Towards Optimizing Large-Scale Multi-Graph Matching in Bioimaging

Supplementary Material

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A. Feasible solution construction: Details

Merge operations

The merge operation between two partial solutions $\mathcal{A} \in \mathbb{Q}^{D_A}$ and $\mathcal{B} \in \mathbb{Q}^{D_B}$ of disjoint $D_A \cap D_B = \emptyset$ object subsets $D_A, D_B \subset [d]$ specified via the matching $E \in \mathcal{A} \times \mathcal{B}$ is formally defined as

$$\begin{aligned} \mathbf{merge}(\mathcal{A}, \mathcal{B}; E) \quad &\coloneqq \quad \{A \cup B \,|\, AB \in E\} \\ &\cup \quad \{A \,|\, \forall B \in \mathcal{B} : AB \notin E\} \\ &\cup \quad \{B \,|\, \forall A \in \mathcal{A} : AB \notin E\}\,. \end{aligned} \tag{6}$$

The merge between an object V^p and a partial solution $\mathcal{Q} \in \mathbb{Q}^D$, $p \notin D \subset [d]$, specified by the matching $E \subset V^p \times \mathcal{Q}$ is a special case

$$\mathbf{merge}(V^p, \mathcal{Q}; E) \coloneqq \mathbf{merge}(\mathcal{V}^p, \mathcal{Q}; E')\,, \tag{7}$$

where $\mathcal{V}^p = \{\{i\} \,|\, i \in V^p\}$ and $E' = \{\{i\} \cup Q \,|\, iQ \in E\}$.

Extensions and parallelization

At the heart of both extensions lie *leaf-labeled*, *ordered* trees with *label set* $[d]$, further referred to as *construction trees*, see Fig. A.1. As described in Sec. 4.2, each `vertex` of a construction tree can be associated with a partial solution (`vertex.solution`) for the objects labeled by its descendant leaves. While leaves are always associated with the trivial solution $\mathcal{V}^p = \{\{i\} \,|\, i \in V^p\}$ of the object they label V^p, $p \in [d]$, solutions are generally the result of (matching and) merging the solutions of their children.

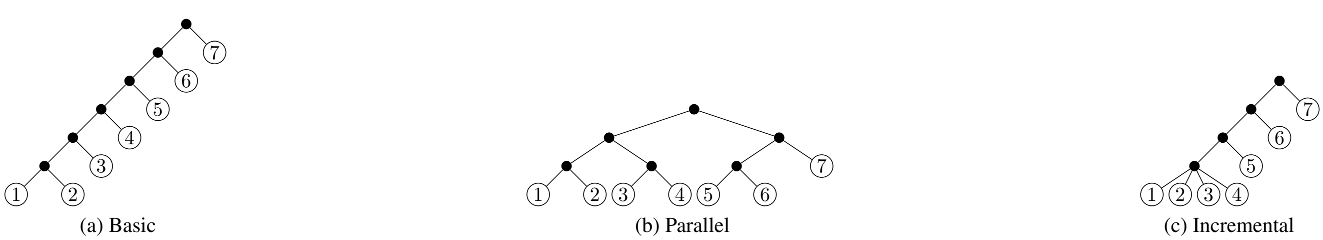

Figure A.1. **Construction Trees.** All three of our proposed construction variants, *basic* (a), *parallel* (b), and *incremental* (c) construction, can be described via construction trees. Shown are example trees for each variant and $d = 7$ objects.

Algorithm A.1: Parallel Solution Construction.

Input: Binary Construction Tree `T`

for $l \in \{$`T.height`, `T.height-1`$, \dots, 1\}$ **do**

  for `vertex` in `T.level`$[l]$ **do in parallel**

    $\mathcal{A} \leftarrow$ `vertex.left.solution`

    $\mathcal{B} \leftarrow$ `vertex.right.solution`

    $E \leftarrow$ Solve GM with costs $C^{\mathcal{A},\mathcal{B}}$ from Eq. (3)

    `vertex.solution` $\leftarrow$ **merge**$(\mathcal{A}, \mathcal{B}; E)$

Algorithm A.2: Parallel GM Local Search.

Input: Solution $\mathcal{Q} \in \mathbb{Q}$

while stopping criterion not met **do**

  for $p \in [d]$ **do in parallel**

    $\mathcal{Q}' \leftarrow \{Q \setminus V^p \mid Q \in \mathcal{Q}\}$

    $E^p \leftarrow$ Solve GM with costs $C^{p,\mathcal{Q}'}$ from Eq. (2)

    $c^p \leftarrow C(\mathbf{merge}(V^p, \mathcal{Q}'; E^p))$

  for $p \in [d]$ in ascending order of c^p **do**

    $\mathcal{Q}' \leftarrow \{Q \setminus V^p \mid Q \in \mathcal{Q}\}$

    $\mathcal{Q}'' \leftarrow \mathbf{merge}(V^p, \mathcal{Q}'; E^p)$

    if $C(\mathcal{Q}'') < C(\mathcal{Q})$ **then**

      $\mathcal{Q} \leftarrow \mathcal{Q}''$

Our *parallel construction* limits itself to *binary* construction trees (see Fig. A.1), which is why it can be implemented with only a GM solver, see Alg. A.1 for pseudocode. The basic construction Alg. 1 can be identified as the special case of binary construction trees with height $d-1$, see Fig. A.1. It allows for no parallelization as only matchings for solutions of vertices at the same level of a construction tree can be computed in parallel.

Arbitrary construction trees are the idea behind *incremental construction*, where solutions for vertices with more than two children require an *MGM solver*, see Fig. A.1. However, in Sec. 4.2 and Sec. 5, we limit ourselves to trees with only one such vertex at level 2, see Fig. A.1. The idea is to use a (potentially) better MGM solver for the first few objects to warm-start the construction with a better partial solution, which we deemed to be the most promising use case.

B. GM local search: Details

Pseudocode for the *sequential* GM local search can be found in Alg. 2. Pseudocode for the *parallel* GM local search can be found in Alg. A.2.

C. Swap local search: Details

C.1. Swap energy change formula

Consider a single swap (Sec. 4.4) performed over an object $i \in [d]$ and two cliques $A, B \in \mathcal{Q}$, yielding a new solution $\mathcal{Q}'$. The induced change in objective value is given by $C(\mathcal{Q}') - C(\mathcal{Q})$, which expands to

$$\left[\sum_{Q,R\in\mathcal{Q}'} \sum_{p,q\in D(Q)\cap D(R)} C^{p,q}_{Q^pQ^q,R^pR^q}\right] - \left[\sum_{Q,R\in\mathcal{Q}} \sum_{p,q\in D(Q)\cap D(R)} C^{p,q}_{Q^pQ^q,R^pR^q}\right]. \tag{8}$$

All cost terms between cliques $Q, R \in (\mathcal{Q} \cap \mathcal{Q}')$ are present in both solution and cancel out. We therefore only consider cost terms that involve at least one of the two cliques $\{A, B\} \subseteq (\mathcal{Q})$ or respectively $\{A', B'\} \subseteq (\mathcal{Q}')$:

$$\begin{aligned}\sum_{Q,R\in\mathcal{Q}} \cdot = \sum_{Q\in\mathcal{Q}}\sum_{R\in\mathcal{Q}} \cdot &= \left(\sum_{Q\in\{A,B\}}\sum_{R\in\mathcal{Q}} \cdot\right) + \left(\sum_{Q\in\mathcal{Q}\setminus\{A,B\}}\sum_{R\in\mathcal{Q}} \cdot\right)\\ &= \left(\sum_{Q\in\{A,B\}}\sum_{R\in\mathcal{Q}} \cdot\right) + \left(\sum_{Q\in\mathcal{Q}\setminus\{A,B\}}\sum_{R\in\{A,B\}} \cdot\right)\\ &\quad + \left(\sum_{Q\in\mathcal{Q}\setminus\{A,B\}}\sum_{R\in\mathcal{Q}\setminus\{A,B\}} \cdot\right).\end{aligned} \tag{9}$$

The last term cancels in $C(\mathcal{Q}') - C(\mathcal{Q})$:

$$\begin{aligned} C(\mathcal{Q}') - C(\mathcal{Q}) = &\left[\sum_{Q \in \{A',B'\}} \sum_{R \in \mathcal{Q}'} \sum_{p,q \in D(Q) \cap D(R)} C^{p,q}_{Q^p Q^q, R^p R^q} \right] \\ &+ \left[\sum_{Q \in \mathcal{Q}' \setminus \{A',B'\}} \sum_{R \in \{A',B'\}} \sum_{p,q \in D(Q) \cap D(R)} C^{p,q}_{Q^p Q^q, R^p R^q} \right] \\ &- \left[\sum_{Q \in \{A,B\}} \sum_{R \in \mathcal{Q}} \sum_{p,q \in D(Q) \cap D(R)} C^{p,q}_{Q^p Q^q, R^p R^q} \right] \\ &- \left[\sum_{Q \in \mathcal{Q} \setminus \{A,B\}} \sum_{R \in \{A,B\}} \sum_{p,q \in D(Q) \cap D(R)} C^{p,q}_{Q^p Q^q, R^p R^q} \right] . \end{aligned} \tag{10}$$

Apart from being limited to two cliques A, B, a single swap is also restricted to a single object $i \in [d]$. It follows, that all cost terms not involving i also cancel out.

Further fixing another object $j \in [d] \setminus \{i\}$, we define the change in cost restricted to i and any other object j as $\delta C^{i,j}_{\text{swap}}(A, B)$:

$$\begin{aligned} \delta C^{i,j}_{\text{swap}}(A, B) \coloneqq &\left[\sum_{Q \in \{A',B'\}} \sum_{R \in \mathcal{Q}'} \sum_{\substack{p \in [D(Q) \cap D(R) \cap \{i\}] \\ q \in [D(Q) \cap D(R) \cap \{j\}]}} C^{p,q}_{Q^p Q^q, R^p R^q} + C^{q,p}_{Q^p Q^q, R^p R^q} \right] \\ &+ \left[\sum_{Q \in \mathcal{Q}' \setminus \{A',B'\}} \sum_{R \in \{A',B'\}} \sum_{\substack{p \in [D(Q) \cap D(R) \cap \{i\}] \\ q \in [D(Q) \cap D(R) \cap \{j\}]}} C^{p,q}_{Q^p Q^q, R^p R^q} + C^{q,p}_{Q^p Q^q, R^p R^q} \right] \\ &- \left[\sum_{Q \in \{A,B\}} \sum_{R \in \mathcal{Q}} \sum_{\substack{p \in [D(Q) \cap D(R) \cap \{i\}] \\ q \in [D(Q) \cap D(R) \cap \{j\}]}} C^{p,q}_{Q^p Q^q, R^p R^q} + C^{q,p}_{Q^p Q^q, R^p R^q} \right] \\ &- \left[\sum_{Q \in \mathcal{Q} \setminus \{A,B\}} \sum_{R \in \{A,B\}} \sum_{\substack{p \in [D(Q) \cap D(R) \cap \{i\}] \\ q \in [D(Q) \cap D(R) \cap \{j\}]}} C^{p,q}_{Q^p Q^q, R^p R^q} + C^{q,p}_{Q^p Q^q, R^p R^q} \right] . \end{aligned} \tag{11}$$

Using Eq. (11), we can write the change in objective as a sum over all other objects $j \in [d] \setminus \{i\}$:

$$C(\mathcal{Q}') - C(\mathcal{Q}) = \sum_{j \in [d] \setminus \{i\}} \delta C^{i,j}_{\text{swap}}(A, B) \,, \tag{12}$$

which is the formulation of Eq. (4).

D. Incomplete to complete reduction

In the following, we formally state and discuss the *polynomial-time reduction* from incomplete to complete MGM, often mentioned without references or further elaboration in related works, *e.g.*, [48, 50, 51]. To transform an incomplete problem with objects V^p, $p \in [d]$, and costs $C^{p,q}$, $p, q \in [d]$, to a complete one with objects of equal cardinalities, we introduce *dummy* vertices, similar to the respective GM reduction in [17]. More precisely, given that $|V|$ stands for the *total* number of vertices in the original incomplete problem, we add $|V| - |V^p| =: |\Delta^p|$ dummy vertices Δ^p to each object V^p, $p \in [d]$. As a result, we construct a complete MGM problem with objects

$$\widetilde{V}^p = V^p \cup \Delta^p, \quad p \in [d] \,, \tag{13}$$

and costs

$$\widetilde{C}^{p,q}_{is,jt} = \begin{cases} C^{p,q}_{is,jt} & \text{, if } i,s,j,t \in V \\ 0 & \text{, else} \end{cases} . \quad (14)$$

In this way, each vertex $i \in V^p$, $p \in [d]$, of the incomplete problem receives a dummy vertex $\delta \in \Delta^q$ in each object $q \in [d] \backslash \{p\}$ of the complete problem. These dummy vertices are used to turn each clique of the incomplete problem maximal *s.t.* they span all objects of the (complete) problem. In turn, omitting matchings to dummy vertices defines the inverse transformation, which leads us to the following

Theorem 2. *Incomplete MGM is polynomial-time reducible to complete MGM,*

The proof is based on the reduction (13)-(14) and given in Suppl. D.1 . However, citing such a transformation, without further reasoning, as a practical way to apply solvers for complete MGM to incomplete problems is *naïve*. The reduction (13)-(14) adds $\sum_{p \in [d]}(|V| - |V^p|) = (d-1)|V|$ dummy vertices, *i.e.*, $d-1$ times the size $|V|$ of the incomplete problem, which renders most existing algorithms impractical by considerably slowing them down.

Yet, the described reduction is *minimal* within the class of *clique-wise* reductions, that is the size of respective complete problems can not be decreased in general:

Definition 1. *A problem class P^A is reducible in polynomial time to the problem class P^B, if (i) there exists a mapping $T_{inst}\colon P^A \to P^B$ and (ii) for any $A \in P^A$ there exists a mapping T_{sol} from the optimal solutions of $T_{inst}(A)$ to the optimal solutions of A, where all mappings are computable in polynomial time.*

Being very general, Definition 1 does not consider several practical issues: First, an optimal solution is rarely known a priori. Therefore, the mapping T_{sol} is usually defined for *all* solutions that can potentially become optimal. In case of the MGM problem these are all feasible multi-matchings, as for each of them costs exists that turn them optimal. Second, if approximate algorithms are used to solve the reduced problem, then the optimum-to-optimum mapping, as required in Definition 1, is insufficient. Instead, the mapping T_{sol} must be *monotone* w.r.t. the objective values of the mapped feasible solutions: Feasible solutions with lower objective values in $T_{inst}(A)$ correspond to feasible solutions with lower objective values in A. These properties are summarized in the following

Definition 2. *We say that there is a* robust *polynomial time* reduction *of a problem class P^A to the problem class P^B, if (i) there exists a mapping $T_{inst}\colon P^A \to P^B$ and (ii) for any $A \in P^A$ there exists a* monotone *mapping T_{sol} from the feasible solutions of $T_{inst}(A)$ to the feasible solutions of A, where all mappings are computable in polynomial time.*

Let now P^A and P^B correspond respectively to the problem classes of incomplete and complete MGM. Any feasible solution of an arbitrary MGM problem A' is given by its vertex partition $\mathcal{Q}^{A'} \in \mathbb{Q}^{A'}$. In the following, we define a subset of robust reductions, where each clique in the partition $\mathcal{Q}^A$, $A \in P^A$, has its counterpart in the *respective* partition $\mathcal{Q}^B$, $B = T_{inst}(A)$:

Definition 3. *Assume $B = T_{inst}(A)$ and $\mathcal{Q}^A := T_{sol}(\mathcal{Q}^B)$. We call the robust reduction $P^A \to P^B$* clique-wise*, if the mapping T_{sol} is induced by a clique-wise mapping $T_{clique} : \mathcal{Q}^B \to \mathcal{Q}^A$* s.t. *$T_{clique}\big|_{\mathcal{Q}^B \backslash T^{-1}_{clique}(\emptyset)}$ is injective, which directly implies $|\mathcal{Q}^A| \leq |\mathcal{Q}^B|$.*

As it follows from the proof of Theorem 2, the reduction (13)-(14) is clique-wise. We are not aware of any non-clique-wise reductions, although cannot exclude their existence. Definition 3 leads to the following important

Theorem 3. *Let $B := T_{inst}(A) \in P^B$ be a complete MGM problem instance that is a clique-wise reduction of an incomplete MGM problem instance $A \in P^A$. Let A have $|V|$ vertices in total. Than* each object *in B has at least $|V|$ vertices.*

Proof. Consider the feasible multi-matching in A which leaves all vertices unmatched. This has $|\mathcal{Q}^A| = |V|$ cliques representing individual vertices. On the other side, the number of (maximal) cliques in any (complete) MGM solution in B is equal to the number of vertices in each of its objects. Hence, the latter cannot be less than $|V|$ due to Definition 3. ☐

D.1. Proof for polynomial reduction from incomplete to complete MGM

Before we prove the polynomial-time reduction from incomplete to complete MGM, we formalize complete MGM analogously to incomplete MGM in Sec. 4.1, denoting complete pendants to incomplete quantities with a tilde, *e.g.*, V^p and $\widetilde{V}^p$.

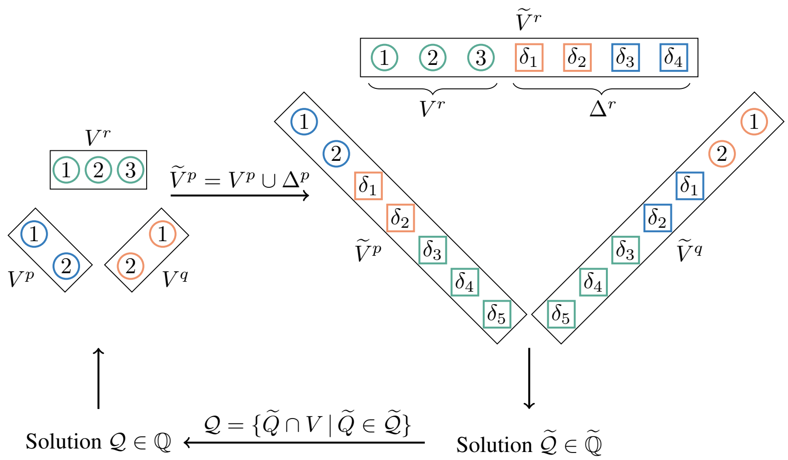

Figure D.1. **Reducing incomplete to complete MGM.** To reduce an incomplete problem with objects V^p, V^q, V^r to a complete problem, one adds dummy vertices Δ^p to each object V^p. Solutions to the incomplete problem can be recovered from solutions to the complete one by removing dummy vertices from cliques. However, this transformation is, without further reasoning, *impractical* since *each* object $\widetilde{V}^p$ of the complete problem is the same size as the *entire* incomplete problem $|V|$.

Instances, hereafter referred to as problems, of incomplete (or complete) problems are fully defined via their objects V^p, $p \in [d]$, (or $\widetilde{V}^p$, $p \in [\widetilde{d}]$,) and costs $C^{p,q}$, $p, q \in [d]$, (or $\widetilde{C}^{p,q}$, $p, q \in [\widetilde{d}]$,) with the difference that objects of the complete instance must have equal cardinalities $\widetilde{V}^p = \widetilde{V}^q =: \widetilde{n} \;\; \forall p, q \in [\widetilde{d}]$. We denote the set of feasible vertex partitions or solutions to the complete problem with $\widetilde{\mathbb{Q}}$. All cliques $\widetilde{Q} \in \widetilde{\mathcal{Q}}$ of solutions $\widetilde{\mathcal{Q}} \in \widetilde{\mathbb{Q}}$ to the complete problem must contain *exactly one* element per object $p \in [\widetilde{d}]$, *i.e.*, $|\widetilde{Q} \cap \widetilde{V}^p| = 1$, which implies that cliques are *maximal* $D(\widetilde{Q}) = [\widetilde{d}]$. Furthermore, since all vertices must be matched, solutions $\widetilde{\mathcal{Q}} \in \widetilde{\mathbb{Q}}$ contain exactly $\widetilde{n}$ cliques $|\widetilde{\mathcal{Q}}| = \widetilde{n}$. The objective for complete MGM is defined in analogy to Eq. (1)

$$\min_{\widetilde{\mathcal{Q}} \in \widetilde{\mathbb{Q}}} \left[\widetilde{C}(\widetilde{\mathcal{Q}}) := \sum_{\widetilde{Q}, \widetilde{R} \in \widetilde{\mathcal{Q}}} \sum_{p,q \in [\widetilde{d}]} \widetilde{C}^{p,q}_{\widetilde{Q}^p \widetilde{Q}^q, \widetilde{R}^p \widetilde{R}^q} \right] . \tag{15}$$

Proof of Theorem 2. As described in Eq. (13) and (14), given an incomplete problem with objects V^p, $p \in [d]$, and costs $C^{p,q}$, $p, q \in [d]$, we construct a complete problem with objects

$$\widetilde{V}^p = V^p \cup \Delta^p, \quad p \in [d] \,, \tag{16}$$

and costs

$$\widetilde{C}^{p,q}_{is,jt} = \begin{cases} C^{p,q}_{is,jt} & \text{, if } i, s, j, t \in V \\ 0 & \text{, else} \end{cases} \,, \tag{17}$$

where Δ^p, $p \in [d]$ are *dummy vertices s.t.* $|\Delta^p| = |V| - |V^p|$, see Fig. D.1. This problem is indeed *complete* since $|\widetilde{V}^p| = |V^p| + |V| - |V^p| = |V| \;\; \forall p \in [d]$. It is also constructed in *polynomial time* since $\sum_{p \in [d]} |V| - |V^p| = (d-1)|V|$ dummy vertices and $\sum_{p \in [d]} \sum_{q \in [d] \setminus \{p\}} (|V|^4 - |V^p|^2 |V^q|^2)$ costs are added to the incomplete problem.

What remains to prove is that the optimal solution of the incomplete problem can be determined in polynomial time by solving the complete one. This follows from the fact that we can *translate* solutions without changing their objective between the incomplete and complete problem by *adding* and *removing* dummy vertices. A solution $\widetilde{\mathcal{Q}} \in \widetilde{\mathbb{Q}}$ for the complete problem is translated to the solution

$$\mathcal{Q} := \{\widetilde{Q} \cap V \,|\, \widetilde{Q} \in \widetilde{\mathcal{Q}}\} \,, \tag{18}$$

for the incomplete problem. We relate the cliques of both solutions via the *well-defined* function $\tau : \widetilde{\mathcal{Q}} \to \mathcal{Q}, \widetilde{Q} \mapsto \widetilde{Q} \cap V$. The solution to the incomplete problem is actually a solution, *i.e.*, $\mathcal{Q} \in \mathbb{Q}$, because it only contains *non-dummy vertices* by construction and *exhaustiveness*, *pairwise disjointness*, and *feasibility* carry over from the solution to the complete problem.

Algorithm D.1: Incomplete to complete solution translation

Input: Incomplete Solution $\mathcal{Q} \in \mathbb{Q}$

$\mathcal{Q} \leftarrow \mathcal{Q} \setminus \{\emptyset\}$

```
// enumerate cliques
```

$\{Q_l\}_{l\in[\,|\mathcal{Q}|\,]} \leftarrow \mathcal{Q}$

$\widetilde{\mathcal{Q}} \leftarrow \{\widetilde{Q}_l\}_{l\in[\,|V|\,]}$ where $\widetilde{Q}_l = Q_l$ if $l \leq |\mathcal{Q}|$ and $\widetilde{Q}_l = \emptyset$ else

for $p \in [d]$ **do**

```
    // enumerate dummy vertices
```

  $\{\delta_l\}_{l\in[\,|V|-|V^p|\,]} \leftarrow \Delta^p$

  Reorder $\widetilde{\mathcal{Q}}$ *s.t.* $|\widetilde{Q}_l \cap V^p| = 1$ if $l \leq |V^p|$

  for $l \in [\,|V| - |V^p|\,]$ **do**

    $\widetilde{Q}_{l+|V^p|} \leftarrow \widetilde{Q}_{l+|V^p|} \cup \delta_l$

Moreover, both objectives are equal

$$\begin{aligned} \widetilde{C}(\widetilde{\mathcal{Q}}) &= \textstyle\sum_{\widetilde{Q},\widetilde{R}\in\widetilde{\mathcal{Q}}} \sum_{p,q\in[d]} \widetilde{C}^{p,q}_{\widetilde{Q}^p\widetilde{Q}^q,\widetilde{R}^p\widetilde{R}^q} \\ &\overset{\text{Eq. (17)}}{=} \textstyle\sum_{\widetilde{Q},\widetilde{R}\in\widetilde{\mathcal{Q}}} \sum_{p,q\in D(\tau(\widetilde{Q}))\cap D(\tau(\widetilde{R}))} C^{p,q}_{\tau(\widetilde{Q})^p\tau(\widetilde{Q})^q,\tau(\widetilde{R})^p\tau(\widetilde{R})^q} \\ &= \textstyle\sum_{Q,R\in\mathcal{Q}} \sum_{p,q\in D(Q)\cap D(R)} C^{p,q}_{Q^pQ^q,R^pR^q} \\ &= C(\mathcal{Q})\,. \end{aligned} \tag{19}$$

For the other direction, Alg. D.1 can be used to translate an incomplete solution $\mathcal{R} \in \mathbb{Q}$ to a complete solution $\widetilde{\mathcal{R}} \in \widetilde{\mathbb{Q}}$, which is indeed a solution because each dummy vertex is added to *exactly one* (possibly empty) clique out of $|V|$ in total, which means cliques are *maximal* and *exhaustiveness*, *pairwise disjointness*, and *feasibility* carry over from the solution to the incomplete problem. Note, however, that this solution is generally not unique because the enumeration of dummy vertices in Alg. D.1 is an unspecified degree of freedom. Now, it holds by construction that $\mathcal{R} = \{\widetilde{R} \cap V \mid \widetilde{R} \in \widetilde{\mathcal{R}}\}$, hence, $C(\mathcal{R}) = \widetilde{C}(\widetilde{\mathcal{R}})$ by Eq. (19).

Therefore, translating any optimal solution to the complete problem via Eq. (18) yields an optimal solution to the incomplete one, which concludes the proof because this translation can be computed in polynomial time.

□ □

Minimality w.r.t. Number of Dummy Vertices

As discussed in Suppl. D, the number of dummy vertices required in Eq. (16) is $d-1$ times the size of the entire incomplete problem and yet *minimal* because the incomplete solution not matching any vertex can only be obtained through a complete solution matching each non-dummy vertex to $(d-1)$ dummy vertices, see Fig. D.2 for an example in the setting of Fig. D.1.

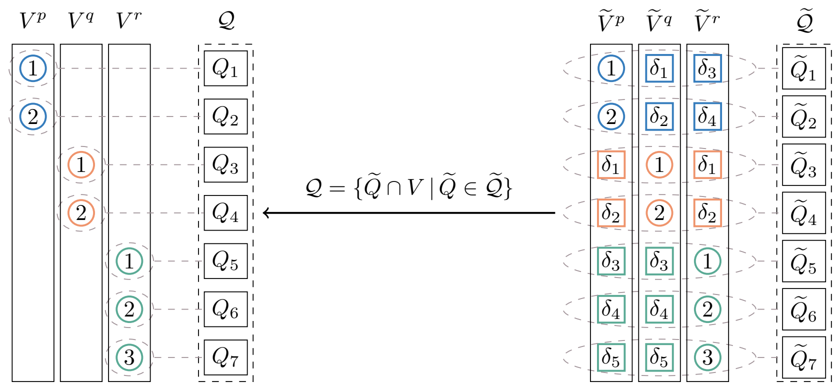

Figure D.2. **Requiring all dummy vertices.** In the reduction from Theorem 2, to obtain the incomplete solution that does not match any vertex (left), all dummy vertices are required by the complete solution (right). This example uses the setting of Fig. D.1. Solutions are illustrated identically to Fig. 4.

E. Experiment setup: Details

Setup Details. All experiments in Sec. 5.2 were run with a single core on an AMD Milan EPYC 7513. Parallel methods in Sec. 5.1 were granted 10 processing cores. As mentioned in Sec. 5, we solve GM subproblems of *GREEDA* using the state-of-the-art GM solver [19, 20]. We run it with the following otherwise default parameters: Batch size 10, 10 greedy generations per batch, number of batches 10. Additionally, we start with a *fixed* randomization seed 42, which turns the otherwise randomized solver [19, 20] into a deterministic one. This is needed to separate the randomization effects of our construction heuristic (see Sec. 4.2) from those of the underlying GM solver. It follows, that we can attribute any randomization effects in our experiments to *GREEDA* itself.

F. Ablation study: Details

Impact of randomization in GREEDA. To assess the influence of randomization, we run each algorithm 100 times using different object orderings. Fig. F.1 shows box plots of the resulting cost distributions. Although the cost distributions significantly differ for different variants of the construction method, this difference vanishes after the local search. In light of this, we find that *incremental* construction's additional runtime (see Fig. 5) is hard to justify. In contrast, *parallel* construction's

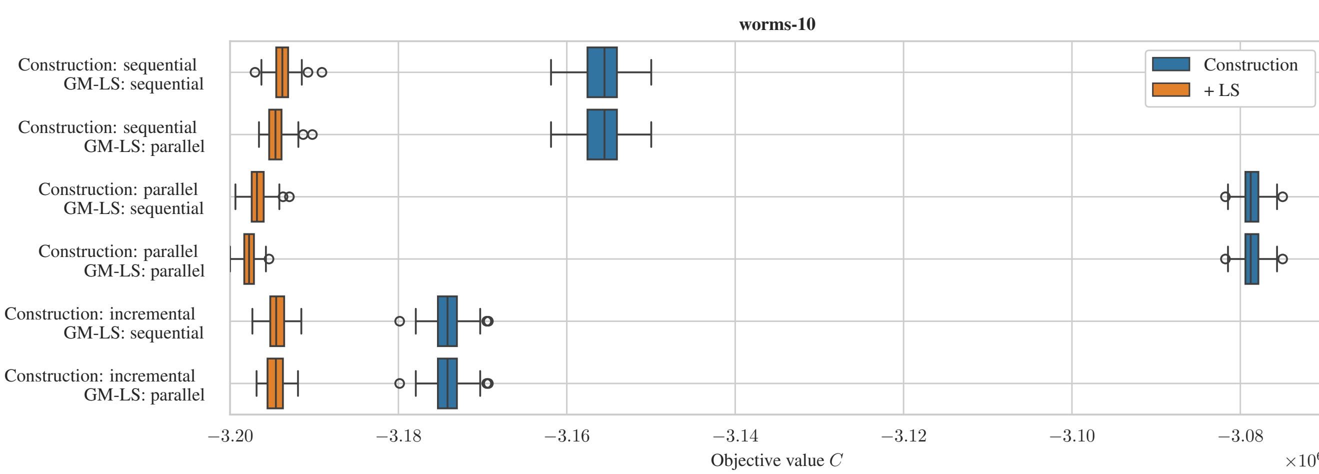

Figure F.1. Boxplots comparing different combinations of construction and GM local search algorithm variants. Results averaged over 10 instances of *worms-10*. Cost distributions over 100 runs.
Result after either *sequential*, *parallel* or *incremental* construction (initial) and after improving with either *sequential* or *parallel* GM local search (+ LS).

improved scaling can be fully utilized because local searches compensate the cost difference to *sequential* construction.

G. Detailed views on Figure 6

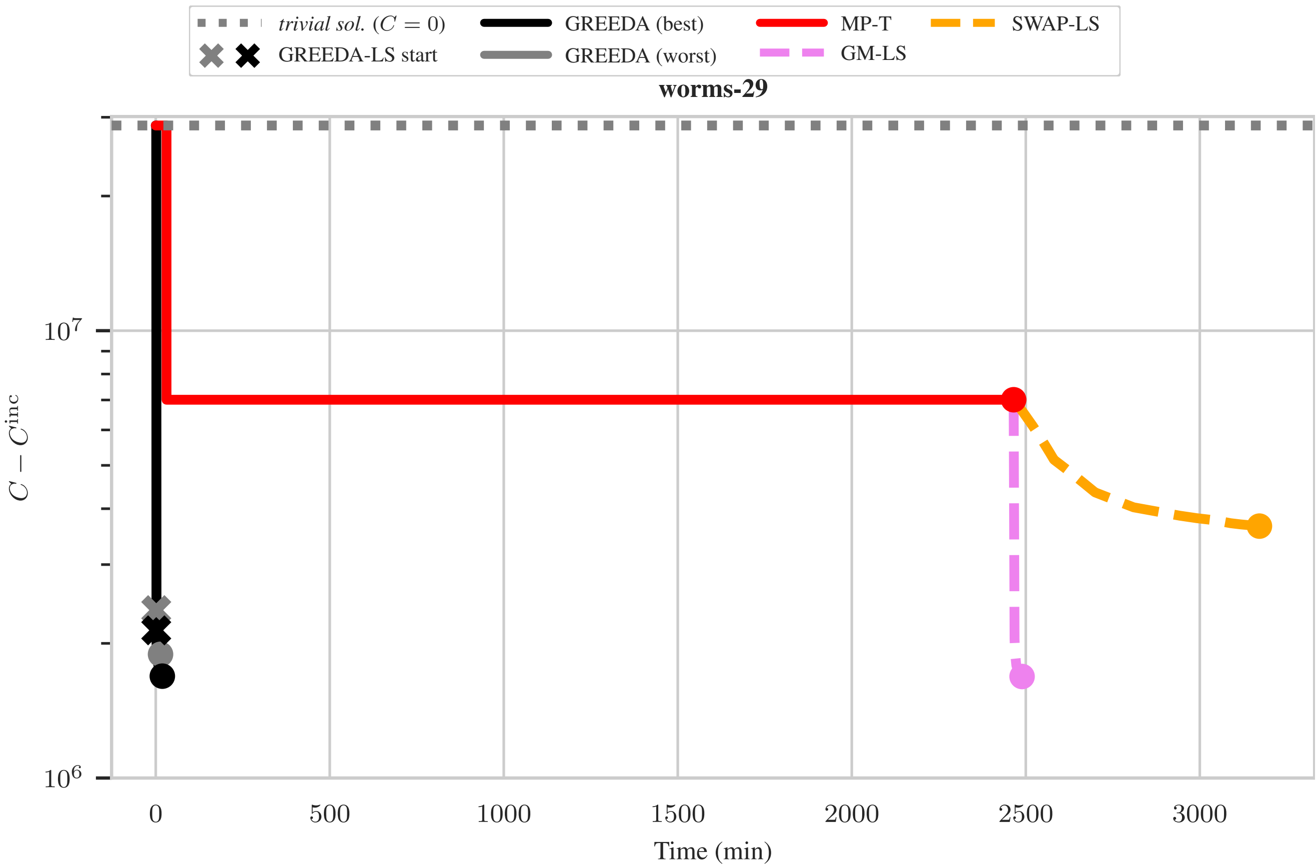

Figure G.1. **Detailed view of Fig. 6a.** Objective with respect to runtime comparison. Objective plotted in log-scale and offset by the *inconsistent* solution's objective C^{inc} obtained from independently solving $d(d-1)/2$ pairwise GM problems. Our proposed algorithm, *GREEDA*, finds an initial solution after only two minutes. In contrast, *MP-T* takes over half an hour and yields a far worse result (see zoom Fig. G.2). Both of our local search methods, *GM-LS* (Sec. 4.3) and *Swap-LS* (Sec. 4.4), can significantly improve on the final result of MP-T.

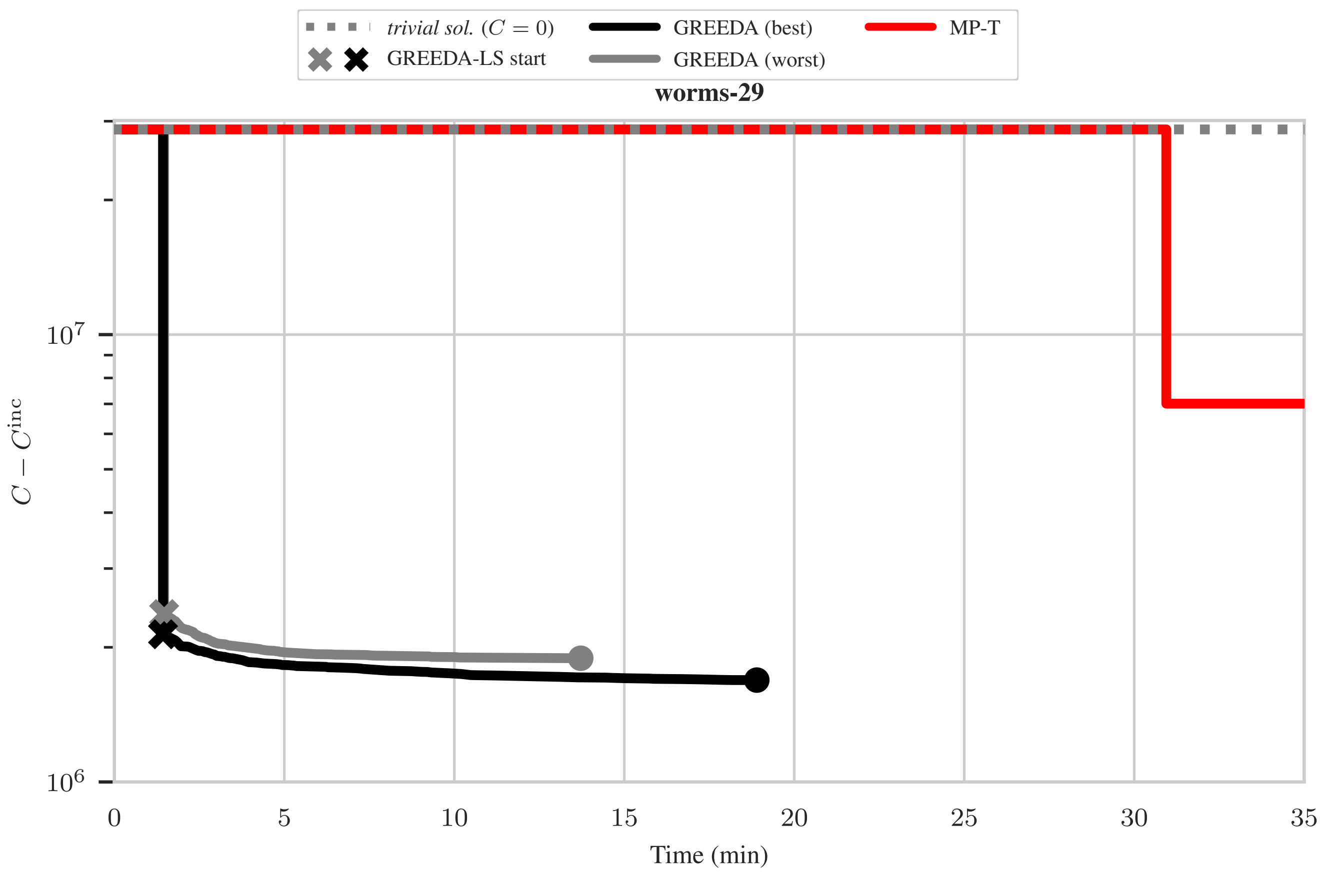

Figure G.2. **Detailed view of Fig. 6b.** Objective with respect to runtime comparison. Objective plotted in log-scale and offset by the *inconsistent* solution's objective C^{inc} obtained from independently solving $d(d-1)/2$ pairwise GM problems. Our proposed algorithm, *GREEDA*, finds an initial solution after only two minutes. In contrast, *MP-T* takes over half an hour and yields a far worse result.

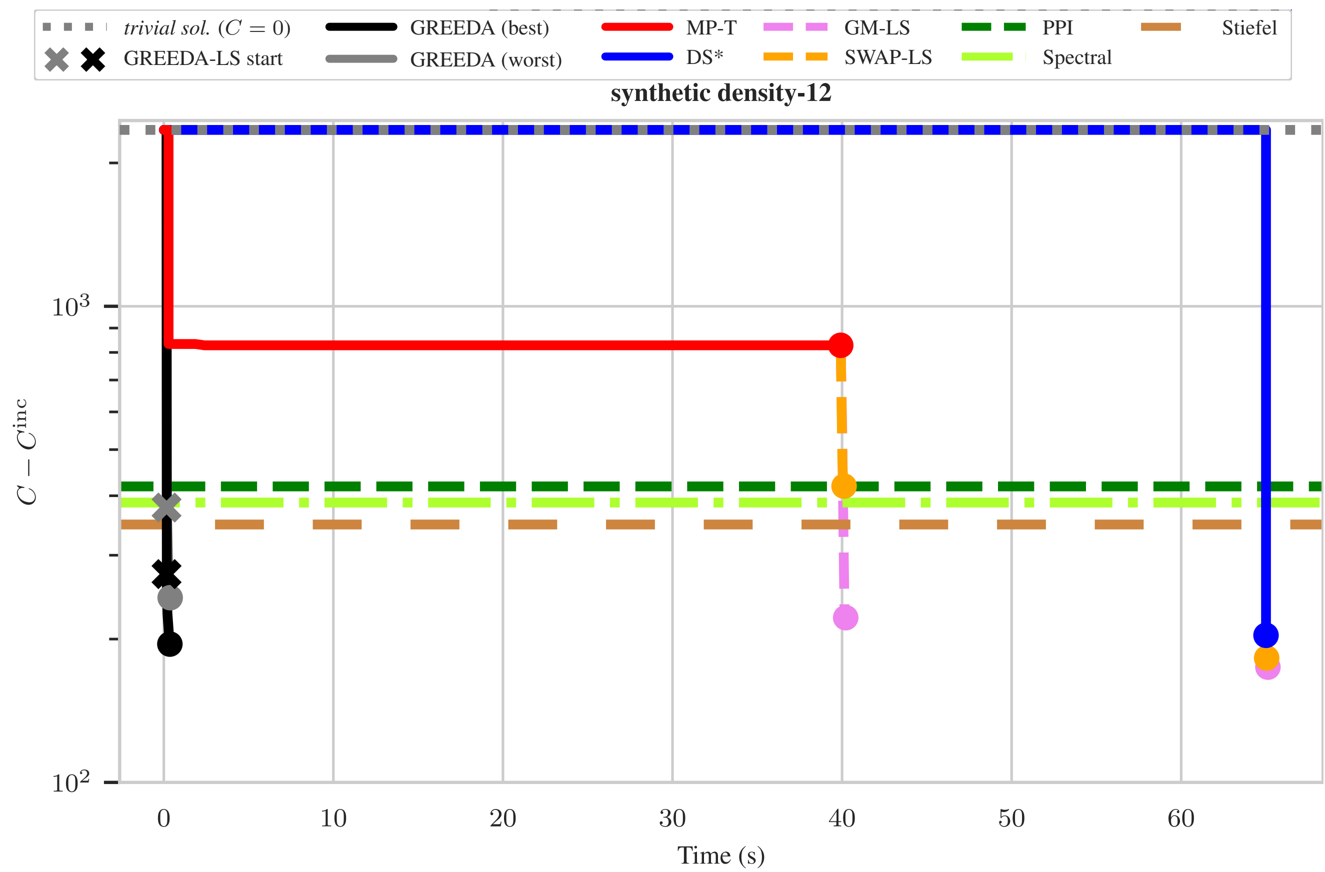

Figure G.3. **Detailed view of Fig. 6c.** Objective with respect to runtime comparison. Objective plotted in log-scale and offset by the *inconsistent* solution's objective C^{inc} obtained from independently solving $d(d-1)/2$ pairwise GM problems. GREEDA returns better solutions than all considered synchronization approaches (*PPI*, *Spectral*, *Stiefel*) and returns a solution significantly faster than *DS**.

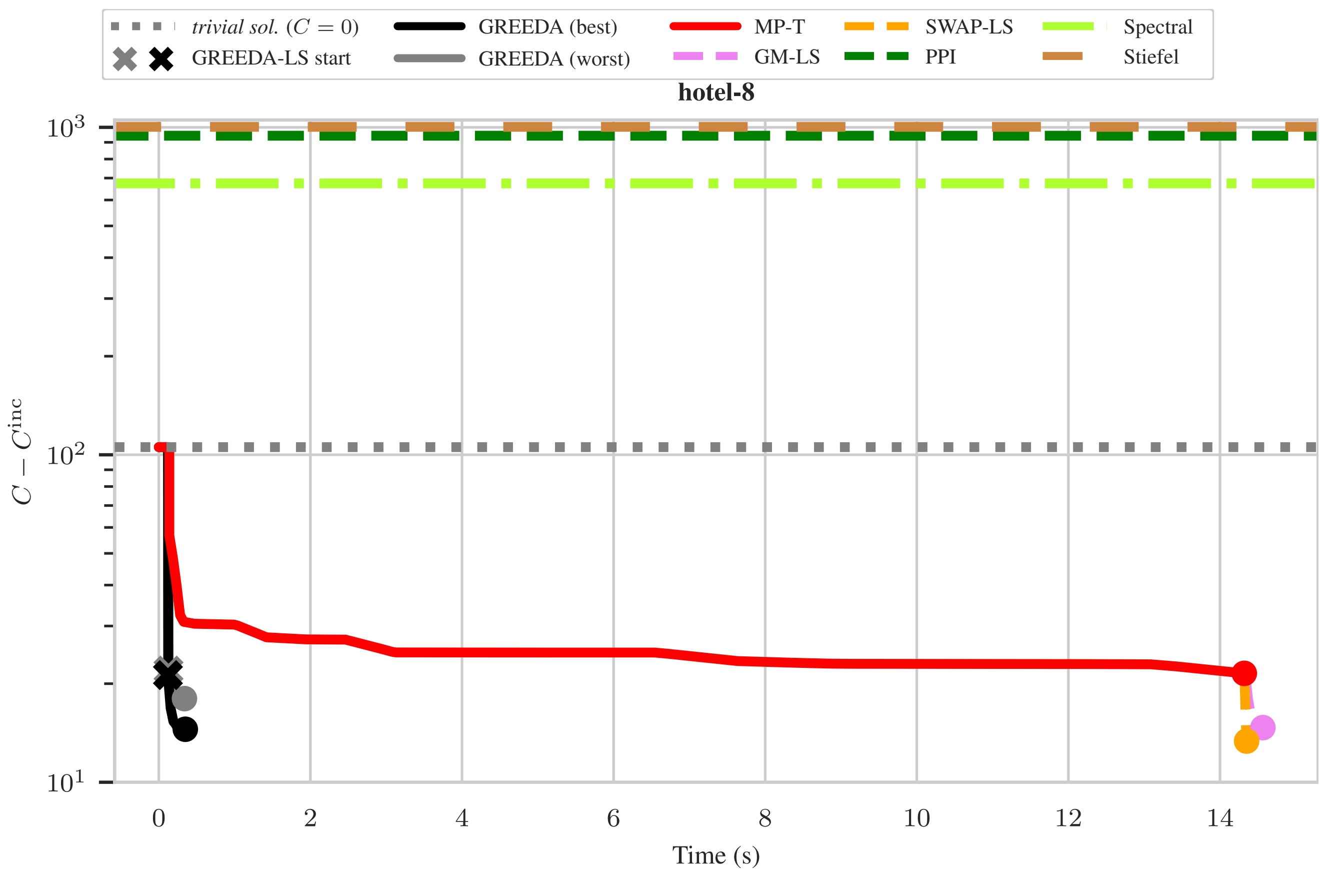

Figure G.4. **Detailed view of Fig. 6d.** Objective with respect to runtime comparison. Objective plotted in log-scale and offset by the *inconsistent* solution's objective C^{inc} obtained from independently solving $d(d-1)/2$ pairwise GM problems. Synchronization methods (*PPI*, *Spectral*, *Stiefel*) return a result that is worse than the *trivial*, nothing matched solution. This is due to *blunders*, which may occur with these methods during their attempt to recover a cycle consistent solution, see Sec. 3.

H. Detailed objective values per instance

In most cases the best and the second best results have been obtained by *GREEDA (best)* and *GREEDA (worst)* computed based on 10 runs of our sequential variant of *GREEDA*. Note that these runs can be executed in parallel in the same time and even their sequential execution often takes less time then a single execution of the competing algorithms. This, in turn, means that achieving even *GREEDA (best)* results is computationally cheaper than execution of the competing algorithms. In the tables, *Our-C* refers to the intermediate result of *GREEDA*, obtained by our sequential construction heuristic of Sec. 4.2 and serves as a point of reference for the influence of the *GM local search* (see Sec. 4.2) and *SWAP local search* (see Sec. 4.4) on the final results of *GREEDA*.

Table H.1. **Synthetic complete (4 objects)**. Objective values per instance.

	Instance									
Method	1	2	3	4	5	6	7	8	9	10
DS* [5]	-496.08	**-546.60**	**-552.41**	**-559.55**	**-558.49**	**-553.06**	**-543.75**	**-547.87**	**-553.29**	**-554.76**
MP-T [41]	**-558.48**	**-546.60**	**-552.41**	**-559.55**	**-558.49**	**-553.06**	**-543.75**	**-547.87**	**-553.29**	**-554.76**
GREEDA (best)	**-558.48**	**-546.60**	**-552.41**	**-559.55**	**-558.49**	**-553.06**	**-543.75**	**-547.87**	**-553.29**	**-554.76**
GREEDA (worst)	**-558.48**	**-546.60**	**-552.41**	**-559.55**	**-558.49**	**-553.06**	**-543.75**	**-547.87**	**-553.29**	**-554.76**
Our-C (best)	**-558.48**	**-546.60**	**-552.41**	**-559.55**	**-558.49**	**-553.06**	**-543.75**	**-547.87**	**-553.29**	**-554.76**
Our-C (worst)	**-558.48**	**-546.60**	**-552.41**	**-559.55**	**-558.49**	**-553.06**	**-543.75**	**-547.87**	**-553.29**	**-554.76**
PPI ($\alpha = -1$) [12]	**-558.48**	**-546.60**	**-552.41**	**-559.55**	**-558.49**	**-553.06**	**-543.75**	**-547.87**	**-553.29**	**-554.76**
PPI ($\alpha = -10$) [12]	**-558.48**	**-546.60**	**-552.41**	**-559.55**	**-558.49**	**-553.06**	**-543.75**	**-547.87**	**-553.29**	**-554.76**
PPI ($\alpha = -50$) [12]	**-558.48**	**-546.60**	**-552.41**	**-559.55**	**-558.49**	**-553.06**	**-543.75**	**-547.87**	**-553.29**	**-554.76**
PPI ($\alpha = 0$) [12]	**-558.48**	**-546.60**	**-552.41**	**-559.55**	**-558.49**	**-553.06**	**-543.75**	**-547.87**	**-553.29**	**-554.76**
Stiefel [8]	**-558.48**	**-546.60**	**-552.41**	**-559.55**	**-558.49**	**-553.06**	**-543.75**	**-547.87**	**-553.29**	**-554.76**
Spectral [6, 34]	**-558.48**	**-546.60**	**-552.41**	**-559.55**	**-558.49**	**-553.06**	**-543.75**	**-547.87**	**-553.29**	**-554.76**

Table H.2. **Synthetic complete (8 objects)**. Objective values per instance.

	Instance									
Method	1	2	3	4	5	6	7	8	9	10
DS* [5]	**-2587.65**	**-2565.68**	**-2585.59**	**-2562.17**	**-2610.60**	**-2593.88**	**-2558.20**	**-2583.81**	**-2592.51**	**-2565.65**
MP-T [41]	-2514.68	**-2565.68**	-2490.44	-2393.16	**-2610.60**	-2464.22	-2429.58	-2456.84	**-2592.51**	**-2565.65**
GREEDA (best)	**-2587.65**	**-2565.68**	**-2585.59**	**-2562.17**	**-2610.60**	**-2593.88**	**-2558.20**	**-2583.81**	**-2592.51**	**-2565.65**
GREEDA (worst)	**-2587.65**	**-2565.68**	**-2585.59**	**-2562.17**	**-2610.60**	**-2593.88**	**-2558.20**	**-2583.81**	**-2592.51**	**-2565.65**
Our-C (best)	**-2587.65**	**-2565.68**	**-2585.59**	**-2562.17**	**-2610.60**	**-2593.88**	**-2558.20**	**-2583.81**	**-2592.51**	**-2565.65**
Our-C (worst)	**-2587.65**	**-2565.68**	**-2585.59**	**-2562.17**	**-2610.60**	**-2593.88**	**-2558.20**	**-2583.81**	**-2592.51**	**-2565.65**
PPI ($\alpha = -1$) [12]	**-2587.65**	**-2565.68**	**-2585.59**	**-2562.17**	**-2610.60**	**-2593.88**	**-2558.20**	**-2583.81**	**-2592.51**	**-2565.65**
PPI ($\alpha = -10$) [12]	**-2587.65**	**-2565.68**	**-2585.59**	**-2562.17**	**-2610.60**	**-2593.88**	**-2558.20**	**-2583.81**	**-2592.51**	**-2565.65**
PPI ($\alpha = -50$) [12]	**-2587.65**	**-2565.68**	**-2585.59**	**-2562.17**	**-2610.60**	**-2593.88**	**-2558.20**	**-2583.81**	**-2592.51**	**-2565.65**
PPI ($\alpha = 0$) [12]	**-2587.65**	**-2565.68**	**-2585.59**	**-2562.17**	**-2610.60**	**-2593.88**	**-2558.20**	**-2583.81**	**-2592.51**	**-2565.65**
Stiefel [8]	**-2587.65**	**-2565.68**	**-2585.59**	**-2562.17**	**-2610.60**	**-2593.88**	**-2558.20**	**-2583.81**	**-2592.51**	**-2565.65**
Spectral [6, 34]	**-2587.65**	**-2565.68**	**-2585.59**	**-2562.17**	**-2610.60**	**-2593.88**	**-2558.20**	**-2583.81**	**-2592.51**	**-2565.65**

Table H.3. **Synthetic complete (12 objects)**. Objective values per instance.

	Instance									
Method	1	2	3	4	5	6	7	8	9	10
DS* [5]	**-6092.97**	**-6039.32**	**-6107.78**	**-6052.55**	**-6121.11**	**-6109.68**	**-6012.00**	**-6082.33**	**-6089.19**	**-6058.51**
MP-T [41]	-5714.89	**-6039.32**	-5958.21	-5784.81	-5911.11	-5998.88	**-6012.00**	**-6082.33**	**-6089.19**	-5887.15
GREEDA (best)	**-6092.97**	**-6039.32**	**-6107.78**	**-6052.55**	**-6121.11**	**-6109.68**	**-6012.00**	**-6082.33**	**-6089.19**	**-6058.51**
GREEDA (worst)	**-6092.97**	**-6039.32**	**-6107.78**	**-6052.55**	**-6121.11**	**-6109.68**	**-6012.00**	**-6082.33**	**-6089.19**	**-6058.51**
Our-C (best)	**-6092.97**	**-6039.32**	**-6107.78**	**-6052.55**	**-6121.11**	**-6109.68**	**-6012.00**	**-6082.33**	**-6089.19**	**-6058.51**
Our-C (worst)	**-6092.97**	**-6039.32**	**-6107.78**	**-6052.55**	**-6121.11**	**-6109.68**	**-6012.00**	**-6082.33**	**-6089.19**	**-6058.51**
PPI ($\alpha = -1$) [12]	**-6092.97**	**-6039.32**	**-6107.78**	**-6052.55**	**-6121.11**	**-6109.68**	**-6012.00**	**-6082.33**	**-6089.19**	**-6058.51**
PPI ($\alpha = -10$) [12]	**-6092.97**	**-6039.32**	**-6107.78**	**-6052.55**	**-6121.11**	**-6109.68**	**-6012.00**	**-6082.33**	**-6089.19**	**-6058.51**
PPI ($\alpha = -50$) [12]	**-6092.97**	**-6039.32**	**-6107.78**	**-6052.55**	**-6121.11**	**-6109.68**	**-6012.00**	**-6082.33**	**-6089.19**	**-6058.51**
PPI ($\alpha = 0$) [12]	**-6092.97**	**-6039.32**	**-6107.78**	**-6052.55**	**-6121.11**	**-6109.68**	**-6012.00**	**-6082.33**	**-6089.19**	**-6058.51**
Stiefel [8]	**-6092.97**	**-6039.32**	**-6107.78**	**-6052.55**	**-6121.11**	**-6109.68**	**-6012.00**	**-6082.33**	**-6089.19**	**-6058.51**
Spectral [6, 34]	**-6092.97**	**-6039.32**	**-6107.78**	**-6052.55**	**-6121.11**	**-6109.68**	**-6012.00**	**-6082.33**	**-6089.19**	**-6058.51**

Table H.4. **Synthetic complete (16 objects)**. Objective values per instance.

	Instance									
Method	1	2	3	4	5	6	7	8	9	10
DS* [5]	**-11080.96**	**-11021.58**	**-11059.12**	**-10953.02**	**-11142.22**	**-11079.59**	**-10965.60**	**-11096.36**	**-11095.05**	**-10995.11**
MP-T [41]	**-11080.96**	**-11021.58**	-10614.98	-10521.96	**-11142.22**	-10815.80	**-10965.60**	-10478.03	**-11095.05**	-10696.66
GREEDA (best)	**-11080.96**	**-11021.58**	**-11059.12**	**-10953.02**	**-11142.22**	**-11079.59**	**-10965.60**	**-11096.36**	**-11095.05**	**-10995.11**
GREEDA (worst)	**-11080.96**	**-11021.58**	**-11059.12**	**-10953.02**	**-11142.22**	**-11079.59**	**-10965.60**	**-11096.36**	**-11095.05**	**-10995.11**
Our-C (best)	**-11080.96**	**-11021.58**	**-11059.12**	**-10953.02**	**-11142.22**	**-11079.59**	**-10965.60**	**-11096.36**	**-11095.05**	**-10995.11**
Our-C (worst)	**-11080.96**	**-11021.58**	**-11059.12**	**-10953.02**	**-11142.22**	**-11079.59**	**-10965.60**	**-11096.36**	**-11095.05**	**-10995.11**
PPI ($\alpha = -1$) [12]	**-11080.96**	**-11021.58**	**-11059.12**	**-10953.02**	**-11142.22**	**-11079.59**	**-10965.60**	**-11096.36**	**-11095.05**	**-10995.11**
PPI ($\alpha = -10$) [12]	**-11080.96**	**-11021.58**	**-11059.12**	**-10953.02**	**-11142.22**	**-11079.59**	**-10965.60**	**-11096.36**	**-11095.05**	**-10995.11**
PPI ($\alpha = -50$) [12]	**-11080.96**	**-11021.58**	**-11059.12**	**-10953.02**	**-11142.22**	**-11079.59**	**-10965.60**	**-11096.36**	**-11095.05**	**-10995.11**
PPI ($\alpha = 0$) [12]	**-11080.96**	**-11021.58**	**-11059.12**	**-10953.02**	**-11142.22**	**-11079.59**	**-10965.60**	**-11096.36**	**-11095.05**	**-10995.11**
Stiefel [8]	**-11080.96**	**-11021.58**	**-11059.12**	**-10953.02**	**-11142.22**	**-11079.59**	**-10965.60**	**-11096.36**	**-11095.05**	**-10995.11**
Spectral [6, 34]	**-11080.96**	**-11021.58**	**-11059.12**	**-10953.02**	**-11142.22**	**-11079.59**	**-10965.60**	**-11096.36**	**-11095.05**	**-10995.11**

Table H.5. **Synthetic deform (4 objects)**. Objective values per instance.

	Instance									
Method	1	2	3	4	5	6	7	8	9	10
DS* [5]	-300.55	-258.25	-201.20	**-335.48**	-229.73	-216.31	-252.86	-272.90	-256.74	**-333.06**
MP-T [41]	-259.75	-243.01	-221.72	-273.21	-244.24	-253.33	-249.54	-261.09	-272.10	-251.08
GREEDA (best)	**-342.83**	**-321.97**	**-293.32**	-332.71	**-320.10**	**-316.27**	**-327.82**	**-337.11**	**-310.94**	**-333.06**
GREEDA (worst)	-320.39	-316.59	-275.08	-325.74	-304.48	-310.91	-313.53	-316.47	-293.91	-308.68
Our-C (best)	**-342.83**	**-321.97**	-290.45	-332.71	-313.84	-315.79	**-327.82**	-336.97	-307.92	**-333.06**
Our-C (worst)	-312.79	-292.71	-268.69	-318.48	-297.31	-301.19	-306.26	-310.90	-288.37	-304.37
PPI ($\alpha = -1$) [12]	-239.50	-275.12	-224.91	-285.34	-234.92	-287.56	-253.50	-263.17	-233.62	-325.36
PPI ($\alpha = -10$) [12]	-239.50	-275.12	-224.91	-285.34	-234.92	-287.56	-253.50	-263.17	-233.62	-325.36
PPI ($\alpha = -50$) [12]	-239.50	-275.12	-224.91	-285.34	-234.92	-287.56	-253.50	-263.17	-233.62	-325.36
PPI ($\alpha = 0$) [12]	-239.50	-275.12	-224.91	-285.34	-234.92	-287.56	-253.50	-263.17	-233.62	-325.36
Stiefel [8]	-246.20	-308.17	-214.10	-281.79	-260.42	-299.52	-271.89	-273.62	-275.13	-331.06
Spectral [6, 34]	-311.32	-305.97	-255.17	-288.32	-293.01	-301.13	-313.60	-323.15	-272.69	-331.06

Table H.6. **Synthetic deform (8 objects)**. Objective values per instance.

	Instance									
Method	1	2	3	4	5	6	7	8	9	10
DS* [5]	-1359.40	-1451.00	-1102.54	-1381.85	**-1510.40**	-1438.76	-1379.50	-1518.18	**-1536.48**	-1473.00
MP-T [41]	-1077.48	-1073.27	-1038.39	-1091.66	-1056.98	-1094.43	-1126.61	-1124.24	-1143.33	-1065.24
GREEDA (best)	**-1523.81**	**-1456.37**	**-1400.88**	**-1458.59**	**-1510.40**	**-1497.54**	**-1501.71**	**-1544.77**	**-1536.48**	**-1484.89**
GREEDA (worst)	**-1523.81**	-1352.34	-1316.56	-1421.30	-1350.41	-1411.94	**-1501.71**	-1383.75	-1455.26	-1395.26
Our-C (best)	**-1523.81**	-1451.67	-1399.80	**-1458.59**	**-1510.40**	-1454.10	**-1501.71**	**-1544.77**	-1524.00	-1447.43
Our-C (worst)	-1332.59	-1316.75	-1250.74	-1309.20	-1304.83	-1314.44	-1348.24	-1334.56	-1302.36	-1303.50
PPI ($\alpha = -1$) [12]	-1206.75	-1320.08	-1002.31	-1170.86	-1141.62	-1149.83	-1287.75	-1084.23	-1369.18	-1234.06
PPI ($\alpha = -10$) [12]	-1206.75	-1320.08	-1002.31	-1170.86	-1141.62	-1149.83	-1287.75	-1084.23	-1369.18	-1234.06
PPI ($\alpha = -50$) [12]	-1206.75	-1320.08	-1002.31	-1170.86	-1141.62	-1149.83	-1287.75	-1084.23	-1369.18	-1234.06
PPI ($\alpha = 0$) [12]	-1206.75	-1320.08	-1002.31	-1170.86	-1141.62	-1149.83	-1287.75	-1084.23	-1369.18	-1234.06
Stiefel [8]	-1290.64	-1225.39	-1041.66	-1257.69	-1301.75	-1139.65	-1282.15	-1166.61	-1421.89	-1259.67
Spectral [6, 34]	-1039.06	-1278.72	-1134.08	-1254.22	-1249.44	-1138.73	-1350.71	-1238.23	-1340.46	-1274.97

Table H.7. **Synthetic deform (12 objects)**. Objective values per instance.

	Instance									
Method	1	2	3	4	5	6	7	8	9	10
DS* [5]	-3472.17	-3390.14	-3313.19	**-3467.06**	**-3498.74**	-3423.23	-3412.43	-3546.99	-3519.20	-3489.32
MP-T [41]	-2501.12	-2553.97	-2463.41	-2502.23	-2338.36	-2531.84	-2623.14	-2587.31	-2657.60	-2504.11
GREEDA (best)	**-3642.08**	**-3477.52**	**-3318.36**	**-3467.06**	**-3498.74**	**-3478.99**	**-3480.48**	**-3650.50**	**-3522.58**	**-3520.58**
GREEDA (worst)	**-3642.08**	**-3477.52**	-3030.20	-3465.70	**-3498.74**	-3147.24	**-3480.48**	-3403.00	-3271.79	**-3520.58**
Our-C (best)	**-3642.08**	-3472.93	-3249.69	-3405.63	-3405.38	-3219.17	-3472.07	**-3650.50**	-3514.62	-3509.51
Our-C (worst)	-3328.76	-2965.79	-2909.93	-3033.99	-2973.73	-3050.96	-2971.48	-3101.67	-3084.10	-3283.28
PPI ($\alpha = -1$) [12]	-3032.24	-3223.24	-2518.60	-2816.83	-3036.67	-2745.14	-2947.76	-3033.10	-2816.07	-3193.91
PPI ($\alpha = -10$) [12]	-3032.24	-3223.24	-2518.60	-2816.83	-3036.67	-2745.14	-2947.76	-3033.10	-2816.07	-3193.91
PPI ($\alpha = -50$) [12]	-3032.24	-3223.24	-2518.60	-2816.83	-3036.67	-2745.14	-2947.76	-3033.10	-2816.07	-3193.91
PPI ($\alpha = 0$) [12]	-3032.24	-3223.24	-2518.60	-2816.83	-3036.67	-2745.14	-2947.76	-3033.10	-2816.07	-3193.91
Stiefel [8]	-3115.34	-3260.37	-2489.57	-2995.29	-3330.71	-2680.03	-3198.72	-3244.04	-3065.90	-3175.29
Spectral [6, 34]	-3100.72	-3270.11	-2562.64	-2841.84	-3156.92	-2520.59	-3105.31	-2849.44	-3086.78	-3177.61

Table H.8. **Synthetic deform (16 objects)**. Objective values per instance.

	Instance									
Method	1	2	3	4	5	6	7	8	9	10
DS* [5]	-6102.13	**-6569.81**	-6150.22	-6267.58	**-6272.24**	-6042.47	-5603.83	-6462.57	-6410.54	-6552.52
MP-T [41]	-4596.00	-4586.45	-4462.58	-4601.28	-4206.68	-4587.89	-4515.56	-4594.57	-4897.13	-4556.19
GREEDA (best)	**-6507.40**	**-6569.81**	**-6151.96**	**-6278.85**	**-6272.24**	**-6349.59**	**-6231.85**	**-6615.35**	**-6411.24**	**-6561.48**
GREEDA (worst)	-6491.59	**-6569.81**	**-6151.96**	-6266.32	-5351.73	-5797.67	-6231.65	-5572.09	-5660.36	**-6561.48**
Our-C (best)	-6486.80	**-6569.81**	-5996.61	-6130.09	-6128.86	-6156.88	-6119.53	**-6615.35**	-6400.92	-6550.48
Our-C (worst)	-5465.99	-5822.27	-5233.77	-5448.51	-4923.51	-5419.15	-5338.57	-5385.89	-5471.18	-5728.51
PPI ($\alpha = -1$) [12]	-5343.40	-6449.73	-4713.40	-4966.05	-6040.01	-4831.64	-5305.82	-5405.26	-5747.75	-6285.36
PPI ($\alpha = -10$) [12]	-5343.40	-6449.73	-4713.40	-4966.05	-6040.01	-4831.64	-5305.82	-5405.26	-5747.75	-6285.36
PPI ($\alpha = -50$) [12]	-5343.40	-6449.73	-4713.40	-4966.05	-6040.01	-4831.64	-5305.82	-5405.26	-5747.75	-6285.36
PPI ($\alpha = 0$) [12]	-5343.40	-6449.73	-4713.40	-4966.05	-6040.01	-4831.64	-5305.82	-5405.26	-5747.75	-6285.36
Stiefel [8]	-5476.12	-6376.02	-4817.78	-5288.46	-6152.68	-4894.31	-5798.80	-6229.24	-5632.25	-6158.10
Spectral [6, 34]	-5156.91	-6418.48	-4710.80	-5116.72	-5398.72	-4808.77	-5721.31	-5519.21	-5406.90	-6120.15

Table H.9. **Synthetic density (4 objects)**. Objective values per instance.

	Instance									
Method	1	2	3	4	5	6	7	8	9	10
DS* [5]	-127.32	-202.04	-195.01	-225.28	-128.95	-152.09	-142.68	-179.58	-204.61	-196.60
MP-T [41]	-126.18	-168.00	-162.30	-175.28	-131.48	-161.30	-170.17	-165.38	-159.68	-193.65
GREEDA (best)	**-179.63**	**-222.06**	**-202.26**	**-232.06**	**-176.74**	**-213.24**	**-208.91**	**-228.06**	**-210.06**	**-232.46**
GREEDA (worst)	-170.13	-206.80	-200.06	**-232.06**	-163.05	-197.62	-194.49	-221.14	-204.68	-226.77
Our-C (best)	-175.86	-219.83	**-202.26**	**-232.06**	-172.82	-208.75	-198.67	**-228.06**	-204.68	**-232.46**
Our-C (worst)	-161.36	-200.74	-193.04	-200.97	-162.69	-193.59	-193.86	-213.62	-190.67	-224.44
PPI ($\alpha = -1$) [12]	-111.38	-197.04	-177.46	-219.11	-115.70	-190.07	-159.78	-199.60	-170.95	**-232.46**
PPI ($\alpha = -10$) [12]	-111.38	-197.04	-177.46	-219.11	-115.70	-190.07	-159.78	-199.60	-170.95	**-232.46**
PPI ($\alpha = -50$) [12]	-111.38	-197.04	-177.46	-219.11	-115.70	-190.07	-159.78	-199.60	-170.95	**-232.46**
PPI ($\alpha = 0$) [12]	-111.38	-197.04	-177.46	-219.11	-115.70	-190.07	-159.78	-199.60	-170.95	**-232.46**
Stiefel [8]	-124.74	-190.48	-169.00	-200.78	-136.40	-190.07	-197.18	-216.94	-163.16	**-232.46**
Spectral [6, 34]	-137.02	**-222.06**	-185.66	-202.48	-148.84	-191.40	-193.77	-212.34	-184.66	**-232.46**

Table H.10. **Synthetic density (8 objects)**. Objective values per instance.

	Instance									
Method	1	2	3	4	5	6	7	8	9	10
DS* [5]	-759.03	-969.81	**-933.74**	-790.95	-773.20	-716.29	-799.89	**-1062.28**	-771.63	-772.49
MP-T [41]	-549.26	-750.97	-685.10	-689.82	-596.83	-616.10	-646.45	-750.82	-649.43	-737.14
GREEDA (best)	**-788.79**	**-970.22**	**-933.74**	**-920.62**	**-816.22**	**-867.25**	**-956.28**	**-1062.28**	**-938.28**	**-990.28**
GREEDA (worst)	-737.58	**-970.22**	-847.47	-807.62	-753.24	-814.47	-895.38	**-1062.28**	-846.58	-952.76
Our-C (best)	-783.28	-969.81	-904.70	-919.80	-792.82	-854.36	-928.66	**-1062.28**	**-938.28**	**-990.28**
Our-C (worst)	-706.34	-818.47	-807.87	-807.62	-706.62	-786.88	-833.84	-907.88	-806.48	-856.05
PPI ($\alpha = -1$) [12]	-547.99	-930.85	-751.57	-834.28	-619.45	-682.50	-717.21	**-1062.28**	-710.25	-869.19
PPI ($\alpha = -10$) [12]	-547.99	-930.85	-751.57	-834.28	-619.45	-682.50	-717.21	**-1062.28**	-710.25	-869.19
PPI ($\alpha = -50$) [12]	-547.99	-930.85	-751.57	-834.28	-619.45	-682.50	-717.21	**-1062.28**	-710.25	-869.19
PPI ($\alpha = 0$) [12]	-547.99	-930.85	-751.57	-834.28	-619.45	-682.50	-717.21	**-1062.28**	-710.25	-869.19
Stiefel [8]	-574.45	-930.85	-830.38	-847.61	-636.18	-638.44	-800.21	**-1062.28**	-803.97	-946.36
Spectral [6, 34]	-586.56	-930.85	-776.62	-853.17	-617.23	-669.89	-853.03	**-1062.28**	-823.40	-916.63

Table H.11. **Synthetic density (12 objects)**. Objective values per instance.

	Instance									
Method	1	2	3	4	5	6	7	8	9	10
DS* [5]	-1930.30	**-2326.68**	-2044.00	-2174.62	-1828.16	-1938.15	**-2162.66**	**-2374.66**	**-2298.66**	**-2350.66**
MP-T [41]	-1312.45	-1685.81	-1554.37	-1487.44	-1444.97	-1422.00	-1462.78	-1522.89	-1523.21	-1785.80
GREEDA (best)	**-1941.46**	**-2326.68**	**-2075.42**	**-2174.66**	**-2006.66**	**-2013.55**	**-2162.66**	**-2374.66**	**-2298.66**	**-2350.66**
GREEDA (worst)	-1751.36	**-2326.68**	-1904.04	**-2174.66**	-1791.08	-1848.43	**-2162.66**	**-2374.66**	**-2298.66**	**-2350.66**
Our-C (best)	-1935.62	**-2326.68**	-1997.62	-2104.45	-1980.57	-1992.10	-2154.84	**-2374.66**	-2265.93	**-2350.66**
Our-C (worst)	-1644.13	-2018.96	-1819.23	-1788.94	-1706.30	-1775.49	-1885.24	-2203.97	-1895.58	-1920.12
PPI ($\alpha = -1$) [12]	-1436.70	**-2326.68**	-1624.70	-2024.10	-1570.67	-1477.44	-1944.32	-2353.32	-2214.01	-2308.66
PPI ($\alpha = -10$) [12]	-1436.70	**-2326.68**	-1624.70	-2024.10	-1570.67	-1477.44	-1944.32	-2353.32	-2214.01	-2308.66
PPI ($\alpha = -50$) [12]	-1436.70	**-2326.68**	-1624.70	-2024.10	-1570.67	-1477.44	-1944.32	-2353.32	-2214.01	-2308.66
PPI ($\alpha = 0$) [12]	-1436.70	**-2326.68**	-1624.70	-2024.10	-1570.67	-1477.44	-1944.32	-2353.32	-2214.01	-2308.66
Stiefel [8]	-1654.88	-2274.60	-1727.16	-2029.04	-1674.76	-1566.04	-2128.23	**-2374.66**	-2233.78	-2321.22
Spectral [6, 34]	-1589.40	-2274.60	-1657.32	-1979.28	-1605.12	-1490.94	-2103.12	**-2374.66**	-2236.16	-2284.01

Table H.12. **Synthetic density (16 objects)**. Objective values per instance.

	Instance									
Method	1	2	3	4	5	6	7	8	9	10
DS* [5]	-3611.85	**-4271.20**	**-3855.68**	-3672.47	**-3599.20**	-3501.50	**-3878.61**	**-4235.20**	**-4319.20**	**-4203.20**
MP-T [41]	-2415.26	-3129.29	-2794.45	-2769.06	-2514.60	-2489.36	-2568.80	-2698.16	-2872.80	-2949.04
GREEDA (best)	**-3615.19**	**-4271.20**	**-3855.68**	**-3934.51**	**-3599.20**	**-3655.20**	**-3878.61**	**-4235.20**	**-4319.20**	**-4203.20**
GREEDA (worst)	-3166.69	**-4271.20**	-3441.45	**-3934.51**	-3235.72	-3237.90	-3449.02	**-4235.20**	**-4319.20**	**-4203.20**
Our-C (best)	-3595.18	-4235.60	-3622.02	-3854.66	-3488.16	-3496.59	-3875.20	**-4235.20**	**-4319.20**	**-4203.20**
Our-C (worst)	-3031.47	-3529.46	-3270.01	-3254.39	-3034.19	-3112.27	-3338.71	-3503.95	-3473.68	-3419.59
PPI ($\alpha = -1$) [12]	-3282.39	**-4271.20**	-3367.93	-3863.73	-2907.88	-2521.89	-3818.20	**-4235.20**	-4308.86	-4160.08
PPI ($\alpha = -10$) [12]	-3282.39	**-4271.20**	-3367.93	-3863.73	-2907.88	-2521.89	-3818.20	**-4235.20**	-4308.86	-4160.08
PPI ($\alpha = -50$) [12]	-3282.39	**-4271.20**	-3367.93	-3863.73	-2907.88	-2521.89	-3818.20	**-4235.20**	-4308.86	-4160.08
PPI ($\alpha = 0$) [12]	-3282.39	**-4271.20**	-3367.93	-3863.73	-2907.88	-2521.89	-3818.20	**-4235.20**	-4308.86	-4160.08
Stiefel [8]	-3236.71	-4231.16	-3285.32	-3844.63	-3118.98	-2865.77	-3840.05	-4220.86	-4277.02	-4178.04
Spectral [6, 34]	-3373.49	-4231.16	-3097.15	-3810.30	-3012.60	-3003.76	-3816.55	**-4235.20**	-4308.86	-4160.08

Table H.13. **Synthetic outlier (4 objects)**. Objective values per instance.

	Instance									
Method	1	2	3	4	5	6	7	8	9	10
MP-T [41]	**-43.28**	**-43.22**	**-43.22**	**-43.28**	**-43.80**	**-43.93**	**-43.22**	**-43.22**	**-43.22**	**-43.22**
GREEDA (best)	**-43.28**	**-43.22**	**-43.22**	**-43.28**	**-43.80**	**-43.93**	**-43.22**	**-43.22**	**-43.22**	**-43.22**
GREEDA (worst)	**-43.28**	**-43.22**	**-43.22**	**-43.28**	**-43.80**	**-43.93**	**-43.22**	**-43.22**	**-43.22**	**-43.22**
Our-C (best)	**-43.28**	**-43.22**	**-43.22**	**-43.28**	**-43.80**	**-43.93**	**-43.22**	**-43.22**	**-43.22**	**-43.22**
Our-C (worst)	-43.22	**-43.22**	**-43.22**	-43.22	**-43.80**	**-43.93**	**-43.22**	**-43.22**	**-43.22**	**-43.22**
PPI ($\alpha = -1$) [12]	151.31	123.79	126.37	114.25	91.40	138.68	107.06	115.97	109.48	95.50
PPI ($\alpha = -10$) [12]	151.31	123.79	126.37	114.25	91.40	138.68	107.06	115.97	109.48	95.50
PPI ($\alpha = -50$) [12]	151.31	123.79	126.37	114.25	91.40	138.68	107.06	115.97	109.48	95.50
PPI ($\alpha = 0$) [12]	151.31	123.79	126.37	114.25	91.40	138.68	107.06	115.97	109.48	95.50
Stiefel [8]	107.49	116.87	123.31	120.32	110.56	119.03	113.74	117.90	111.75	110.41
Spectral [6, 34]	59.17	72.86	59.69	31.01	37.28	67.33	76.65	62.67	64.79	70.23

Table H.14. **Synthetic outlier (8 objects)**. Objective values per instance.

	Instance									
Method	1	2	3	4	5	6	7	8	9	10
MP-T [41]	-177.99	-118.01	-191.68	-163.66	-198.75	-36.83	-116.89	-164.24	-160.04	-165.19
GREEDA (best)	**-202.26**	**-201.70**	**-202.40**	**-202.31**	**-202.67**	**-204.36**	**-203.10**	**-202.33**	**-201.70**	**-201.75**
GREEDA (worst)	**-202.26**	**-201.70**	**-202.40**	**-202.31**	**-202.67**	**-204.36**	**-203.10**	**-202.33**	**-201.70**	**-201.75**
Our-C (best)	-202.21	**-201.70**	**-202.40**	-202.26	**-202.67**	-204.21	**-203.10**	**-202.33**	**-201.70**	-201.70
Our-C (worst)	-202.21	**-201.70**	**-202.40**	-201.70	-202.28	-203.75	**-203.10**	**-202.33**	**-201.70**	-201.70
PPI ($\alpha = -1$) [12]	553.66	546.93	634.10	572.27	541.50	477.10	549.16	531.47	586.99	568.80
PPI ($\alpha = -10$) [12]	553.66	546.93	634.10	572.27	541.50	477.10	549.16	531.47	586.99	568.80
PPI ($\alpha = -50$) [12]	553.66	546.93	634.10	572.27	541.50	477.10	549.16	531.47	586.99	568.80
PPI ($\alpha = 0$) [12]	553.66	546.93	634.10	572.27	541.50	477.10	549.16	531.47	586.99	568.80
Stiefel [8]	513.21	538.86	532.00	523.60	493.64	406.92	574.01	530.65	627.48	718.77
Spectral [6, 34]	323.28	313.44	348.01	243.40	221.18	176.10	222.89	300.65	378.95	246.62

Table H.15. **Synthetic outlier (12 objects)**. Objective values per instance.

	Instance									
Method	1	2	3	4	5	6	7	8	9	10
MP-T [41]	-418.16	-450.80	-451.43	-423.76	-457.68	-221.10	-449.73	-429.57	-385.82	-371.59
GREEDA (best)	**-476.28**	**-475.47**	**-476.74**	**-476.47**	**-477.54**	**-478.24**	**-477.09**	**-476.56**	**-475.89**	**-475.49**
GREEDA (worst)	**-476.28**	**-475.47**	**-476.74**	**-476.47**	**-477.54**	**-478.24**	**-477.09**	**-476.56**	**-475.89**	**-475.49**
Our-C (best)	-476.23	-475.44	-476.71	-476.29	-477.43	-477.79	**-477.09**	**-476.56**	**-475.89**	-475.44
Our-C (worst)	-475.95	-475.44	-476.14	-475.84	-477.04	-477.49	-476.84	-476.07	**-475.89**	-475.44
PPI ($\alpha = -1$) [12]	1320.46	1271.77	1219.17	1264.68	1284.87	1247.82	1253.24	1240.70	1280.23	1350.59
PPI ($\alpha = -10$) [12]	1320.46	1271.77	1219.17	1264.68	1284.87	1247.82	1253.24	1240.70	1280.23	1350.59
PPI ($\alpha = -50$) [12]	1320.46	1271.77	1219.17	1264.68	1284.87	1247.82	1253.24	1240.70	1280.23	1350.59
PPI ($\alpha = 0$) [12]	1320.46	1271.77	1219.17	1264.68	1284.87	1247.82	1253.24	1240.70	1280.23	1350.59
Stiefel [8]	1280.12	1540.30	1544.65	1191.32	1240.35	1184.11	1226.88	1203.56	1348.05	1575.70
Spectral [6, 34]	423.55	855.65	659.90	710.45	512.36	426.29	880.14	691.17	1049.95	683.05

Table H.16. **Synthetic outlier (16 objects)**. Objective values per instance.

	Instance									
Method	1	2	3	4	5	6	7	8	9	10
MP-T [41]	-767.65	-698.38	-779.73	-815.39	-840.60	-482.10	-807.77	-783.80	-790.96	-689.79
GREEDA (best)	**-866.71**	**-866.75**	**-866.49**	**-867.03**	**-867.90**	**-867.24**	**-866.42**	**-867.69**	**-864.88**	**-867.05**
GREEDA (worst)	**-866.71**	**-866.75**	**-866.49**	**-867.03**	**-867.90**	**-867.24**	**-866.42**	**-867.69**	**-864.88**	**-867.05**
Our-C (best)	-866.60	-866.70	-866.46	-866.80	-867.79	-866.64	-866.16	-867.42	**-864.88**	-867.00
Our-C (worst)	-865.54	-865.76	-865.89	-865.69	-866.59	-865.78	-865.84	-866.52	-864.43	-867.00
PPI ($\alpha = -1$) [12]	2313.98	2232.43	2239.33	2312.93	2207.55	2133.70	2331.69	2300.79	2284.11	2276.66
PPI ($\alpha = -10$) [12]	2313.98	2232.43	2239.33	2312.93	2207.55	2133.70	2331.69	2300.79	2284.11	2276.66
PPI ($\alpha = -50$) [12]	2313.98	2232.43	2239.33	2312.93	2207.55	2133.70	2331.69	2300.79	2284.11	2276.66
PPI ($\alpha = 0$) [12]	2313.98	2232.43	2239.33	2312.93	2207.55	2133.70	2331.69	2300.79	2284.11	2276.66
Stiefel [8]	2273.46	2787.82	2377.35	2276.56	2185.84	2196.56	2310.61	2261.99	2571.14	2639.39
Spectral [6, 34]	1186.75	1469.42	1240.60	1063.78	1067.85	1082.20	1328.48	1360.12	845.68	1888.81

Table H.17. **Hotel (4 objects)**. Objective values per instance.

	Instance									
Method	1	2	3	4	5	6	7	8	9	10
MP-T [41]	-9.54	**-25.75**	**-25.06**	**-25.78**	-10.94	-6.67	**-20.20**	-25.98	-14.94	**-26.96**
GREEDA (best)	**-13.87**	**-25.75**	**-25.06**	**-25.78**	**-11.58**	**-8.10**	**-20.20**	**-26.12**	**-16.67**	**-26.96**
GREEDA (worst)	-10.90	**-25.75**	**-25.06**	**-25.78**	-10.20	-6.86	**-20.20**	-22.63	-16.38	**-26.96**
Our-C (best)	**-13.87**	**-25.75**	**-25.06**	**-25.78**	-11.36	**-8.10**	**-20.20**	-25.98	**-16.67**	**-26.96**
Our-C (worst)	-9.21	**-25.75**	-23.36	-21.85	-10.20	-6.54	-11.44	-16.07	-9.41	**-26.96**
PPI ($\alpha = -1$) [12]	267.09	201.60	177.05	204.26	302.49	284.35	146.02	146.19	247.50	183.10
PPI ($\alpha = -10$) [12]	267.09	201.60	177.05	204.26	302.49	284.35	146.02	146.19	247.50	183.10
PPI ($\alpha = -50$) [12]	267.09	201.60	177.05	204.26	302.49	284.35	146.02	146.19	247.50	183.10
PPI ($\alpha = 0$) [12]	267.09	201.60	177.05	204.26	302.49	284.35	146.02	146.19	247.50	183.10
Stiefel [8]	250.43	168.45	165.85	147.66	250.39	237.95	235.13	152.52	224.35	147.68
Spectral [6, 34]	157.07	138.86	124.54	86.58	200.18	197.37	227.20	83.68	102.20	57.94

Table H.18. **Hotel (8 objects)**. Objective values per instance.

	Instance									
Method	1	2	3	4	5	6	7	8	9	10
MP-T [41]	-70.24	**-147.92**	-72.00	-88.25	-43.82	-58.13	-100.78	-55.59	-44.61	**-145.83**
GREEDA (best)	**-97.33**	**-147.92**	**-81.10**	**-96.33**	**-61.84**	**-69.24**	**-101.97**	**-81.95**	**-65.87**	**-145.83**
GREEDA (worst)	**-97.33**	**-147.92**	-74.12	-85.84	-44.30	-60.78	-85.07	-62.22	-37.57	**-145.83**
Our-C (best)	-97.20	**-147.92**	-77.98	-94.99	-57.50	-63.07	**-101.97**	-78.86	-51.56	**-145.83**
Our-C (worst)	-64.27	-147.50	-49.91	-57.91	-27.31	-51.41	-83.91	-52.09	-32.04	-135.25
PPI ($\alpha = -1$) [12]	735.56	871.59	861.77	986.47	850.57	800.57	728.11	834.52	998.48	705.68
PPI ($\alpha = -10$) [12]	735.56	871.59	861.77	986.47	850.57	800.57	728.11	834.52	998.48	705.68
PPI ($\alpha = -50$) [12]	735.56	871.59	861.77	986.47	850.57	800.57	728.11	834.52	998.48	705.68
PPI ($\alpha = 0$) [12]	735.56	871.59	861.77	986.47	850.57	800.57	728.11	834.52	998.48	705.68
Stiefel [8]	910.64	727.24	994.98	869.29	947.13	1120.48	876.04	983.31	925.62	624.15
Spectral [6, 34]	488.61	589.10	538.22	628.80	706.74	684.14	230.58	726.49	891.15	202.24

Table H.19. **Hotel (12 objects)**. Objective values per instance.

	Instance									
Method	1	2	3	4	5	6	7	8	9	10
MP-T [41]	-157.08	**-341.76**	-185.34	-188.09	-95.82	-101.53	**-277.02**	-239.21	-143.85	**-327.81**
GREEDA (best)	**-232.45**	**-341.76**	**-195.50**	**-211.12**	**-140.78**	**-131.32**	**-277.02**	**-241.25**	**-189.06**	**-327.81**
GREEDA (worst)	-229.80	**-341.76**	-190.47	-187.11	-83.55	-111.16	**-277.02**	-189.33	-124.03	**-327.81**
Our-C (best)	-230.63	**-341.76**	-191.28	-203.14	-132.29	-120.09	**-277.02**	-225.48	-187.64	**-327.81**
Our-C (worst)	-141.24	-334.44	-122.75	-146.23	-59.94	-84.67	-227.96	-135.49	-83.07	-274.08
PPI ($\alpha = -1$) [12]	1778.67	1739.01	2009.06	1933.65	2160.84	2257.74	1933.82	1652.54	1896.90	1941.30
PPI ($\alpha = -10$) [12]	1778.67	1739.01	2009.06	1933.65	2160.84	2257.74	1933.82	1652.54	1896.90	1941.30
PPI ($\alpha = -50$) [12]	1778.67	1739.01	2009.06	1933.65	2160.84	2257.74	1933.82	1652.54	1896.90	1941.30
PPI ($\alpha = 0$) [12]	1778.67	1739.01	2009.06	1933.65	2160.84	2257.74	1933.82	1652.54	1896.90	1941.30
Stiefel [8]	1997.70	1680.36	2253.16	2288.69	2607.97	2622.23	1916.57	1741.68	1813.59	1616.33
Spectral [6, 34]	1276.50	1006.73	1175.67	1642.52	1908.62	1778.91	1142.10	1180.81	1320.92	728.80

Table H.20. **Hotel (16 objects)**. Objective values per instance.

Method	Instance									
	1	2	3	4	5	6	7	8	9	10
MP-T [41]	-186.95	**-608.31**	-364.29	-376.98	-248.73	-206.48	-495.75	-490.30	-205.27	**-603.05**
GREEDA (best)	**-377.28**	**-608.31**	**-390.82**	**-396.30**	**-261.07**	**-285.30**	**-501.24**	**-495.07**	**-292.80**	**-603.05**
GREEDA (worst)	-360.91	-607.96	-368.46	-339.09	-179.45	-225.34	-475.42	-475.32	-189.07	**-603.05**
Our-C (best)	-364.89	**-608.31**	-372.96	-383.80	-221.75	-274.43	-483.31	-469.22	-255.85	**-603.05**
Our-C (worst)	-229.88	-547.66	-325.79	-292.44	-107.99	-174.33	-408.72	-378.82	-130.73	-569.49
PPI ($\alpha = -1$) [12]	3190.40	2938.13	3464.07	3583.64	3517.20	4243.95	3087.30	2851.73	3628.93	3023.43
PPI ($\alpha = -10$) [12]	3190.40	2938.13	3464.07	3583.64	3517.20	4243.95	3087.30	2851.73	3628.93	3023.43
PPI ($\alpha = -50$) [12]	3190.40	2938.13	3464.07	3583.64	3517.20	4243.95	3087.30	2851.73	3628.93	3023.43
PPI ($\alpha = 0$) [12]	3190.40	2938.13	3464.07	3583.64	3517.20	4243.95	3087.30	2851.73	3628.93	3023.43
Stiefel [8]	3909.72	3129.24	3478.68	3784.96	4647.44	4764.61	3095.22	2957.87	3712.68	3014.59
Spectral [6, 34]	2294.19	2333.57	2282.51	1666.33	3180.32	2690.89	1937.30	1019.42	2048.70	1992.59

Table H.21. **House (4 objects)**. Objective values per instance.

Method	Instance									
	1	2	3	4	5	6	7	8	9	10
MP-T [41]	-31.07	-19.17	**-33.91**	**-32.64**	-11.09	**-25.03**	**-24.46**	**-40.35**	-19.34	-12.63
GREEDA (best)	**-31.18**	**-20.40**	**-33.91**	**-32.64**	**-13.36**	**-25.03**	**-24.46**	**-40.35**	**-23.09**	**-15.90**
GREEDA (worst)	**-31.18**	-19.95	**-33.91**	**-32.64**	-8.86	**-25.03**	**-24.46**	**-40.35**	**-23.09**	**-15.90**
Our-C (best)	**-31.18**	**-20.40**	**-33.91**	**-32.64**	**-13.36**	**-25.03**	**-24.46**	**-40.35**	**-23.09**	**-15.90**
Our-C (worst)	-28.33	-18.42	-31.63	-25.05	-8.68	-20.98	-21.11	-37.61	-15.64	-13.23
PPI ($\alpha = -1$) [12]	136.23	125.16	180.12	92.32	251.83	177.24	180.57	207.87	197.94	246.69
PPI ($\alpha = -10$) [12]	136.23	125.16	180.12	92.32	251.83	177.24	180.57	207.87	197.94	246.69
PPI ($\alpha = -50$) [12]	136.23	125.16	180.12	92.32	251.83	177.24	180.57	207.87	197.94	246.69
PPI ($\alpha = 0$) [12]	136.23	125.16	180.12	92.32	251.83	177.24	180.57	207.87	197.94	246.69
Stiefel [8]	137.69	166.61	119.83	124.77	198.25	73.96	172.71	127.88	211.53	192.28
Spectral [6, 34]	86.19	70.10	29.19	83.02	229.73	96.58	105.49	71.50	151.85	126.87

Table H.22. **House (8 objects)**. Objective values per instance.

Method	Instance									
	1	2	3	4	5	6	7	8	9	10
MP-T [41]	-146.75	**-145.68**	-101.38	-112.32	**-102.53**	-144.82	-89.98	-116.46	-132.30	-79.20
GREEDA (best)	**-147.52**	**-145.68**	**-113.22**	**-116.08**	**-102.53**	**-145.90**	**-93.64**	**-116.73**	**-134.16**	**-89.10**
GREEDA (worst)	-146.75	-143.81	-102.60	-106.78	-95.21	-137.45	**-93.64**	-113.78	**-134.16**	-74.07
Our-C (best)	**-147.52**	-144.15	-113.14	**-116.08**	-101.92	-145.37	**-93.64**	-115.83	**-134.16**	-85.23
Our-C (worst)	-136.40	-129.43	-96.37	-103.74	-51.77	-128.21	-68.64	-84.97	-91.82	-68.64
PPI ($\alpha = -1$) [12]	584.87	607.67	849.19	631.56	763.97	645.70	930.20	806.16	618.87	704.57
PPI ($\alpha = -10$) [12]	584.87	607.67	849.19	631.56	763.97	645.70	930.20	806.16	618.87	704.57
PPI ($\alpha = -50$) [12]	584.87	607.67	849.19	631.56	763.97	645.70	930.20	806.16	618.87	704.57
PPI ($\alpha = 0$) [12]	584.87	607.67	849.19	631.56	763.97	645.70	930.20	806.16	618.87	704.57
Stiefel [8]	573.68	601.70	766.95	586.92	745.64	645.58	945.36	722.57	650.21	978.02
Spectral [6, 34]	263.27	234.54	419.22	490.95	190.60	279.34	707.91	419.61	421.87	378.20

Table H.23. **House (12 objects)**. Objective values per instance.

Method	Instance									
	1	2	3	4	5	6	7	8	9	10
MP-T [41]	**-363.43**	-390.65	-249.22	-246.86	-231.75	-351.03	-228.90	**-355.49**	-277.57	-198.67
GREEDA (best)	**-363.43**	**-391.51**	**-260.92**	**-249.83**	**-236.53**	**-353.46**	**-257.68**	**-355.49**	**-283.75**	**-223.04**
GREEDA (worst)	**-363.43**	**-391.51**	**-260.92**	-220.06	-216.38	-352.24	-248.19	**-355.49**	-165.53	-194.62
Our-C (best)	**-363.43**	-390.92	-246.56	-242.82	-231.12	-349.11	-242.73	**-355.49**	-282.67	-213.06
Our-C (worst)	-159.87	-342.97	-143.01	-170.58	-111.24	-319.59	-201.58	-268.65	-143.74	-130.61
PPI ($\alpha = -1$) [12]	1319.68	1342.21	1634.18	1640.06	1968.44	1460.50	1814.31	1470.12	1392.24	2063.46
PPI ($\alpha = -10$) [12]	1319.68	1342.21	1634.18	1640.06	1968.44	1460.50	1814.31	1470.12	1392.24	2063.46
PPI ($\alpha = -50$) [12]	1319.68	1342.21	1634.18	1640.06	1968.44	1460.50	1814.31	1470.12	1392.24	2063.46
PPI ($\alpha = 0$) [12]	1319.68	1342.21	1634.18	1640.06	1968.44	1460.50	1814.31	1470.12	1392.24	2063.46
Stiefel [8]	1350.64	1353.99	1437.91	1730.19	2224.61	1377.13	1878.36	1429.20	1318.21	2157.14
Spectral [6, 34]	496.81	753.09	495.64	910.74	1066.84	466.91	1749.49	975.07	610.82	1279.84

Table H.24. **House (16 objects)**. Objective values per instance.

Method	Instance									
	1	2	3	4	5	6	7	8	9	10
MP-T [41]	-667.38	**-715.49**	-486.26	-469.58	-451.45	**-649.10**	-440.82	**-638.35**	-392.39	-397.59
GREEDA (best)	**-667.56**	**-715.49**	**-502.76**	**-489.41**	**-479.29**	**-649.10**	**-480.40**	**-638.35**	**-441.27**	**-443.21**
GREEDA (worst)	-665.67	-714.79	-462.39	-423.45	-419.96	-643.48	-460.30	-630.82	-356.24	-324.86
Our-C (best)	-659.33	-709.27	-491.07	-479.01	-476.99	-633.75	-454.16	-637.70	-396.78	-416.62
Our-C (worst)	-571.72	-664.26	-227.00	-325.90	-270.26	-558.03	-321.19	-563.36	-285.76	-238.97
PPI ($\alpha = -1$) [12]	2412.77	2263.70	2553.10	2806.94	3280.38	2491.27	2836.06	2891.44	3259.36	3199.15
PPI ($\alpha = -10$) [12]	2412.77	2263.70	2553.10	2806.94	3280.38	2491.27	2836.06	2891.44	3259.36	3199.15
PPI ($\alpha = -50$) [12]	2412.77	2263.70	2553.10	2806.94	3280.38	2491.27	2836.06	2891.44	3259.36	3199.15
PPI ($\alpha = 0$) [12]	2412.77	2263.70	2553.10	2806.94	3280.38	2491.27	2836.06	2891.44	3259.36	3199.15
Stiefel [8]	2365.88	2284.58	3148.88	2859.80	3700.07	2539.10	2753.59	2758.95	3088.10	3436.32
Spectral [6, 34]	1049.60	1372.92	1969.43	1776.11	2113.20	1096.22	1711.94	1770.04	1466.61	2185.79

Table H.25. **Worms (3 objects)**. Objective values per instance.

Method	Instance									
	0	1	2	3	4	5	6	7	8	9
MP-T [41]	-177433.51	-173780.52	-178503.66	-169398.37	-169689.28	-167291.05	-177953.84	-170321.01	-175987.63	-168465.39
GREEDA (best)	**-217522.65**	**-215454.88**	**-215199.17**	**-210777.35**	**-210598.58**	**-210471.70**	**-215829.86**	**-212856.99**	**-221534.89**	**-215555.11**
GREEDA (worst)	-217174.97	-215340.78	-214761.36	-210394.74	-210261.78	-210214.41	-215578.46	-212215.16	-221284.18	-215273.66
Our-C (best)	-216737.76	-214834.72	-214259.10	-209395.67	-209642.54	-209430.28	-215358.17	-211806.00	-221186.58	-214813.92
Our-C (worst)	-216316.70	-213922.96	-213087.39	-207506.61	-207791.70	-208384.29	-214206.09	-209991.58	-220331.84	-214093.14
PPI ($\alpha = -1$) [12]	inf	inf	inf	inf	inf	inf	inf	inf	inf	inf
PPI ($\alpha = -10$) [12]	inf	inf	inf	inf	inf	inf	inf	inf	inf	inf
PPI ($\alpha = -50$) [12]	inf	inf	inf	inf	inf	inf	inf	inf	inf	inf
PPI ($\alpha = 0$) [12]	inf	inf	inf	inf	inf	inf	inf	inf	inf	inf
Stiefel [8]	inf	inf	inf	inf	inf	inf	inf	inf	inf	inf
Spectral [6, 34]	inf	inf	inf	inf	inf	inf	inf	inf	inf	inf

Table H.26. **Worms (4 objects)**. Objective values per instance.

Method	Instance									
	0	1	2	3	4	5	6	7	8	9
MP-T [41]	-342771.63	-332156.36	-330145.63	-335375.92	-341906.33	-338015.52	-340481.69	-331096.25	-335178.64	-348453.65
GREEDA (best)	**-432492.80**	**-423476.43**	**-425229.64**	**-422593.38**	**-431920.67**	**-432793.85**	**-432705.38**	**-426378.82**	**-432107.82**	**-430375.59**
GREEDA (worst)	-432073.67	-421898.06	-423420.34	-420989.75	-430182.44	-432363.66	-432086.18	-425606.18	-431079.91	-428961.96
Our-C (best)	-430647.09	-421341.83	-422762.46	-419303.03	-429888.09	-431220.07	-430554.94	-423626.22	-430758.66	-427982.19
Our-C (worst)	-427726.22	-416653.33	-419032.44	-417471.66	-427884.47	-427672.21	-428571.72	-418944.09	-427830.61	-424674.73
PPI ($\alpha = -1$) [12]	inf	inf	inf	inf	inf	inf	inf	inf	inf	inf
PPI ($\alpha = -10$) [12]	inf	inf	inf	inf	inf	inf	inf	inf	inf	inf
PPI ($\alpha = -50$) [12]	inf	inf	inf	inf	inf	inf	inf	inf	inf	inf
PPI ($\alpha = 0$) [12]	inf	inf	inf	inf	inf	inf	inf	inf	inf	inf
Stiefel [8]	inf	inf	inf	inf	inf	inf	inf	inf	inf	inf
Spectral [6, 34]	inf	inf	inf	inf	inf	inf	inf	inf	inf	inf

Table H.27. **Worms (5 objects)**. Objective values per instance.

Method	Instance									
	0	1	2	3	4	5	6	7	8	9
MP-T [41]	-556779.60	-552338.16	-557822.95	-549829.07	-552029.53	-553601.42	-548888.82	-555146.05	-560515.26	-549732.95
GREEDA (best)	**-719704.60**	**-708814.31**	**-718663.16**	**-717930.04**	**-716916.33**	**-717935.18**	**-711323.30**	**-717136.96**	**-717023.66**	**-716040.58**
GREEDA (worst)	-717091.92	-706369.95	-717586.67	-715652.13	-714795.00	-716531.28	-710229.48	-714687.79	-715276.26	-714827.11
Our-C (best)	-714792.05	-704546.87	-715085.48	-713518.53	-711539.26	-714801.03	-706867.91	-712212.62	-712143.43	-713571.86
Our-C (worst)	-710415.98	-694913.56	-709632.08	-707393.05	-708157.42	-708991.37	-699549.48	-707875.75	-706455.41	-706458.75
PPI ($\alpha = -1$) [12]	inf	inf	inf	inf	inf	inf	inf	inf	inf	inf
PPI ($\alpha = -10$) [12]	inf	inf	inf	inf	inf	inf	inf	inf	inf	inf
PPI ($\alpha = -50$) [12]	inf	inf	inf	inf	inf	inf	inf	inf	inf	inf
PPI ($\alpha = 0$) [12]	inf	inf	inf	inf	inf	inf	inf	inf	inf	inf
Stiefel [8]	inf	inf	inf	inf	inf	inf	inf	inf	inf	inf
Spectral [6, 34]	inf	inf	inf	inf	inf	inf	inf	inf	inf	inf

Table H.28. **Worms (6 objects)**. Objective values per instance.

	Instance									
Method	0	1	2	3	4	5	6	7	8	9
MP-T [41]	-817795.98	-823492.65	-820057.88	-812306.05	-824559.41	-826352.93	-810666.26	-811507.99	-826202.06	-815589.18
GREEDA (best)	**-1067304.57**	**-1077004.01**	**-1075811.41**	**-1076387.19**	**-1074800.27**	**-1064503.77**	**-1073901.60**	**-1074173.43**	**-1073257.14**	**-1071755.00**
GREEDA (worst)	-1063079.91	-1074564.62	-1072893.48	-1072947.77	-1072166.69	-1060482.88	-1070949.03	-1070075.01	-1070690.52	-1068628.77
Our-C (best)	-1059720.15	-1068549.92	-1066918.03	-1068388.95	-1066873.23	-1056711.64	-1065051.97	-1066236.98	-1066991.58	-1061486.98
Our-C (worst)	-1043200.40	-1062631.30	-1059814.54	-1060413.68	-1058084.06	-1048126.78	-1055387.50	-1056446.79	-1060124.02	-1057892.91
PPI ($\alpha = -1$) [12]	inf	inf	inf	inf	inf	inf	inf	inf	inf	inf
PPI ($\alpha = -10$) [12]	inf	inf	inf	inf	inf	inf	inf	inf	inf	inf
PPI ($\alpha = -50$) [12]	inf	inf	inf	inf	inf	inf	inf	inf	inf	inf
PPI ($\alpha = 0$) [12]	inf	inf	inf	inf	inf	inf	inf	inf	inf	inf
Stiefel [8]	inf	inf	inf	inf	inf	inf	inf	inf	inf	inf
Spectral [6, 34]	inf	inf	inf	inf	inf	inf	inf	inf	inf	inf

Table H.29. **Worms (7 objects)**. Objective values per instance.

	Instance									
Method	0	1	2	3	4	5	6	7	8	9
MP-T [41]	-1145530.78	-1112774.86	-1131316.02	-1144590.80	-1155885.55	-1132631.56	-1111282.47	-1125492.38	-1130527.77	-1142180.35
GREEDA (best)	**-1493077.98**	**-1494048.49**	**-1503654.25**	**-1501596.32**	**-1502135.93**	**-1505313.81**	**-1490538.17**	**-1503215.68**	**-1503014.88**	**-1500384.80**
GREEDA (worst)	-1488654.88	-1487775.72	-1499078.04	-1497588.08	-1499376.00	-1498845.55	-1486296.94	-1498616.65	-1498512.79	-1496526.31
Our-C (best)	-1481409.09	-1482692.62	-1489699.43	-1486764.21	-1487636.05	-1490226.47	-1474078.87	-1490370.49	-1490304.31	-1490768.53
Our-C (worst)	-1458616.09	-1472658.34	-1479293.72	-1479554.24	-1480319.13	-1482641.75	-1452803.09	-1479271.55	-1478941.71	-1479651.13
PPI ($\alpha = -1$) [12]	inf	inf	inf	inf	inf	inf	inf	inf	inf	inf
PPI ($\alpha = -10$) [12]	inf	inf	inf	inf	inf	inf	inf	inf	inf	inf
PPI ($\alpha = -50$) [12]	inf	inf	inf	inf	inf	inf	inf	inf	inf	inf
PPI ($\alpha = 0$) [12]	inf	inf	inf	inf	inf	inf	inf	inf	inf	inf
Stiefel [8]	inf	inf	inf	inf	inf	inf	inf	inf	inf	inf
Spectral [6, 34]	inf	inf	inf	inf	inf	inf	inf	inf	inf	inf

Table H.30. **Worms (8 objects)**. Objective values per instance.

	Instance									
Method	0	1	2	3	4	5	6	7	8	9
MP-T [41]	-1503720.31	-1494037.63	-1497606.43	-1490221.38	-1486414.94	-1490000.28	-1503775.54	-1474272.93	-1519216.32	-1520142.14
GREEDA (best)	**-1999990.91**	**-1986823.36**	**-1986189.67**	**-1999684.79**	**-1999285.29**	**-1996646.15**	**-1998268.17**	**-1993949.62**	**-1996852.07**	**-1997574.28**
GREEDA (worst)	-1992823.58	-1984938.43	-1979986.23	-1994474.11	-1991122.07	-1984644.28	-1991278.38	-1988300.54	-1989832.26	-1991442.98
Our-C (best)	-1983115.67	-1969740.36	-1963576.49	-1980485.48	-1975734.54	-1973229.56	-1975999.15	-1976667.68	-1975758.62	-1981135.85
Our-C (worst)	-1963785.96	-1958977.97	-1937320.72	-1968778.22	-1966493.99	-1962392.04	-1963083.38	-1966771.17	-1962385.20	-1960000.60
PPI ($\alpha = -1$) [12]	inf	inf	inf	inf	inf	inf	inf	inf	inf	inf
PPI ($\alpha = -10$) [12]	inf	inf	inf	inf	inf	inf	inf	inf	inf	inf
PPI ($\alpha = -50$) [12]	inf	inf	inf	inf	inf	inf	inf	inf	inf	inf
PPI ($\alpha = 0$) [12]	inf	inf	inf	inf	inf	inf	inf	inf	inf	inf
Stiefel [8]	inf	inf	inf	inf	inf	inf	inf	inf	inf	inf
Spectral [6, 34]	inf	inf	inf	inf	inf	inf	inf	inf	inf	inf

Table H.31. **Worms (9 objects)**. Objective values per instance.

	Instance									
Method	0	1	2	3	4	5	6	7	8	9
MP-T [41]	-1922118.45	-1959860.94	-1923370.31	-1895073.04	-1887813.20	-1902582.11	-1942511.58	-1929583.00	-1949715.28	-1937657.25
GREEDA (best)	**-2566492.95**	**-2565434.05**	**-2563374.38**	**-2564475.34**	**-2581582.12**	**-2559492.63**	**-2582885.43**	**-2560993.89**	**-2583695.60**	**-2559715.21**
GREEDA (worst)	-2555124.31	-2555821.16	-2553940.96	-2554864.99	-2574644.72	-2553518.01	-2575748.07	-2550273.58	-2576919.87	-2550658.06
Our-C (best)	-2536758.10	-2540212.39	-2541863.25	-2532166.33	-2553265.78	-2530686.73	-2552016.58	-2528785.26	-2556311.05	-2534129.29
Our-C (worst)	-2519708.64	-2524335.17	-2515474.85	-2516729.16	-2535255.07	-2505732.98	-2532253.95	-2512300.14	-2544425.96	-2513488.69
PPI ($\alpha = -1$) [12]	inf	inf	inf	inf	inf	inf	inf	inf	inf	inf
PPI ($\alpha = -10$) [12]	inf	inf	inf	inf	inf	inf	inf	inf	inf	inf
PPI ($\alpha = -50$) [12]	inf	inf	inf	inf	inf	inf	inf	inf	inf	inf
PPI ($\alpha = 0$) [12]	inf	inf	inf	inf	inf	inf	inf	inf	inf	inf
Stiefel [8]	inf	inf	inf	inf	inf	inf	inf	inf	inf	inf
Spectral [6, 34]	inf	inf	inf	inf	inf	inf	inf	inf	inf	inf

Table H.32. **Worms (10 objects)**. Objective values per instance.

Method	Instance									
	0	1	2	3	4	5	6	7	8	9
MP-T [41]	-2396051.47	-2399287.91	-2384910.77	-2418170.64	-2379910.84	-2389996.36	-2371701.10	-2437000.82	-2351875.27	-2401302.08
GREEDA (best)	**-3220291.02**	**-3200220.09**	**-3201979.16**	**-3198230.71**	**-3197242.83**	**-3215285.84**	**-3197181.82**	**-3216187.68**	**-3194085.42**	**-3197234.58**
GREEDA (worst)	-3206480.07	-3191007.10	-3189750.06	-3188921.13	-3190156.39	-3206885.10	-3187830.10	-3201529.71	-3188905.18	-3184930.80
Our-C (best)	-3187281.05	-3159168.93	-3166865.99	-3160159.50	-3165609.07	-3183557.18	-3158197.61	-3180563.37	-3155481.63	-3161826.38
Our-C (worst)	-3159966.43	-3145862.89	-3132114.22	-3145457.87	-3140330.42	-3162280.05	-3140378.97	-3153200.24	-3133200.50	-3126978.40
PPI ($\alpha = -1$) [12]	inf	inf	inf	inf	inf	inf	inf	inf	inf	inf
PPI ($\alpha = -10$) [12]	inf	inf	inf	inf	inf	inf	inf	inf	inf	inf
PPI ($\alpha = -50$) [12]	inf	inf	inf	inf	inf	inf	inf	inf	inf	inf
PPI ($\alpha = 0$) [12]	inf	inf	inf	inf	inf	inf	inf	inf	inf	inf
Stiefel [8]	inf	inf	inf	inf	inf	inf	inf	inf	inf	inf
Spectral [6, 34]	inf	inf	inf	inf	inf	inf	inf	inf	inf	inf

Table H.33. **Worms (29 objects)**. Objective values per instance.

Method	Instance									
	0	-	-	-	-	-	-	-	-	-
MP-T [41]	-21748249.19	-	-	-	-	-	-	-	-	-
GREEDA (best)	**-27115925.70**	-	-	-	-	-	-	-	-	-
GREEDA (worst)	-26947041.10	-	-	-	-	-	-	-	-	-
Our-C (best)	-26674999.96	-	-	-	-	-	-	-	-	-
Our-C (worst)	-26232974.36	-	-	-	-	-	-	-	-	-
PPI ($\alpha = -1$) [12]	inf	-	-	-	-	-	-	-	-	-
PPI ($\alpha = -10$) [12]	inf	-	-	-	-	-	-	-	-	-
PPI ($\alpha = -50$) [12]	inf	-	-	-	-	-	-	-	-	-
PPI ($\alpha = 0$) [12]	inf	-	-	-	-	-	-	-	-	-
Stiefel [8]	inf	-	-	-	-	-	-	-	-	-
Spectral [6, 34]	inf	-	-	-	-	-	-	-	-	-

I. List of Mathematical Symbols

Symbol	Definition	Page
V^1, V^2, V^p	Vertex sets/objects that need to be matched	1, 4
$C^{p,q}_{is,jt}$	Single cost factor between objects V^p and V^q	1, 4
V	Union over all vertex sets of the matching problem	2, 4
d	number of objects	2
$[d]$	Set of object indices. Abbreviation for the interval $[1, d] \cap \mathbb{N}$	4
$\overline{E}$	Edge set, specifying all possible matchings	4
$\overline{G}$	d-partite Graph	4
$C(i, s, j, t)$	Cost function	4
$C_{is,jt}$	Matrix representation of cost factor $C(i, s, j, t)$	4
E	Subset of edges constituting a matching	4
$\mathcal{Q}$	Vertex/clique partition; A feasible solution	4
Q	Clique	4
$\mathbb{Q}$	Set of all feasible solutions/vertex partitions	4
$\mathbb{Q}^D$	Set of all feasible solutions/vertex partitions, restriced to set of objects D	4
$D(Q)$	set of object indices covered by clique Q	4
Q^p	single vertex, belonging to object p, contained in clique Q	4
$C(\mathcal{Q})$	Total cost of clique partition/solution $\mathcal{Q}$	4