi = \deg \beta = d + 1$. Since C_0 was arbitrary, this also holds for cameras moving along a line with constant speed. □

9.6. Minimal problems for several linear RS_1 cameras

This section and the next provide proofs for all claims in Section 6. The following lemma is an extended version of Lemma 15.

Lemma 41. *Let $(R, \mathcal{C}, C) \in \mathcal{P}_1^{cs}$ be of type I and let $(a : b : 0)$ be such that $\mathcal{C}^\infty = (a : b : 0) \cdot R$. Consider the associated picture-taking map $\Phi\colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ and a line $L \subset \mathbb{P}^3$. Then, we have:*

- *either $\overline{\Phi(L)}$ is a conic through the points $(0 : 1 : 0)$ and $(a : b : 0)$,*
- *or $\overline{\Phi(L)}$ is the line through the points $(0 : 1 : 0)$ and $(a : b : 0)$.*

Conversely, for a generic conic c through $(0 : 1 : 0)$ and $(a : b : 0)$, there exists a one-dimensional family of lines $L \subseteq \mathbb{P}^3$ with $c = \overline{\Phi(L)}$. These lines rule a smooth quadric surface in $\mathbb{P}^3$.

Proof. Up to a change of coordinates, as in the proof of Proposition 10, we may assume that $R = I_3$ and $\mathcal{C}$ is the line parametrised by

$$C\colon \mathbb{P}^1 \dashrightarrow \mathbb{P}^3,\ (v : t) \mapsto (av : bv : 0 : t)$$

for some real parameters a, b with $(a : b) \neq 0$. Note that this is indeed an element of $\mathcal{P}_1^{cs}$ because $\mathcal{C} \neq \mathcal{C}^\infty = (a : b : 0 : 0) = C(1 : 0)$. By (2), we obtain $\Sigma \vee (v : t) = (-t : 0 : v : av)$. The proof of this lemma proceeds by first observing that in this setting the picture-taking map has a particularly elegant description. More precisely, the projection matrix is given by

$$P(v : t) = \begin{pmatrix} t & 0 & 0 & -av \\ 0 & t & 0 & -bv \\ 0 & 0 & t & 0 \end{pmatrix}.$$

Let $X = (X_1 : X_2 : X_3 : X_0) \in \mathbb{P}^3$. We aim to determine the time at which the camera sees X. For this, we need to check which rolling plane contains X, i.e., we need to solve

$$-tX_1 + vX_3 + avX_0 = 0.$$

However, we see immediately that $t = v \cdot \frac{X_3 + aX_0}{X_1}$ for $X_1 \neq 0$. Thus, we obtain for the picture-taking map $\Phi\colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$,

$$(X_1 : X_2 : X_3 : X_0) \mapsto (v\frac{X_3 + aX_0}{X_1} X_1 - avX_0 : \\ v\frac{X_3 + aX_0}{X_1} X_2 - bvX_0 : v\frac{X_3 + aX_0}{X_1} X_3).$$

We observe that v is a common factor of all entries, i.e. we obtain

$$\Phi\colon X \mapsto ((X_3 + aX_0)X_1 - aX_1X_0 : \\ (X_3 + aX_0)X_2 - bX_1X_0 : (X_3 + aX_0)X_3). \tag{33}$$

We now fix two points $\alpha, \beta \in \mathbb{P}^3$ with $\alpha = (\alpha_1 : \alpha_2 : \alpha_3 : \alpha_0)$ and $\beta = (\beta_1 : \beta_2 : \beta_3 : \beta_0)$, and consider the line $L = \alpha \vee \beta$ they span. The following MACAULAY2 computation

```
QQ[x0,x1,x2,x3,y0,y1,y2,
a0,a1,a2,a3,b0,b1,b2,b3,s,t,a,b]
I=ideal(y1-((x3+a*x0)*x1-a*x1*x0),
y2-((x3+a*x0)*x2-b*x1*x0),
     y0-(x3+a*x0)*x3,x0-(s*a0+t*b0),
     x1-(s*a1+t*b1),x2-(s*a2+t*b2),
     x3-(s*a3+t*b3))
J=eliminate(I,{x0,x1,x2,x3,s,t})
```

shows that $\Phi(L)$ is cut out by the quadratic polynomial

$$b(\alpha_0\beta_3 - \alpha_3\beta_0)y_1^2 + (\alpha_2\beta_1 - \alpha_1\beta_2)y_0^2 \\ + a(\alpha_3\beta_0 - \alpha_0\beta_3)y_1y_2 + (a\alpha_0\beta_2 - a\alpha_2\beta_0 - b\alpha_0\beta_1 \\ + b\alpha_1\beta_0 + \alpha_3\beta_2 - \alpha_2\beta_3)y_1y_0 + (\alpha_1\beta_3 - \alpha_3\beta_1)y_2y_0.$$

For the rest of the discussion, we denote the coefficient of y_iy_j by c_{ij}. We first note that $c_{22} = 0$, thus any conic that is an image of a line passes through the point $(0 : 1 : 0)$. This has already been observed in Proposition 13. Moreover, we observe that $ac_{11} + bc_{12} = 0$. Therefore any such conic passes through the point $(a : b : 0)$. The following continuation of our previous MACAULAY2 calculation shows that these are the only two conditions satisfied by a generic image conic:

```
M=first entries gens J
f=M_(00)
T=QQ[a0,a1,a2,a3,b0,b1,b2,b3,
a,b][y0,y1,y2]
g=sub(f,T)
N=last coefficients(g,Monomials=>
{y1^2,y2^2,y0^2,y1*y2,y1*y0,y2*y0})
T1=QQ[a0,a1,a2,a3,b0,b1,b2,b3,a,b,
z0,z1,z2,z3,z4,z5]
Nnew=sub(N,T1)
Check=ideal(z0-Nnew_(0,0),z1-Nnew_(1,0),
z2-Nnew_(2,0),z3-Nnew_(3,0),
```

```
z4-Nnew_(4,0),z5-Nnew_(5,0))
N1=eliminate(Check,{a0,a1,a2,a3,
b0,b1,b2,b3})
```

Thus, for a generic conic c through the two points $(0:1:0)$ and $(a:b:0)$, there is a line L such that $\overline{\Phi(L)} = c$. Since the space of such conics is three-dimensional and $\dim \mathrm{Gr}(1, \mathbb{P}^3) = 4$, there has to be in fact a one-dimensional family of such lines. It can be obtained as follows: Fix a generic line L with $\overline{\Phi(L)} = c$. Consider the set of all lines that meet $\mathcal{C}$, K, and L. These are all camera rays that Φ contracts to points on the conic c. These camera rays rule a smooth quadric surface since the lines $\mathcal{C}$, K, and L are generically pairwise skew. The other ruling of that quadric contains K, $\mathcal{C}$, and L; and therefore, any line L' in that second ruling will also satisfy $\overline{\Phi(L')} = c$. $\square$

In terms of the joint-camera map, minimality means that $\Phi^{(m)}$ in (19) is dominant and has generically finite fibers. A necessary condition for a reconstruction problem to be minimal is for it to be *balanced*.

Definition 42. The Reconstruction Problem 14 is called *balanced* if domain and codomain of its associated joint-camera map $\Phi^{(m)}$ in (19) have the same dimensions.

To determine the dimensions of the varieties $\mathfrak{X}$ and $\mathcal{Y}$, we count the elements of a general arrangement X in $\mathfrak{X}$ in the following way:

1. Count all *free points*: in any collection of points, some may be *dependent* on others, defined as being collinear with two other points. Each minimal set of independent points (which we call free points) has the same cardinality, which we denote by p_0.
2. Count all *third collinear points*: whenever more than two points are collinear, we pick three of them (always including the free points if present) and we count the dependent points among the three as third collinear points. Write their amount as p_3.
3. Count all *further collinear points*: These are the points that are collinear with at least three already-counted points. Write their amount as p_4. Note that $p_0 + p_3 + p_4 = p$.
4. Count all *free* lines, i.e., the lines that do not contain any points. Write their amount as ℓ_0.
5. Count all *lines incident to* exactly *one point*. Write their amount as ℓ_1.
6. Count all *lines incident to* exactly *two points*. Write their amount as ℓ_2.

In this way, we counted all elements of X and associated the following combinatorial data to it:

p_0	free points
p_3	third collinear points
p_4	further collinear points
ℓ_0	free lines
ℓ_1	lines incident to one point
ℓ_2	lines incident to two points.

Recall that in addition $p_\infty \in \{0, 1\}$ indicates whether we know the point $(a:b:0)$ from Lemma 41. We observe the following implication:

$$p_\infty = 0 \quad \Rightarrow \quad \ell_0 = \ell_1 = \ell_2 = p_4 = 0. \tag{34}$$

Indeed, if $\ell_i > 0$, then there is an image conic that determines the point $(a:b:0)$. Also, if $p_4 > 0$, then there are four collinear points in space. Their image points, together with $(0:1:0)$, determine a unique image conic, which gives us again the point $(a:b:0)$.

Now, we are able to compute the dimension of the source and target spaces of the joint-camera map $\Phi^{(m)}$:

Lemma 43. *We have* $\dim \mathcal{P} = 8$, $\dim G = 7$, *and*

$$\begin{aligned}\dim \mathfrak{X} &= 3p_0 + p_3 + p_4 + 4\ell_0 + 2\ell_1,\\ \dim \mathcal{Y} &= p_\infty + 2p_0 + 2p_3 + p_4 + 3\ell_0 + 2\ell_1 + \ell_2.\end{aligned}$$

Proof. For the dimensions of the camera space and the group, see Section 4.3 and Remark 1. In $\mathbb{P}^3$, once a line is fixed by two points, the further points on this line have only one degree of freedom. In $\mathbb{P}^2$, lines become conics, and these are only fixed after their third image point is counted. This explains the differentiation between p_3 and p_4. Apart from these considerations, computing $\dim \mathfrak{X}$ and $\dim \mathcal{Y}$ is straightforward. $\square$

Corollary 44. *The Reconstruction Problem 14 is balanced if and only if*

$$\begin{aligned}&3p_0 + p_3 + p_4 + 4\ell_0 + 2\ell_1 - 7\\ &= m(p_\infty + 2p_0 + 2p_3 + p_4 + 3\ell_0 + 2\ell_1 + \ell_2 - 8).\end{aligned} \tag{35}$$

Proof. This follows immediately from (19) and Lemma 43. $\square$

Now, we classify the minimal problems for at least two cameras. In Section 9.7, we show that there are no minimal problems for a single camera.

Proposition 45. *There are finitely many balanced problems for* $m > 1$.

Proof. All variables in (35) are non-negative integers. These become bounded from above if we were to require both sides of (35) to be zero or negative. Thus, there are only finitely many sequences I =

$(p_0, p_3, p_4, \ell_0, \ell_1, \ell_2, p_\infty)$ where both sides of (35) can be zero or negative. But a computer search reveals that this cannot happen. Thus, we may assume that the right-hand side of (35) is positive and write

$$m - 1 = \frac{p_0 - p_3 + \ell_0 - \ell_2 + 1 - p_\infty}{2p_0 + 2p_3 + p_4 + 3\ell_0 + 2\ell_1 + \ell_2 - 8 + p_\infty}.$$

Since the denominator in the right-hand side is positive, a necessary condition for m to be an integer is that this denominator is smaller than or equal to the numerator. This condition is equivalent to

$$p_0 + 3p_3 + p_4 + 2\ell_0 + 2\ell_1 + 2\ell_2 + 2p_\infty \leq 9.$$

Thus, there are only finitely many choices for the combinatorial datum $I = (p_0, p_3, p_4, \ell_0, \ell_1, \ell_2, p_\infty)$. For each such sequence I, there are only finitely many reconstruction problems. □

A computer search reveals the full list of sequences $I = (p_0, p_3, p_4, \ell_0, \ell_1, \ell_2, p_\infty)$ satisfying (35) for some $m > 1$. Here, we notice that some sequences give rise to two balanced problems due to combinatorial ambiguity. For instance, the two lines of the configuration $I = (3, 0, 0, 0, 1, 1, 1)$ could intersect at one of the three points, or not. To account for this, we introduce a further combinatorial datum $\gamma \in \{0, 1, \ldots, \ell_1 + \ell_2\}$ that counts the maximal number of lines going through any free point in the configuration. As it turns out, this resolves all ambiguities in our list of balanced problems.

Theorem 46. *Let $m > 1$. The Reconstruction Problem 14 is balanced if and only if its combinatorial signature $I = (p_0, p_3, p_4, \ell_0, \ell_1, \ell_2, \gamma, p_\infty)$ is contained in the following list:*

$m = 2$	$m = 2$	$m = 2$
$(1,0,0,0,3,0,3,1)$	$(3,0,0,0,1,1,1,1)$	$(4,1,0,0,0,0,0,1)$
$(1,0,0,1,2,0,2,1)$	$(3,0,0,0,1,1,2,1)$	$(5,0,0,0,0,1,1,1)$
$(1,0,0,2,1,0,1,1)$	$(3,0,0,0,2,0,1,1)$	$(5,0,0,0,1,0,1,1)$
$(1,0,0,3,0,0,0,1)$	$(3,0,0,0,2,0,2,1)$	$(5,0,0,1,0,0,0,1)$
$(2,1,0,0,1,0,1,1)$	$(3,0,0,1,0,1,1,1)$	$(7,0,0,0,0,0,0,1)$
$(2,1,0,1,0,0,0,1)$	$(3,0,0,1,1,0,1,1)$	$(3,2,0,0,0,0,0,0)$
$(2,1,2,0,0,0,0,1)$	$(3,0,0,2,0,0,0,1)$	$(6,1,0,0,0,0,0,0)$
$(3,0,0,0,0,2,2,1)$	$(3,1,1,0,0,0,0,1)$	$(9,0,0,0,0,0,0,0)$

$m = 3$	$m = 4$	$m = 5$
$(2,0,0,0,2,0,1,1)$	$(1,0,0,2,0,0,0,1)$	$(4,0,0,0,0,0,0,1)$
$(2,0,0,0,2,0,2,1)$	$(3,0,0,0,1,0,1,1)$	
$(2,0,0,1,0,1,1,1)$	$(5,0,0,0,0,0,0,0)$	
$(3,0,0,1,0,0,0,1)$		
$(3,1,0,0,0,0,0,1)$		
$(4,1,0,0,0,0,0,0)$		

The three problems printed in gray, that is, $(1,0,0,0,3,0,3,1)$, $(2,1,2,0,0,0,0,1)$, and $(3,1,1,0,0,0,0,1)$, are non-minimal. All other 31 problems are minimal.

Proof. This is the list of all balanced problems, obtained from augmenting I by the additional combinatorial datum γ as described. To show that the problems printed in black are minimal, we carefully construct the joint camera-map map $\Phi^{(m)}$ for each balanced problem in the computer algebra system `Macaulay2` [20] and test that its Jacobian at a randomly chosen point is invertible. This shows that the derivative of $\Phi^{(m)}$ is surjective almost everywhere, and thus that $\Phi^{(m)}$ is dominant. The balancedness condition ensures that the fibers of such a dominant $\Phi^{(m)}$ are generically finite. The non-minimality of the three problems printed in gray is shown in Lemmas 47 and 48 below. □

Lemma 47. *There is no minimal problem with $m > 1$ and $p_4 > 0$ (i.e., four or more collinear points in space).*

Proof. We show that the joint-camera map $\Phi^{(m)}$ in (19) cannot be dominant whenever $m > 1$ and $\mathfrak{X}$ contains four collinear points. We assume for contradiction that there exists such a dominant map. Hence, for generic $Y_1, \ldots, Y_m \in \mathcal{Y}$, there are m linear RS_1 cameras and an arrangement X in $\mathfrak{X}$ such that Y_i is the picture of X under the i-th camera. The arrangement X contains four collinear points. We denote their image points on the i-th image Y_i by $x_{i,1}, x_{i,2}, x_{i,3}, x_{i,4}$. Together with the point $(0:1:0)$, they form a conic c_i in the i-th image plane. That conic is the image of the line $L \subset \mathbb{P}^3$ that is spanned by the four collinear points in space. Restricting the i-th camera map Φ_i (see (7)) to the line L, yields a birational map from L to the image conic c_i.

Now we consider the birational projection π_i of the conic c_i away from the point $(0:1:0)$ onto the line l_i that is spanned by $(1:0:0)$ and $(0:0:1)$. We denote the image of the point $x_{i,j}$ under this projection by $y_{i,j} \in l_i$. The composed map $\pi_i \circ \Phi_i|_L$ is an isomorphism from the line L onto the line l_i. Therefore, $\pi_2 \circ \Phi_2|_L \circ (\pi_1 \circ \Phi_1|_L)^{-1}$ is an isomorphism between the lines l_1 and l_2 that maps the four points $y_{1,1}, \ldots, y_{1,4}$ onto the four points $y_{2,1}, \ldots, y_{2,4}$. Hence, the cross-ratio of the four points $y_{1,1}, \ldots, y_{1,4}$ has to equal the cross-ratio of $y_{2,1}, \ldots, y_{2,4}$. This contradicts that the pictures Y_1 and Y_2 were chosen generically. □

Lemma 48. *There is no minimal problem with $m > 1$, a single point, and without free lines.*

Proof. An arrangement X in $\mathfrak{X}$ consists of a single point and possibly some lines passing through that point. For such an arrangement X and an arbitrary m-tuple of linear RS_1 cameras, we show that – even after modding out the group G – there are infinitely many m-tuples of cameras that yield the same m pictures. This implies that the generic fiber of $\Phi^{(m)}$ cannot be finite, and so the reconstruction problem cannot be minimal.

We begin by modding out the group G. To fix the translation, we assume that the single point in $\mathbb{P}^3$ is at the origin in $\mathbb{R}^3$, i.e., $(0 : 0 : 0 : 1)$. To fix the global scaling, we assume that the Euclidean distance in $\mathbb{R}^3$ between the origin and the starting position $C_2(0)$ of the second camera is one. Finally, we fix the rotation of the first camera to be $R_1 = I_3$.

In particular, we have not assumed anything about the linear constant-speed movement $C_1 : \mathbb{R}^1 \to \mathbb{R}^3$ of the first camera. Since X consists of a single point with some adjacent lines, at any given time $x \in \mathbb{R}^1$, the arrangement X looks the same from any camera center on the ray starting from the origin and passing through $C_1(x)$. In other words, when positioned at $C_1(x)$, one can move back and forth along the line spanned by $C_1(x)$ and the origin without changing the picture of X at time x. Therefore, we can modify the first camera by multiplying the map C_1 with any non-zero scalar without changing the image of X under the first camera. □

All minimal problems listed in Theorem 46 are depicted in Figure 4. To conclude the proof of Theorem 17, we need to exclude the possibility of minimal problems for a single camera. We will explain this in Section 9.7. Beforehand, we explain our degree computations reported in Figure 4.

Remark 49. For each of the finitely many minimal problems, we used the numerical algebraic geometry package `HomotopyContinuation.jl` [11] to compute their respective degrees, i.e. the number of solutions of a general instance of a reconstruction problem with that combinatorial type. More precisely, we used the monodromy method provided in that package. This method can strictly speaking only provide lower bounds for the degree, but multiple runs can be tried and if the results stays the same across runs, this indicates that the number computed is the actual degree. In our computations, small degrees (<2000) remained constant as described above throughout six monodromy runs, while larger degrees tended to differ run-by-run. In any case, we report in Figure 4 the largest computed value, indicating the larger degrees with an asterisk. A plus indicates that the computation had to be interrupted after running more than 5 days. We used interval certification to guarantee that for each computed value there are indeed at least that many solutions. This holds even for the interrupted computations.

9.7. No minimal problem for a single linear RS_1 camera

Our next aim is to prove that no minimal problems arise from a single camera. Throughout this section, we assume that $m = 1$. Then, by Corollary 44, the Reconstruction Problem 14 is balanced if and only if

$$p_0 + \ell_0 + 1 = p_\infty + p_3 + \ell_2.$$

Lemma 50. *For $m = 1$, there is no minimal problem with ℓ_0 positive.*

Proof. If $p_\infty = 0$, the assertion follows from (34). Otherwise, Lemma 41 tells us that there exists a one-dimensional family of lines in $\mathbb{P}^3$ mapping to a generic image conic in $\mathbb{P}^2$ via Φ, i.e., a free image conic has infinitely many solutions. □

Therefore, we may assume that $\ell_0 = 0$ and obtain

$$p_0 + 1 = p_\infty + p_3 + \ell_2 \tag{36}$$

as numerical condition for our balanced problems. We observe that this equation only involves free points, third collinear points, and lines incident to two points. If $p_\infty = 0$, the remaining features do not appear by (34). If $p_\infty = 1$, they do not affect minimality:

Lemma 51. *For any minimal problem with $m = 1$ and $p_\infty = 1$, one can add or remove further collinear points (corresponding to p_4) and lines incident to one point (corresponding to ℓ_1) without changing minimality.*

Proof. Since every minimal problem is balanced and neither p_4 nor ℓ_1 appear in the balanced equation (36), it is sufficient to show that the existence of solutions is not affected by adding or removing further collinear points or lines incident to one point.

The existence of solutions is not affected by removing data, thus this part is clear. In order to show that adding such points and lines does not affect minimality, we consider the following set-up:

Let $\widetilde{\mathfrak{X}}$ be the set of arrangements obtained from arrangements in $\mathfrak{X}$ after forgetting all further collinear points and all lines incident to exactly one point. Similarly, we consider $\widetilde{\mathcal{Y}}$ obtained from $\mathcal{Y}$. Thus, we obtain forgetful maps

$$\alpha\colon \mathcal{Y} \to \widetilde{\mathcal{Y}} \quad \text{and} \quad \beta\colon \mathfrak{X} \to \widetilde{\mathfrak{X}}.$$

Analogously to the joint camera map $\Phi^{(1)}$ in (19), we now obtain a map

$$\tilde{\Phi}^{(1)}\colon (\mathcal{P} \times \widetilde{\mathfrak{X}})/G \dashrightarrow \widetilde{\mathcal{Y}}. \tag{37}$$

We need to show that for generic $\tilde{y} \in \widetilde{\mathcal{Y}}$ and generic $y \in \mathcal{Y}$ with $\alpha(y) = \tilde{y}$, any solution $\tilde{S} \in (\mathcal{P} \times \widetilde{\mathfrak{X}})/G$ with $\tilde{\Phi}^{(1)}(\tilde{S}) = \tilde{y}$ can be uniquely extended to a solution $S \in (\mathcal{P} \times \mathfrak{X})/G$ with $\Phi^{(1)}(S) = y$. By the same arguments as in the proof of Lemma 41, we see that, for a generic line $L \subset \mathbb{P}^3$ with image conic c, the map $\Phi_{|L}\colon L \to c$ is birational. Therefore, when c is the conic passing through three points that are the images of three collinear points in $\mathbb{P}^3$ and the two known points at infinity from Lemma 41 and L is the solution of c in $\tilde{S}$, then any further point on c corresponds to a unique point on L. Thus, adding further collinear points does not affect minimality.

In order to add a line incident to exactly one point, we consider a generic point $x \in \mathbb{P}^2$ with fixed pre-image $X \in \mathbb{P}^3$ and fix a generic image conic c passing through x. We claim there is a unique line $L \subset \Phi^{-1}(c)$ passing through X with $\Phi(L) = c$. Recall from Lemma 41 that the lines mapping to c rule a smooth quadric surface Q_c. Any smooth quadric surface is doubly ruled, i.e., there are two lines passing through each point. The pre-image of any point on c is a line in $\mathbb{P}^3$ that is contracted, i.e, one of the lines in the quadric Q_c passing through X is the pre-image of x. The other line through X is not contracted and therefore maps to c. This is the desired unique line. $\square$

Due to Lemmas 50 and 51, we may from now on assume that $\ell_0 = \ell_1 = p_4 = 0$. This means that the generic arrangement in $\mathfrak{X}$ only contains free points, third collinear points, and lines incident to exactly two points. We encode these incidences in a graph:

Definition 52. Let X be a generic arrangement in $\mathfrak{X}$. We consider the points and lines in X in $\mathbb{R}^3$. This point-line arrangement induces a graph $\mathfrak{G} = \mathfrak{G}(p, \ell, \mathcal{I})$ whose vertices are the points and whose edges are the line segments connecting points.

For a balanced problem with $\ell_0 = \ell_1 = p_4 = 0$, this graph has $p_0 + p_3$ many vertices and its number of edges is $\ell_2+2p_3 = p_0+p_3+1-p_\infty$, where the latter equality follows from (36). Thus, if $p_\infty = 1$, the graph has the same amount of edges as vertices; if $p_\infty = 0$, it has one more edge. For every connected component $\mathfrak{C}$ of the graph $\mathfrak{G}$, we denote by $\mathfrak{X}_\mathfrak{C}$ the set of arrangements obtained from arrangements in $\mathfrak{X}$ after forgetting all point and lines, except those in the component $\mathfrak{C}$.

Proposition 53. *Let $\mathfrak{C}$ be a connected component of the graph $\mathfrak{G}$. For a generic $X \in \mathfrak{X}_\mathfrak{C}$, there is a one-dimensional family of arrangements X' in $\mathfrak{X}_\mathfrak{C}$ such that $\Phi(X) = \Phi(X')$.*

The statement above is the key insight for why there are no minimal problems for a single camera. Before we can prove the proposition, we need the following helpful tool. Recall that the congruence associated with a linear RS_1 camera consists of all lines meeting both $\mathcal{C}$ and K. Therefore, for a generic image point $x \in \mathbb{P}^2$, the line $\overline{\Phi^{-1}(x)}$ meets both lines $\mathcal{C}$ and K.

Lemma 54. *Let c be the image conic of a generic line in $\mathbb{P}^3$ under Φ. Moreover, let $x, x' \in c$ be generic points, and $L := \overline{\Phi^{-1}(x)}, L' := \overline{\Phi^{-1}(x')}$. Then, there is an isomorphism $\gamma\colon L \to L'$ such that $\overline{\Phi(X \vee \gamma(X))} = c$ for almost all $X \in L$. Furthermore, we have that $\gamma(L \cap \mathcal{C}) = L' \cap \mathcal{C}$ and $\gamma(L \cap K) = L' \cap K$.*

Proof. Recall that the lines that Φ maps onto the conic c rule a smooth quadric surface Q_c, which is therefore doubly ruled. A generic point $X \in L$ is met by two lines on the quadric Q_c. One of the lines is L itself. Let $\tilde{L}$ be the other line. Then $\tilde{L}$ intersects L' in a single point X'. We define $\gamma(X) := X'$. This is clearly a well-defined isomorphism from L to L'. For those X outside the base locus of Φ, the line $\tilde{L}$ is mapped by Φ onto the conic c. This proves the first assertion.

For the second assertion, we start by recalling that the quadric Q_c contains both lines $\mathcal{C}$ and K. Note that both $\mathcal{C}$ and K are in Φ's base locus. If $X \in L$ is the point of intersection between L and $\mathcal{C}$, then the two lines on the quadric Q_c that pass through X are L and $\mathcal{C}$. By our definition of γ, we have that $\gamma(X)$ is the point of intersection between L' and $\mathcal{C}$. The same reasoning applies to the point of intersection between L and K. $\square$

Proposition 53. Every vertex V in the graph $\mathfrak{C}$ corresponds to a point V in the arrangement X, and thus gives rise to a line $L_V := \overline{\Phi^{-1}(\Phi(V))}$. Every edge E between two vertices V_1 and V_2 corresponds to an image conic $\overline{\Phi(V_1 \vee V_2)}$, and thus gives rise to an isomorphism $\gamma_E : L_{V_1} \to L_{V_2}$ as in Lemma 54. By construction of γ_E as seen in the proof of Lemma 54, we see that γ_E sends V_1 to V_2.

We start by assuming that the component $\mathfrak{C}$ is a spanning tree. In that case, we can start at any vertex V_1 and pick a generic point V_1' on the line L_{V_1}. For any neighboring vertex V_2 connected to V_1 by the edge E, we define $V_2' := \gamma_E(V_1')$. By Lemma 54, the lines $V_1 \vee V_2$ and $V_1' \vee V_2'$ have the same image under Φ. Continuing like this throughout the whole component $\mathfrak{C}$, we construct an arrangement X' with $\Phi(X) = \Phi(X')$. By varying V_1' on the line L_{V_1}, we see that this family of arrangements X' is one-dimensional.

If $\mathfrak{C}$ contains a cycle, then we consider any vertex V_1 on the cycle and, by composing all the edge maps γ_E along the cycle, we obtain an automorphism $\gamma_{V_1} : L_{V_1} \to L_{V_1}$. Since L_{V_1} is a projective line, every such automorphism has either exactly two fixpoints or is the identity. We already know from Lemma 54 that γ_{V_1} has the two fixpoints $L_{V_1} \cap \mathcal{C}$ and $L_{V_1} \cap K$. Since V_1 is a point in the original arrangement X, the point V_1 must also be a fixpoint of γ_{V_1}, which shows that γ_{V_1} must be the identity. Since this holds for every cycle in the connected graph $\mathfrak{C}$, we can – as in the case of spanning trees – start at any vertex V_1 and with any point V_1' on the line L_{V_1} and then use the graph maps γ_E to define an arrangement X' with the same image under Φ as the original arrangement X. $\square$

Corollary 55. *For $m = 1$, the Reconstruction Problem 14 is never minimal.*

Proof. We can understand the domain $(\mathcal{P} \times \mathfrak{X})/G$ of the joint-camera map $\Phi^{(1)}$ by modding out the group G from the camera space $\mathcal{P}$. Indeed, modulo the group G, every equivalence class of cameras in (18) is represented by a tuple $(I_3, (0:0:0:1) \vee (a:b:0:0), C : (v:t) \mapsto (av :$

$bv : 0 : t))$, where $a^2 + b^2 = 1$. We consider a generic such camera representative and a generic arrangement $X \in \mathfrak{X}$. Let Φ be the picture-taking map of that camera. By Lemmas 50 and 51, we may assume that $\ell_0 = \ell_1 = p_4 = 0$. Then, by Proposition 53, there is an η-dimensional family of arrangements $X' \in \mathfrak{X}$ with $\Phi(X') = \Phi(X)$, where η is the number of connected components of $\mathfrak{G}$. Hence, the dimension of the fiber of $\Phi(X)$ under the joint-camera map $\Phi^{(1)}$ is at least $\eta > 0$, and so the reconstruction problem cannot be minimal. $\square$

This concludes the proof of Theorem 17.

9.8. Straight-Cayley cameras

In this section, we prove the statements from Section 7.

Proposition 18. To find the curve on which the camera center in (22) moves, we first substitute a for $u - u_0$ and then eliminate a from (22). This is what the following `Macaulay2` [20] computation does:

```
restart
C2R = c -> matrix {
{c_(0,0)^2-c_(1,0)^2-c_(2,0)^2+1,
2*c_(0,0)*c_(1,0)+2*c_(2,0),
    2*c_(0,0)*c_(2,0)-2*c_(1,0)},
{2*c_(0,0)*c_(1,0)-2*c_(2,0),
-c_(0,0)^2+c_(1,0)^2-c_(2,0)^2+1,
    2*c_(1,0)*c_(2,0)+2*c_(0,0)},
{2*c_(0,0)*c_(2,0)+2*c_(1,0),
2*c_(1,0)*c_(2,0)-2*c_(0,0),
    -c_(0,0)^2-c_(1,0)^2+c_(2,0)^2+1}

Rng = QQ[a,c_1..c_3,o_1..o_3,t_1..t_3,
v_1..v_3]
C = genericMatrix(Rng,c_1,3,1)
O = genericMatrix(Rng,o_1,3,1)
T = genericMatrix(Rng,t_1,3,1)
V = genericMatrix(Rng,v_1,3,1)
Rd = C2R(a*O)
-- Non-normalized Cayley matrix of rotations
d = 1+a^2*o_1^2+a^2*o_2^2+a^2*o_3^2
-- The denominator to normalize R
-- Equations for the camera center C
-- in the world coordinate system
e = flatten entries(d*C +
transpose(Rd)*(a*V+T))
-- Avoid the components for d=0
eMinusD = saturate(ideal(e),ideal(d))
-- Eliminate a
Rne = QQ[flatten entries (vars Rng)_{1..12}]
curve = sub(eliminate({a},eMinusD),Rne)
```

The last line computes the ideal of the curve $\mathcal{C}$ along which the camera center moves. By plugging in random values for the parameters o_i, t_j, v_k, we see that the ideal describes a twisted cubic curve in the variables c_1, c_2, c_3.

The projection matrix $P(a)$ and the rolling planes map $\Sigma^\vee(a)$ can be computed as follows:

```
P = Rd|(d*(a*V+T))
Sigma = matrix{{1,0,-a}}*P
```

Again, by plugging in random values for the parameters o_i, t_j, v_k, we see that the rolling planes map $\Sigma^\vee$ is of degree four in a. This means that for a generic space point X, there are four solutions in a such that X is contained in the rolling plane $\Sigma(a)$. Each such solution a gives rise to an image point $P(a) \cdot X$. $\square$

Proof of Theorem 19. We continue to work with the `Macaulay2` code from above. First, we can check for every choice of parameters that either $\Sigma(0) \cap \Sigma(1)$ or $\Sigma(0) \cap \Sigma(2)$ is a line (and not a plane):

```
M1small = sub(Sigma, a => 0)
|| sub(Sigma, a => 1)
I1small = sub(minors(2,M1small), Rne)
M2small = sub(Sigma, a => 0)
|| sub(Sigma, a => 2)
I2small = sub(minors(2,M2small), Rne)
radical(I1small+I2small) == ideal 1_Rne
```

Hence, if all rolling planes are supposed to intersect in a line K, then K has to be one of those two lines. This means that we have to express the condition that $\Sigma(0) \cap \Sigma(i)$ (for $i \in \{1, 2\}$) being a line implies that it is contained in all rolling planes $\Sigma(a)$. These conditions are encoded in the following ideals `I1` and `I2`:

```
M1 = M1small || Sigma
M2 = M2small || Sigma
I1 = minors(3,M1)
I2 = minors(3,M2)
```

Since the conditions are supposed to hold for *all* choices of a, when viewing the elements in `I1+I2` as polynomials in a, all of their coefficients have to be zero. We collect all those coefficients in an ideal `I` and decompose it into prime ideals:

```
Rng2 = Rne[a]
L = flatten entries gens sub(I1+I2, Rng2)
I = sub(ideal flatten apply(L, eq ->
     flatten entries last
     coefficients eq), Rne)
dec = decompose I
```

This yields 5 prime ideals:

$$\begin{aligned}
I_1 &:= \langle v_3, o_3, o_2+1\rangle,\\
I_2 &:= \langle v_3, o_1, o_2^2+o_3^2+o_2\rangle,\\
I_3 &:= \langle v_3, o_3t_1-o_1t_3+o_1v_1, 2o_2^2+2o_3^2+3o_2+1,\\
&\qquad 2o_1^2+o_2+1, 2o_2t_1^2+t_1^2-t_3^2+2t_3v_1-v_1^2,\\
&\qquad 2o_1o_2t_1+o_1t_1-o_3t_3+o_3v_1\rangle,\\
I_4 &:= \langle t_3-v_1, o_3, 2o_2+1, 4o_1^2+1\rangle,\\
I_5 &:= \langle v_3, t_3-v_1, t_1, o_3, o_1\rangle.
\end{aligned}$$

The ideals I_1, I_2, I_5 appear in Theorem 19. The ideal I_4 does not have any real solutions due to its last quadratic generator. Finally, all real solutions of I_3 are already contained in the zero sets of I_1 and I_2. We can see the latter claim by using the generator $2o_1^2+o_2+1 \in I_3$: After substituting $o_2 \mapsto -2o_1^2-1$, the generator $2o_2^2+2o_3^2+3o_2+1$ becomes $8o_1^4+2o_1^2+2o_3^2$, whose only real solutions are $o_1 = 0 = o_3$. In that case, o_2 becomes -1, and all real solutions of I_3 are already solutions of I_1 and I_2. This concludes the proof of the first part of Theorem 19.

To see that the camera moves along a twisted cubic curve for a generic choice of parameters in each of the three components described by I_1, I_2, I_5, it is sufficient to verify this at a single example for each component. This is done in Examples 21 to 23.

When imposing the conditions in ideal I_1 on the rolling planes map $\Sigma^\vee$, we see as follows that it becomes linear:

```
Sigma1 = sub(Sigma, {v_3 => 0, o_3 => 0,
o_2 => -1})
factor1 = gcd flatten entries Sigma1
1/factor1*Sigma1
```

This reveals for parameters in the zero set of I_1 that $\Sigma^\vee$ is the birational map $a \mapsto (1, 0, a, -at_3+av_1+t_1)$. Since $\Sigma^\vee$ is linear in a, such a RS camera has order one.

Similarly, for the ideal I_2, the code

```
use Rng
Sigma2 = sub(Sigma, {v_3 => 0,
o_1 => 0})
factor2 = first flatten entries
sub(Sigma2, Rng/ideal(o_2^2+o_3^2+o_2))
1/factor2*Sigma2
```

shows that $\Sigma^\vee$, when restricted to parameters in the zero set of I_2, is the linear map $a \mapsto (1, 2o_3a, -2o_2a-a, -at_3+av_1+t_1)$, and the RS camera has order one.

For parameters in the zero set of I_5, an analogous computation as for I_1 shows that $\Sigma^\vee$ becomes the cubic map

$$a \mapsto (1-a^2o_2(o_2+2),\ 0,\ a(a^2o_2^2-2o_2-1),\ 0). \quad (38)$$

There are two ways how this map can become linear: One possibility is $o_2 = 0$, but then the parameters would be contained in the solution set of I_2. If $o_2 \neq 0$, the map (38) can only become linear if its quadratic entry divides its cubic entry. This would mean that $1-a^2o_2(o_2+2)$ and $a^2o_2^2-2o_2-1$ are equal up to scaling by a constant that is allowed to depend on o_2 but not on a. This constant would have to be $-2o_2-1$, so we would get

$$-2o_2-1+a^2o_2(o_2+2)(2o_2+1) = a^2o_2^2-2o_2-1$$

for all a, i.e., $(o_2+2)(2o_2+1) = o_2$. The latter is equivalent to $(o_2+1)^2 = 0$, i.e., $o_2 = -1$, but in that case the parameters would be already contained in the zero locus of I_1. Hence, for generic parameters in the zero set of I_5, the rolling planes map $\Sigma^\vee$ is cubic, and it is linear if and only if the parameters are also contained in the zero locus of either I_1 or I_2. □

Proof of Proposition 20. For parameters in the zero locus of I_1, we can compute Φ as follows: Given a space point X, we begin by finding the unique a such that X is contained in the plane $\Sigma(a)$:

```
RX = QQ[gens Rng,x_1..x_3,x_0]
X = matrix{toList(x_1..x_3)|{x_0}}
linX = (sub(Sigma1,RX)*transpose(X))_(0,0)
aX0 = - sub(linX, a => 0)
aX1 = (linX + aX0)/a
aX = aX0 / aX1
```

Then, we can plug that parameter a into the projection matrix P and use it to take a picture of X:

```
PX = sub(sub(P,RX), {v_3 => 0, o_3 => 0,
o_2 => -1, a => aX})
Phi = (aX1)^3*PX*transpose(X)
```

The resulting map has entries of degree four in X. It is enough to observe in an example that those entries do not have any common factor; for that, see Example 21. We can proceed analogously for the ideal I_2. □

10. Diagram of the relationships between parameter spaces of rolling shutter cameras

The following diagram shows the relationships of the parameter spaces of RS cameras we discussed in this paper, as well as the mathematical objects that go into their respective definitions.

$$
\begin{tikzcd}
K,\mathcal{C},\Sigma_\infty^\vee,\lambda \arrow[r, dashed] & \mathcal{P}_{I,d,\delta} \arrow[r] \arrow[d] & \mathcal{P}^{\mathrm{cs}}_{I,1,\delta} \arrow[d] & \\
R,K,\mathcal{C} \arrow[r, dashed] & \mathcal{P}_{I,d} \arrow[r] & \mathcal{P}^{\mathrm{cs}}_{I,1} \arrow[dr, hook] & \\
R,K,C \arrow[r, dashed] & \mathcal{P}_{II,d} \arrow[r] & \mathcal{P}^{\mathrm{cs}}_{II,1} \arrow[r, hook] & \mathcal{P}^{\mathrm{cs}}_{1} \\
K,C,\Sigma_\infty^\vee,h,p \arrow[r, dashed] & \mathcal{P}_{II,d,\delta} \arrow[r] \arrow[u] & \mathcal{P}^{\mathrm{cs}}_{II,1,\delta} \arrow[u] &
\end{tikzcd}
$$

Dashed arrows indicate using building blocks to construct parameter spaces. Normal arrows indicate passing to a special case. Hooked arrows are set inclusions.

11. Details for Figure 5

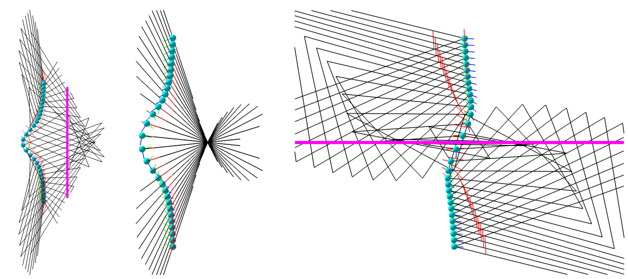

Figure 6. Example 21. The camera center C (cyan) moves along a twisted cubic $\mathcal{C}$. All rolling planes Σ (black) intersect in the line $K = (0 : X_2 : X_0 : X_0)$ (magenta). $\mathcal{C}$ intersects K in the two non-real conjugate points $(0 : i : 1 : 1)$ and $(0 : -i : 1 : 1)$.

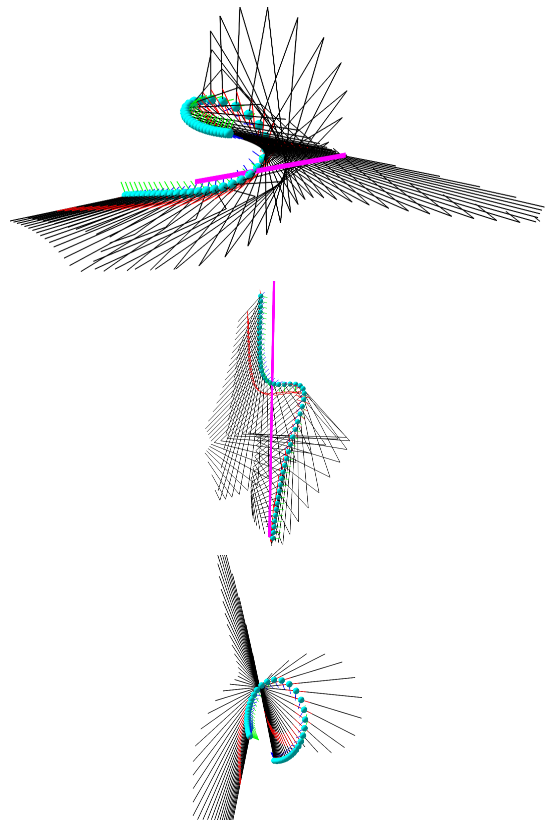

Figure 7. Example 22. The camera center C (cyan) moves along a twisted cubic $\mathcal{C}$. All rolling planes Σ (black) intersect in the line $K = (0 : X_0 : X_3 : X_0)$ (magenta). $\mathcal{C}$ intersects K in $(0 : 0 : 1 : 0)$ and $(0 : 1 : -1 : 1)$.

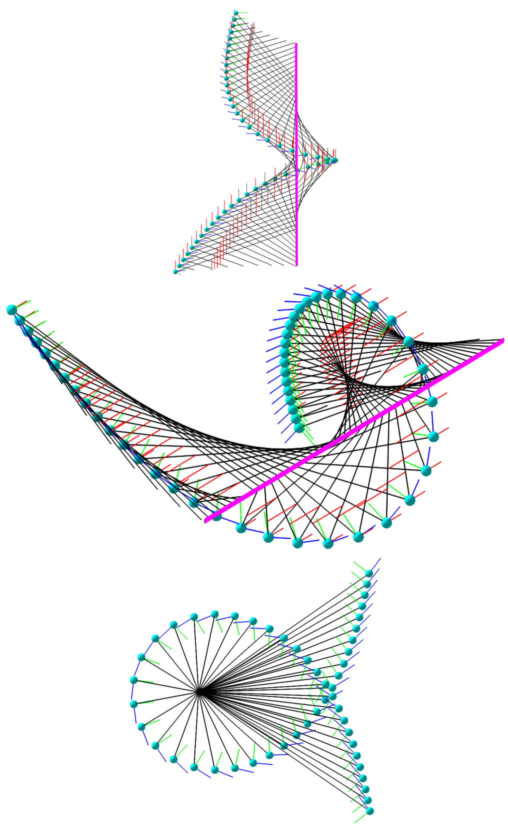

Figure 8. Example 23. The camera center C (cyan) moves along a twisted cubic $\mathcal{C}$. All rolling planes Σ (black) intersect in the line $K = (0 : X_2 : 0 : X_0)$ (magenta). $\mathcal{C}$ does not intersect K.